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WEIGHTED LORENTZ NORM INEQUALITIES FOR THE ONE-SIDED HARDY-LITTLEWOOD MAXIMAL FUNCTIONS AND FOR THE MAXIMAL ERGODIC OPERATOR

P. ORTEGA SALVADOR

ABSTRACT. In this paper we characterize weighted Lorentz norm inequalities for the one sided Hardy-Littlewood maximal function

$$M^+f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f|.$$

Similar questions are discussed for the maximal operator associated to an invertible measure preserving transformation of a measure space.

1. **Introduction and results.** Let M^+ be the maximal operator defined by

$$M^+f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f|. \tag{1.1}$$

Weighted weak type and Lebesgue-norm inequalities for M^+ have been studied in [9], [6], [5] and [1]. The following characterizations have been proved.

THEOREM A. *Let u, v be nonnegative measurable functions. The operator M^+ is of weak type $(1,1)$ with respect to the measures $u\,dx$ and $v\,dx$ if and only if $(u,v) \in {A_1}^+$, i.e., there is a $C > 0$ such that $M^-u \le Cv$ a.e., where M^- is the left maximal operator defined analogously.*

THEOREM B. *Let $1 < p < \infty$. The operator M^+ is of weak type (p,p) with respect to the measures $u\,dx$ and $v\,dx$ if and only if $(u,v) \in {A_p}^+$, i.e., there is a $C > 0$ such that*

$$\int_a^b u \Big(\int_b^c v^{1-p'} \Big)^{p-1} \le C(c-a)^p$$

for all a, b, $c \in \mathbf{R}$ with $a < b < c$, p' being the conjugate exponent of p.

THEOREM C. *Let $1 < p < \infty$ and let w be a nonnegative measurable function. The following statements are equivalent:*

i) The operator M^+ is of weak type (p,p) with respect to the measure $w\,dx$.

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ii) The operator M^+ is bounded in $L_p(w\,dx)$.
iii) The weight w satisfies $A_p{}^+$.

The purpose of this paper is to extend the above results to weighted $L_{p,q}$ spaces. The Lorentz space $L_{p,q}(v\,dx)$ consists of those functions f for which $\|f\|_{p,q;v} < \infty$, where

$$\|f\|_{p,q;v} = \left(q\int_0^\infty \left(\int_{\{x \mid |f(x)|>y\}} v\,d\mu\right)^{q/p} y^{q-1}\,dy\right)^{1/q} \quad \text{if } 1 \le q < \infty \text{ and}$$

$$\|f\|_{p,\infty;v} = \sup_{y>0} y\left(\int_{\{x \mid |f(x)|>y\}} v\,d\mu\right)^{1/p}.$$

A detailed exposition about the $L_{p,q}$ spaces may be found in [3].

In connection with the weighted Lorentz norm inequalities for the one sided maximal function M^+ we introduce the so called $A_{p,q}{}^+$ condition for pairs (u, v) of nonnegative weights:

DEFINITION 1. A pair (u, v) of nonnegative measurable functions satisfies the condition $A_{p,q}{}^+$ (or belongs to the class $A_{p,q}{}^+$), $1 < p < \infty$ and $1 \le q < \infty$ or $p = q = 1$, if there exists a $C > 0$ such that

$$\|\chi_{(a,b)}\|_{p,q;u}\,\|\chi_{(b,c)}v^{-1}\|_{p',q';v} \le C(c-a)$$

for all $a, b, c \in \mathbf{R}$ with $a < b < c$.

It is clear that when $p = q \ge 1$, $A_{p,q}{}^+$ and $A_p{}^+$ coincide.

The extension of the weak type results (Theorems A and B) can be found in our first theorem.

THEOREM 1. *Let $1 \le q \le p < \infty$. Let (u, v) be a pair of nonnegative measurable functions. The following statements are equivalent:*
i) The pair (u, v) satisfies $A_{p,q}{}^+$.
ii) There is a constant $C > 0$ such that

$$\|M^+ f\|_{p,\infty;u} \le C\|f\|_{p,q;v}$$

for every $f \in L_{p,q}(v)$.

