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A Szpilrajn–Marczewski Type Theorem for Concentration Dimension on Polish Space

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Abstract. Let X be a Polish space. We will prove that

$$\dim_T X = \inf\{\dim_L X' : X' \text{ is homeomorphic to } X\},$$

where $\dim_L X$ and $\dim_T X$ stand for the concentration dimension and the topological dimension of X, respectively.

1 Introduction

In [11] a new concept of dimension of measures, defined by means of the Lévy concentration function (see [7]), has been investigated. This dimension, called concentration dimension, has some important properties. It is related to the mass distribution principle (see [3]), it is relatively easy to calculate and it is also strongly related to the Hausdorff dimension. More precisely, the Hausdorff dimension is greater than or equal to the concentration dimension. Moreover, the Hausdorff dimension of a compact set K is equal to the supremum of the lower concentration dimension of measures μ where the supremum is taken over all probability measures μ such that $\operatorname{supp} \mu \subset K$.

The connection between the Hausdorff dimension and the topological dimension was made evident in the case of $\mathbb{R}^n$ space by V. G. Nöbeling (see [14]) and in a more general setting by Szpilrajn in 1937 (see [9, 16]). Similar connections between the concentration dimension and the topological dimension have been established in the case of locally compact metric spaces [12]. In this paper we will generalize these results to the case of Polish spaces. Note also that the relation between the Hausdorff dimension and the packing dimension was studied in [10] while the generic properties of the concentration dimension have been investigated in [13].

2 Notation, Preliminaries and Auxiliary Results

Throughout this paper (X, ρ) denotes a Polish (*i.e.,* separable complete metric) space. By $B(x, r)$ (resp., $B^o(x, r)$, $S(x, r)$) we denote the closed ball (resp., the open ball and

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the sphere) in X with center at x and radius r. By $\dim_H X$ and $\dim_T X$ we denote the Hausdorff dimension and the topological dimension of X, respectively.

By $\mathcal{B}(X)$ we denote the σ-algebra of Borel subsets of X and by $\mathcal{M}(X)$ the family of all finite Borel measures on X. Moreover, by $\mathcal{M}_1(X)$ we denote the family of all $\mu \in \mathcal{M}(X)$ such that $\mu(X) = 1$ and by $\mathcal{M}_{\leq 1}(X)$ the family of all measures $\mu \in \mathcal{M}(X)$ such that $0 < \mu(X) \leq 1$.

Given a measure $\mu \in \mathcal{M}_1(X)$ we define the *lower* and *upper concentration dimension* of μ by the formulas

$$\underline{\dim}_L \mu = \liminf_{r \to 0} \frac{\log Q_\mu(r)}{\log r},$$

$$\overline{\dim}_L \mu = \limsup_{r \to 0} \frac{\log Q_\mu(r)}{\log r},$$

where

$$Q_\mu(r) = \sup\{\mu(A) : \operatorname{diam} A \leq r, A \in \mathcal{B}(X)\} \quad \text{for } r > 0.$$

Recall that Q_μ is the well-known Lévy concentration function frequently used in the theory of random variables (see [7]).

The *concentration dimension* of X is defined by the formula

$$\dim_L X = \sup_{\mu \in \mathcal{M}_1(X)} \underline{\dim}_L \mu \tag{1}$$

Finally, recall that $\dim_H \mu$ for $\mu \in \mathcal{M}_1(X)$ denotes the Hausdorff dimension of μ, *i.e.*, $\dim_H \mu = \inf\{\dim_H A : A \in \mathcal{B}(X) \text{ and } \mu(A) = 1\}$.

Given an arbitrary function $f\colon A \to [0, \infty]$, where A is a Borel subset of $\mathbb{R}$, we denote by $\mathcal{F}_f$ the set of all Borel measurable functions $\phi\colon A \to [0, \infty]$ such that $\phi(\lambda) \geq f(\lambda)$ for $\lambda \in A$. By the upper integral of f we mean the value

$$\overline{\int_A} f(\lambda)\, d\lambda = \inf_{\phi \in \mathcal{F}_f} \int_A \phi(\lambda)\, d\lambda.$$

The following result can be found in [11].

Proposition 1 *For every $\mu \in \mathcal{M}_1(X)$ we have*

$$\dim_H \mu \geq \underline{\dim}_L \mu.$$

Moreover

$$\dim_H X \geq \dim_L X.$$

The following property of outer measures will be useful for futher considerations.

Lemma 2 *Let μ be a nontrivial outer measure. Then there exists a compact set $K \subset X$ such that $\mu(K) > 0$.*

Proof For $A \subset X$ and $\delta > 0$ define

$$\mu_\delta(A) = \inf\Big\{ \sum_{n=1}^{\infty} \mu(U_n) : A \subset \bigcup_{n=1}^{\infty} U_n, \quad \text{where } U_n \text{ are closed sets with } \operatorname{diam} U_n \le \delta \Big\}$$

and

$$\mu_0(A) = \lim_{\delta \to 0} \mu_\delta(A).$$

It is easy to check that μ_0 is a nontrivial outer metric measure (see [15]). Therefore μ_0 restricted to all Borel sets is a measure. From Ulam's theorem (see [1]) it follows that there exists a compact set K such that $\mu_0(K) > 0$. Hence there exists $\delta_0 > 0$ such that $\mu_{\delta_0}(K) > 0$. Consequently, since K is compact, there exists $x \in K$ such that $\mu(K \cap B(x, \delta_0/2)) > 0$. ■

3 Results

We are in a position to formulate the crucial result for our work. It is similar in spirit to Frostman's lemma which says that if $\mathcal{H}^\alpha(K) > 0$, where $\mathcal{H}^\alpha$ denotes the α-Hausdorff measure and $K \subset \mathbb{R}^d$ is a closed set, then there exists a nonzero Borel measure μ supported on K such that $\mu(D) \le (\operatorname{diam} D)^\alpha$ for all Borel sets D (see [6]). A proof that is much simpler than Frostman's original proof (based on the MaxFlow–MinCut theorem) can be found in [5]. Our approach depends on Banach limits and the Riesz representation theorem (for futher discussion see [8]).

Proposition 3 *Suppose that* $\dim_T X \ge d$, *where* $d \in \mathbb{N} \cup \{0\}$. *Then there exists a Borel measure* $\mu \in \mathcal{M}_{\le 1}(X)$ *such that*

$$\mu(B(x, r)) \le r^d \quad \text{for every } x \in X, r > 0. \tag{2}$$

Proof We use an induction argument with respect to d. For $d = 0$ condition (2) obviously holds for every measure $\mu \in \mathcal{M}_{\le 1}(X)$. Assume that the statement of Proposition 3 holds for $d = k$. We will prove that it holds for $d = k + 1$. By the definition of topological dimension (see [2]) there exists $x_0 \in X$ and $\lambda_0 > 0$ such that $\dim_T S(x_0, \lambda) \ge k$ for every $\lambda \in (0, \lambda_0]$. Without any loss of generality we can assume that $\lambda_0 < 1$. Fix arbitrary $\lambda \in (0, \lambda_0]$ and set $X_\lambda = S(x_0, \lambda)$. By the induction hypothesis there exists a nontrivial Borel measure $\tilde{\mu}_\lambda$ on X_λ such that

$$\tilde{\mu}_\lambda(X_\lambda) \le 1 \quad \text{and} \quad \tilde{\mu}_\lambda(B_\lambda(x, r)) \le r^k \tag{3}$$

for every $x \in X_\lambda$ and $r > 0$, where $B_\lambda(x, r)$ stands for the closed ball in the space X_λ with the center at $x \in X_\lambda$ and radius r.

For every $\lambda \in (0, \lambda_0]$ fix a measure $\tilde{\mu}_\lambda \in \mathcal{M}_{\le 1}(X_\lambda)$ satisfying condition (3) and then define the measure $\mu_\lambda \colon \mathcal{B}(X) \to [0, 1]$ by the formula

$$\mu_\lambda(A) = \tilde{\mu}_\lambda(A \cap X_\lambda) \quad \text{for } A \in \mathcal{B}(X).$$

Clearly $\mu_\lambda \in \mathcal{M}_{\leq 1}(X)$, supp $\mu_\lambda \subset S(x_0, \lambda)$ and

$$\mu_\lambda(B(x,r)) \leq 2^k r^k \quad \text{for every } x \in X, r > 0. \tag{4}$$

Now define the function $\varphi\colon \mathcal{B}(X) \to \mathbb{R}$ by the formula

$$\varphi(A) = \overline{\int}_{(0,\lambda_0)} \mu_\lambda(A)\, d\lambda \quad \text{for } A \in \mathcal{B}(X).$$