In the single weight function case, $u = v$, and $p, q > 1$, we can solve completely the problem of characterizing the good weights for the weak type inequality of M^+. Moreover, we see that in this case the weighted weak type inequality is equivalent to the Lorentz strong type inequality and that $A_{p,q}{}^+$ becomes equivalent to $A_p{}^+$. These facts are collected in the next theorem.

THEOREM 2. *Let $1 < p < \infty$ and $1 < q \le \infty$. The following statements are equivalent:*
i) The function w satisfies $A_{p,q}{}^+$.
ii) The function w satisfies $A_p{}^+$.

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iii) There is a $C > 0$ *such that*

$$\|M^+f\|_{p,q;w} \leq C\|f\|_{p,q;w}$$

for every $f \in L_{p,q}(w)$.

iv) There is a $C > 0$ *such that*

$$\|M^+f\|_{p,\infty;w} \leq C\|f\|_{p,q;w}$$

for every $f \in L_{p,q}(w)$.

The proofs of Theorems 1 and 2 are included, respectively, in Sections 2 and 3. In the proofs we adapt the arguments of [2] using extensively the techniques of [5]. It is interesting to note that our results represent an extension of the L_p theory of one-sided weights, but we do not use the L_p theory of weights.

The discrete versions of the preceding theorems allows us to improve the results in [7], where the author studied weighted inequalities for the two-sided ergodic maximal operator associated to an invertible measure preserving transformation in order to prove that the uniform boundedness of the averages in a reflexive $L_{p,q}$ space implies a.e. convergence.

Throughout the paper, $w(E)$ denotes the integral of w over the set E, C is a positive constant not necessarily the same at each occurrence and p' is the conjugate exponent of p.

2. **Proof of Theorem 1.** i) $\Rightarrow$ ii). We shall need the following lemma:

LEMMA 1 ([2]). *Let* $1 \leq q \leq p < \infty$ *and* $\{E_j\}_{j\in \mathbf{N}}$ *be a sequence of sets such that*

$$\sum_{j\in N} \chi_{E_j}(x) \leq B.$$

Then,

$$\sum_{j\in N} \|\chi_{E_j} f\|_{p,q}^p \leq B\|f\|_{p,q}^p.$$

The case $p = q = 1$ is solved in [6] and [5]. Suppose $p > 1$ and let f be a nonnegative function supported on a bounded interval. Let $\lambda > 0$ and let us consider the bounded open set $O_\lambda = \{x : M^+f(x) > \lambda\}$. Let (a, b) be a connected component of O_λ. Then, for every $x \in (a, b)$

$$\int_x^b f > \lambda(b - x). \tag{2.1}$$

Let $\{x_k\}$ be the sequence defined by $x_0 = a$ and x_{k+1} be the number in (x_k, b) satisfying

$$\int_{x_k}^{x_{k+1}} f = \int_{x_{k+1}}^b f. \tag{2.2}$$

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The sequence $\{x_k\}$ is strictly increasing and converges to b. Let x_{k-1}, x_k and x_{k+1} three consecutive terms of the sequence $\{x_k\}$. It is clear by (2.2) that

$$\int_{x_{k-1}}^{b} f = 4\int_{x_k}^{x_{k+1}} f. \tag{2.3}$$

It follows from (2.1), (2.3), the Hölder inequality (in Lorentz spaces) and $A_{p,q}{}^{+}$ that

$$\lambda^p \leq \Big(\frac{4}{b-x_{k-1}}\int_{x_k}^{x_{k+1}} f\Big)^p \leq \frac{4^p}{(b-x_{k-1})^p}\|f\chi_{(x_k,x_{k+1})}\|_{p,q;v}^p\,\|\chi_{(x_k,x_{k+1})}v^{-1}\|_{p',q';v}^p$$
$$\leq C\|f\chi_{(x_k,x_{k+1})}\|_{p,q;v}^p\Big(\int_{x_{k-1}}^{x_k} u\Big)^{-1},$$

i.e.,

$$\int_{x_{k-1}}^{x_k} u \leq \frac{C}{\lambda^p}\|f\chi_{(x_k,x_{k+1})}\|_{p,q;v}^p. \tag{2.4}$$