Clearly $\varphi(\varnothing) = 0$ and $\varphi(X \setminus B(x_0, \lambda_0)) = 0$. Moreover, from the definition of upper integrals, it follows that

$$\varphi(B(x_0, \lambda_0)) > 0$$

and

$$\varphi\Big(\bigcup_{i=1}^{\infty} A_i\Big) \leq \sum_{i=1}^{\infty} \varphi(A_i) \quad \text{for } A_i \subset X, i \in \mathbb{N}.$$

Now consider the function $\tilde{\mu}\colon 2^X \to \mathbb{R}$ given by

$$\tilde{\mu}(E) = \inf\big\{\varphi(A) : A \in \mathcal{B}(X), A \supset E\big\}.$$

It is routine to see that $\tilde{\mu}$ is an outer measure and $\tilde{\mu}(B(x_0, \lambda_0)) > 0$. By Lemma 2 there exists a compact set $K \subset B(x_0, \lambda_0)$ such that $\tilde{\mu}(K) > 0$. Obviously,

$$\overline{\int}_{(0,\lambda_0)} \mu_\lambda(K)\, d\lambda > 0. \tag{5}$$

For $n \in \mathbb{N}$ and $i \in \{1, \dots, n\}$ we define

$$\alpha_{n,i} = \sup\Big\{\mu_\lambda(K) : \lambda \in \Big(\tfrac{(i-1)\lambda_0}{n}, \tfrac{i\lambda_0}{n}\Big]\Big\}.$$

Let

$$\nu_n = \frac{\lambda_0}{n} \sum_{i=1}^{n} \mu_{n,i} \quad \text{for } n \in \mathbb{N}, \tag{6}$$

where $\mu_{n,i} = \mu_{\lambda_{n,i}}$ with $\lambda_{n,i} \in \Big(\frac{(i-1)\lambda_0}{n}, \frac{i\lambda_0}{n}\Big]$ and such that

$$\mu_{\lambda_{n,i}}(K) \geq \frac{\alpha_{n,i}}{2}. \tag{7}$$

By (6) and (7) we have

$$2\nu_n(K) = \frac{2\lambda_0}{n} \sum_{i=1}^{n} \mu_{n,i}(K) \geq \frac{\lambda_0}{n} \sum_{i=1}^{n} \alpha_{n,i}. \tag{8}$$

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Consider the function $\psi\colon (0,\lambda_0] \to (0,\infty)$ given by

$$\psi(\lambda) = \sum_{i=1}^{n} \alpha_{n,i} \cdot 1_{\left(\frac{(i-1)\lambda_0}{n}, \frac{i\lambda_0}{n}\right]}(\lambda).$$

Clearly ψ is Borel measurable and $\psi(\lambda) \geq \mu_\lambda(K)$ for $\lambda \in (0,\lambda_0]$. Thus by (8), the definition of the upper integral, and (5), we have

$$2\nu_n(K) \geq \frac{\lambda_0}{n} \sum_{i=1}^{n} \alpha_{n,i} = \int_0^{\lambda_0} \psi(\lambda)\, d\lambda \geq \overline{\int}_{(0,\lambda_0)} \mu_\lambda(K)\, d\lambda > 0. \tag{9}$$

Define the positive linear functional $\Lambda\colon C(K) \to \mathbb{R}$ by the formula

$$\Lambda(f) = \mathbb{L}\left(\left(\int_K f\, d\nu_n\right)\right) \quad \text{for } f \in C(K),$$

where $\mathbb{L}$ is a Banach limit (see [4]) and $C(K)$ stands for the space of continuous functions $f\colon K \to \mathbb{R}$. By the Riesz representation theorem there exists a unique measure μ_* such that

$$\Lambda(f) = \int_K f\, d\mu_* \quad \text{for } f \in C(K).$$

From inequality (9) it follows that $\Lambda \neq 0$ and consequently $\mu_* \neq 0$. To finish the proof it suffices to verify that the measure $\mu = \mu_*/2^{k+1}$ satisfies condition (2) with $d = k+1$. To this end, fix an arbitrary $x \in X$ and $r > 0$ and consider the ball $B(x,r)$. For $n \in \mathbb{N}$ define

$$\underline{i}(n) = \min J_n \quad \text{and} \quad \overline{i}(n) = \max J_n,$$

where

$$J_n = \left\{1 \leq i \leq n : B(x,r) \cap S(x_0, \lambda_{n,i}) \neq \varnothing\right\}.$$

If $J_n = \varnothing$, we admit $\underline{i}(n) = \overline{i}(n) = 0$. It can be verified that

$$\frac{\lambda_0}{n}\left(\overline{i}(n) - \underline{i}(n)\right) \leq 2r + \frac{\lambda_0}{n}. \tag{10}$$

Further, by (6) and the construction of measure $\mu_{n,i}$ we have

$$\nu_n(B(x,r)) = \frac{\lambda_0}{n} \sum_{i=1}^{n} \mu_{n,i}(B(x,r)) = \frac{\lambda_0}{n} \sum_{i=\underline{i}(n)}^{\overline{i}(n)} \mu_{n,i}(B(x,r))$$

and now, using (4) and (10) we obtain

$$\nu_n(B(x,r)) \leq \frac{\lambda_0}{n} 2^k r^k \left(\overline{i}(n) - \underline{i}(n) + 1\right) \leq 2^{k+1} r^{k+1} + \frac{\lambda_0}{n} 2^{k+1} r^k. \tag{11}$$

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Fix $\eta \in (0, r)$ and let $f \in C(K)$ with $|f| \leq 1$ be such that $f(y) = 1$ for $y \in B(x, r-\eta) \cap K$ and $f(y) = 0$ for $y \notin B(x, r) \cap K$. Then

$$\mu_*(B(x, r-\eta)) \leq \Lambda(f) = \mathbb{L}\left(\left(\int_K f\, d\nu_n\right)\right) \leq \limsup_{n\to\infty} \nu_n(B(x, r)).$$

Consequently, by (11) we have

$$\mu_*(B(x, r-\eta)) \leq \limsup_{n\to\infty}\left(2^{k+1}r^{k+1} + \frac{\lambda_0}{n}2^{k+1}r^k\right) = 2^{k+1}r^{k+1},$$

and since $\eta \in (0, r)$ and $r > 0$ were arbitrary, we have

$$\mu_*(B(x, r)) \leq 2^{k+1}r^{k+1} \quad \text{for all } r > 0.$$

Keeping in mind the definition of μ we obtain

$$\mu(B(x, r)) \leq r^{k+1}.$$

Since $x \in X$ was arbitrary, the proof is complete. ■

Proposition 4 *Let X be a Polish space with $\dim_T X < \infty$. Then there exists a measure $\mu_* \in \mathcal{M}_1(X)$ such that*

$$\underline{\dim}_L \mu_* \geq \dim_T X.$$

Proof We can assume that $X \neq \varnothing$. Set $d = \dim_T X$. By Proposition 3 there exists a measure $\mu \in \mathcal{M}_{\leq 1}(X)$ such that $\mu(B(x, r)) \leq r^d$ for every $x \in X$ and $r > 0$. Define $\mu_* = \mu/\mu(X)$. Clearly $\mu_* \in \mathcal{M}_1(X)$ and

$$\mu_*(B(x, r)) \leq (\mu(X))^{-1}r^d \quad \text{for every } x \in X, r > 0.$$

Hence

$$Q_{\mu_*}(r) \leq (\mu(X))^{-1}r^d \quad \text{for } r > 0$$

and consequently

$$\underline{\dim}_L \mu_* = \liminf_{r\to 0} \frac{\ln Q_{\mu_*}(r)}{\ln r} \geq \liminf_{r\to 0} \frac{d \ln r - \ln \mu(X)}{\ln r} = d. \quad ■$$

Corollary 5 *Let X be a Polish space. Then*

$$\dim_L X \geq \dim_T X.$$

Proof In the case $\dim_T X < \infty$, the assertion follows immediately from Proposition 4. If $\dim_T X = \infty$, then from Proposition 3 it follows that for every $n \in \mathbb{N}$ there exists $\mu_n \in \mathcal{M}_1(X)$ such that $\mu_n(B(x, r)) \leq r^n$ for arbitrary $x \in X$ and $r > 0$. Hence $\tilde{\mu}_n = \mu_n/\mu_n(X)$ satisfies

$$\underline{\dim}_L \tilde{\mu}_n \geq n$$

and consequently

$$\dim_L X = \infty. \quad ■$$

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Corollary 6 (Szpilrajn, [16]) *Let X be a Polish space. Then*

$$\dim_H X \geq \dim_T X.$$

Proof From inequality $\dim_H X \geq \dim_H \mu$, $\mu \in \mathcal{M}_1(X)$, Proposition 3 and the definition of the concentration dimension of X it follows that $\dim_H X \geq \dim_L X$. From this and Corollary 5 the statement follows. ■

Proposition 7 *If* $\dim_T X = \infty$, *then there exists* $\mu \in \mathcal{M}_1(X)$ *such that* $\dim_H \mu = \infty$.