Summing up in k and applying Lemma 1 to the sequence of disjoint sets $\{(x_k,x_{k+1})\}$ yields

$$\int_a^b u \leq \frac{C}{\lambda^p}\sum_k \|\chi_{(x_k,x_{k+1})}\chi_{(a,b)}f\|_{p,q;v}^p \leq \frac{C}{\lambda^p}\|\chi_{(a,b)}f\|_{p,q;v}^p. \tag{2.5}$$

Finally, since inequality (2.5) holds for every connected component of O_λ, a new application of Lemma 1 allows us to write

$$\int_{O_\lambda} u = \sum_j \int_{I_j} u \leq \frac{C}{\lambda^p}\sum_j \|\chi_{I_j}f\|_{p,q;v}^p \leq \frac{C}{\lambda^p}\|f\|_{p,q;v}^p.$$

ii) $\Rightarrow$ i). Let a, b, c be real numbers with $a < b < c$. Let $\{s_n\}$ be an increasing sequence of simple, measurable, nonnegative functions with support of finite v-measure (this ensures $\|s_n\|_{p',q';v} < \infty$) which converges pointwise to $\chi_{(b,c)}v^{-1}$. For every n, there is $f_n \geq 0$ with $\|f_n\|_{p,q;v} = 1$ such that

$$\|s_n\|_{p',q';v} \leq C\int_{\mathbf{R}} f_n s_n v \leq C\int_b^c f_n v^{-1}v = C\int_b^c f_n.$$

Then, if $x \in (a,b)$,

$$M^+f_n(x) \geq \frac{1}{c-x}\int_x^c f_n > \frac{1}{c-a}\int_b^c f_n \geq \frac{C\|s_n\|_{p',q';v}}{c-a}.$$

Therefore

$$(a,b) \subset \Big\{x \in \mathbf{R} \mid M^+f_n(x) > \frac{C\|s_n\|_{p',q';v}}{c-a}\Big\}.$$

Applying inequality ii), we obtain

$$\int_a^b u \leq \frac{C(c-a)^p}{\|s_n\|_{p',q';v}^p}\|f_n\|_{p,q;v},$$

i.e.,

$$\Big(\int_a^b u\Big)^{1/p}\|s_n\|_{p',q';v} \leq C(c-a).$$

Letting n tend to infinity, we get $A_{p,q}{}^{+}$.

REMARK 1. The proof of ii) $\Rightarrow$ i) does not require $q \leq p$.

3. **Proof of Theorem 2.** The proof requires several lemmas.

LEMMA 2. *Let* $1 < p < \infty$ *and* $1 < q \leq \infty$. *If* $w \in A_{p,q}{}^{+}$, *then* $w \in A_{p,1}{}^{+}$.

PROOF. It is immediate because of the inequality

$$\|\chi_{(b,c)} w^{-1}\|_{p',\infty;w} \leq \|\chi_{(b,c)} w^{-1}\|_{p',q';w}.$$

LEMMA 3. *Let* $p > 1$. *Then* $w \in A_{p,1}{}^{+}$ *if and only if there is* $C > 0$ *such that*

$$\frac{|E|}{c-a} \leq C\left(\frac{w(E)}{w(a,b)}\right)^{1/p} \tag{3.1}$$

for all $a, b, c \in \mathbf{R}$ *with* $a < b < c$ *and for every measurable set* E *contained in* (b, c).