Proof Let $(\mu_n)_{n\geq 1}$, $\mu_n \in \mathcal{M}_1(X)$, be such that $\underline{\dim}_L \mu_n \geq n$. Such measures exist by virtue of Proposition 3. Define

$$\mu = \sum_{n=1}^{\infty} \mu_n / 2^n$$

and observe that

$$\dim_H \mu \geq \dim_H \mu_n \quad \text{for } n \in \mathbb{N}.$$

Indeed, fix $A \in \mathcal{B}(X)$ such that $\mu(A) = 1$. Clearly $\mu_n(A) = 1$ for arbitrary $n \in \mathbb{N}$. Thus

$$\dim_H A \geq \dim_H \mu_n \geq \underline{\dim}_L \mu_n \quad \text{for } n \in \mathbb{N}$$

and consequently $\dim_H A \geq n$. Since $A \in \mathcal{B}(X)$ with $\mu(A) = 1$ was arbitrary, hence $\dim_H \mu \geq n$. In turn, since $n \in \mathbb{N}$ was arbitrary, it follows that $\dim_H \mu = \infty$. ■

Theorem 8 *Let X be a Polish space. Then*

$$\dim_T X = \inf\{\dim_L X' : X' \text{ is homeomorphic to } X\}.$$

Proof Set $d = \dim_T X$. We can assume that $d < \infty$. By Proposition 4 for every X' homeomorphic to X, we have

$$\dim_L X' \geq d. \tag{12}$$

On the other hand, it follows from [9, Theorem VII.5] that if we let X' range over all the spaces homeomorphic to a given space X, then

$$\inf\{\dim_H X'\} = d. \tag{13}$$

The assertion of Theorem 8 follows immediately from Proposition 1 and relations (12) and (13). ■

Finally we will show that the assumption $\dim_T X < \infty$ in Proposition 4 cannot be dropped. Indeed, we have the following counterexample.

Counterexample Let $((\hat{X}_n, \hat{\rho}_n))_{n\geq 1}$ be a sequence of compact metric spaces such that $\dim_T \hat{X}_n = n$. From Theorem 8 it follows that for every $n \in \mathbb{N}$ there exists a space (X_n, ρ_n) homeomorphic to $(\hat{X}_n, \hat{\rho}_n)$ such that

$$\dim_L X_n \leq n + 1. \tag{14}$$

Without loss of generality we can assume that $\rho_n(x, y) < \frac{1}{2}$ for $x, y \in X_n$. Set

$$X = \bigcup_{n=1}^{\infty} X_n$$

and define $\rho\colon X \times X \to [0, 1]$ by the formula

$$\rho(x, y) = \begin{cases} \rho_n(x, y) & \text{if } x, y \in X_n \text{ for some } n \in \mathbb{N}, \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to check that ρ is a metric on X, $\dim_T X = \infty$ and $\{X_n : n \in \mathbb{N}\}$ is a family of closed disjoint subsets of (X, ρ).

We claim that $\underline{\dim}_L \mu < \infty$ for arbitrary $\mu \in \mathcal{M}_1(X)$. Suppose, for a contradiction, that $\underline{\dim}_L \mu = \infty$ for some $\mu \in \mathcal{M}_1(X)$. Since

$$1 = \mu(X) = \mu\Big(\bigcup_{n=1}^{\infty} X_n\Big) = \sum_{n=1}^{\infty} \mu(X_n),$$

there exists $n_0 \in \mathbb{N}$ such that $\mu(X_{n_0}) > 0$. Set $X_0 = X_{n_0}$ and consider the measure $\hat{\mu} \in \mathcal{M}_1(X_0)$ given by

$$\hat{\mu}(A) = \mu(A)/\mu(X_0) \quad \text{for } A \in \mathcal{B}(X_0).$$

For $x_0 \in X_0$ and $r \in (0, 1)$ let $B_{X_0}(x_0, r)$ and $B_X(x_0, r)$ stand for the balls in X_0 and X, respectively. Clearly

$$\hat{\mu}(B_{X_0}(x_0, r)) = \frac{\mu\big(B_X(x_0, r) \cap X_0\big)}{\mu(X_0)} \leq \frac{\mu\big(B_X(x_0, r)\big)}{\mu(X_0)} \leq \frac{Q_\mu(2r)}{\mu(X_0)}.$$

It follows that $\underline{\dim}_L \hat{\mu} \geq \underline{\dim}_L \mu = \infty$ and consequently $\dim_L X_{n_0} = \infty$, which contradicts (14).

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