PROOF. Let us suppose $w \in A_{p,1}{}^{+}$. Let $a, b, c \in \mathbf{R}$ with $a < b < c$ and let E be a measurable subset of (b, c). Then, Hölder's inequality and $A_{p,1}{}^{+}$ give (3.1) in the following way:

$$\begin{aligned} |E| = \int_E ww^{-1} = \int_b^c \chi_E ww^{-1} &\leq \|\chi_E\|_{p,1;w} \|\chi_{(b,c)} w^{-1}\|_{p',\infty;w} \\ &\leq C\|\chi_E\|_{p,1;w} \frac{c-a}{\|\chi_{(a,b)}\|_{p,1;w}} = C\left(\frac{w(E)}{w(a,b)}\right)^{1/p} (c-a). \end{aligned}$$

Conversely, suppose we have (3.1) and let $a, b, c \in \mathbf{R}$ with $a < b < c$, $y > 0$ and $E_y = \{x \in (b, c) \mid w^{-1}(x) > y\}$. Then

$$yw(E_y) = \int_{E_y} yw \leq \int_{E_y} w^{-1} w = |E_y| \leq C(c-a)\left(\frac{w(E_y)}{w(a,b)}\right)^{1/p},$$

i.e.,

$$w(E_y) \leq C \frac{(c-a)^{p'}}{y^{p'}\big(w(a,b)\big)^{p'/p}}.$$

If we have taken into account that $\|\chi_{(a,b)}\|_{p,1;w} = \big(w(a,b)\big)^{1/p}$, the above inequality can be written as

$$\|\chi_{(a,b)}\|_{p,1;w}^{p'} y^{p'} \int_{\{x \in (b,c) \mid w^{-1}(x) > y\}} w \leq C(c-a)^{p'}. \tag{3.2}$$

Since (3.2) holds for every $y > 0$, taking the supremum over $y > 0$ we obtain $A_{p,1}{}^{+}$.

REMARK 2. Condition $A_{p,1}{}^{+}$ for a weight w is also studied by the author in [8], where it is shown that it is equivalent to the restricted weak type (p, p) inequality for M^{+} with respect to the measure $w\,dx$.

LEMMA 4. *Let $p > 1$, $1 \leq q \leq \infty$ and $w \in A_{p,q}{}^{+}$. There are $C > 0$ and $\beta > 0$ such that*

$$w\big(\{x \in (a,b) \mid w^{-1}(x) > \beta\lambda\}\big) \geq Cw(a,b) \tag{3.3}$$

for every $\lambda > 0$ and every bounded interval (a,b) with

$$\lambda = \frac{b-a}{w(a,b)} \leq \frac{b-x}{w(x,b)} \quad \textit{for every } x \in (a,b). \tag{3.4}$$

PROOF. Let $\lambda > 0$, $0 < \beta < 1/12$ and (a,b) satisfying (3.4). Let $\{x_i\}$ be the sequence defined by $x_0 = a$ and x_{i+1} be the middle point of (x_i, b).

Let, for $i \geq 1$, $E_i' = \big\{x \in (x_i, x_{i+1}) \mid w^{-1}(x) \leq \frac{4\beta(x_{i+1}-x_i)}{w(x_{i-1},b)}\big\}$. For every $i \geq 1$ we have by definition of E_i',

$$\frac{|E_i'|}{x_{i+1}-x_{i-1}} = \int_{E_i'} \frac{1}{x_{i+1}-x_{i-1}}\,dx \leq \int_{E_i'} \frac{4\beta w(x)}{w(x_{i-1},b)}\,dx \leq 4\beta. \tag{3.5}$$

Let $E_i = (x_i, x_{i+1}) - E_i'$. Then, (3.5) gives

$$\begin{aligned} |E_i| &= x_{i+1} - x_i - |E_i'| \geq x_{i+1} - x_i - 4\beta(x_{i+1} - x_{i-1}) \\ &= x_{i+1} - x_i - 12\beta(x_{i+1} - x_i) = (1 - 12\beta)(x_{i+1} - x_i). \end{aligned} \tag{3.6}$$

Since $w \in A_{p,1}{}^{+}$, by Lemma 3 we have

$$\frac{|E_i|}{x_{i+1}-x_{i-1}} \leq C\left(\frac{w(E_i)}{w(x_{i-1},x_i)}\right)^{1/p} \tag{3.7}$$

for $i \geq 1$, and (3.7) together with (3.6) and the definition of $\{x_i\}$ give

$$w(E_i) \geq C(1-12\beta)^p w(x_{i-1},x_i). \tag{3.8}$$

Summing for $i \geq 1$ we obtain

$$\sum_{i=1}^{\infty} w(E_i) \geq C(1-12\beta)^p w(a,b). \tag{3.9}$$

On the other hand,

$$\bigcup_{i\geq 1} E_i \subset \bigcup_{i\geq 1} \{x \in (x_i, x_{i+1}) \mid w^{-1}(x) > \beta\lambda\} = \{x \in (x_1, b) \mid w^{-1}(x) > \beta\lambda\}.$$

This relation and (3.9) give

$$w\big(\{x \in (x_1,b) \mid w^{-1}(x) > \beta\lambda\}\big) \geq \sum_{i=1}^{\infty} w(E_i) \geq C(1-12\beta)^p w(a,b),$$

which implies (3.3).

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LEMMA 5. *Let* $1 < p < \infty$, $1 < q < \infty$ *and* $w \in A_{p,q}{}^{+}$. *There is a* $C > 0$ *such that*

$$\|\chi_{(a,b)} w^{-1}\|_{p',q';w} \le C(b-a)^{1/p'} \big(M_w^{+}(\chi_{(a,b)} w^{-1})(a)\big)^{1/p} \tag{3.10}$$

for every bounded interval (a, b).

PROOF. Let (a, b) be a bounded interval and let us consider the sequence $\{x_k\}$, where $x_0 = b$ and x_{k+1} is the middle point of (a, x_k). For the estimation of the (p', q')-norm of the function $\chi_{(a,b)} w^{-1}$ we will use duality and the fact that the sequence $\{x_k\}$ provides a partition of (a, b) into disjoint intervals. More precisely, there is $f \ge 0$ with $\|f\|_{p,q;w} \le 1$ such that

$$\|\chi_{(a,b)} w^{-1}\|_{p',q';w} \le C \int_a^b f w^{-1} w = C \sum_{k=0}^{\infty} \int_{x_{k+1}}^{x_k} f w^{-1} w. \tag{3.11}$$

Hölder's inequality, the condition $A_{p,q}{}^{+}$ and the definition of the sequence $\{x_k\}$ allow us to dominate the right-hand side of (3.11) in the following way:

$$\begin{aligned}\sum_{k=0}^{\infty} \int_{x_{k+1}}^{x_k} f w^{-1} w &\le C \sum_{k=0}^{\infty} \|f\|_{p,q;w} \, \|\chi_{(x_{k+1},x_k)} w^{-1}\|_{p',q';w} \le C \sum_{k=0}^{\infty} \frac{x_k - a}{\big(w(a, x_{k+1})\big)^{1/p}} \\ &= C \sum_{k=0}^{\infty} \Big(\frac{x_k - a}{w(a, x_{k+1})}\Big)^{1/p} (x_k - a)^{1/p'} \\ &= 2C \sum_{k=0}^{\infty} \Big(\frac{x_{k+1} - a}{w(a, x_{k+1})}\Big)^{1/p} (x_{k+1} - a)^{1/p'}.\end{aligned} \tag{3.12}$$

Taking into account the definitions of $M_w{}^{+}(\chi_{(a,b)} w^{-1})(a)$ and $\{x_k\}$, the last term in (3.12) is smaller than

(3.13)

$$\begin{aligned} C\big(M_w^{+}(\chi_{(a,b)} w^{-1})(a)\big)^{1/p} \sum_{k=0}^{\infty} (x_{k+1} - a)^{1/p'} \\ = C\big(M_w^{+}(\chi_{(a,b)} w^{-1})(a)\big)^{1/p} (b-a)^{1/p'} \sum_{k=0}^{\infty} 2^{-\frac{k+1}{p'}} \\ = C(b-a)^{1/p'} \big(M_w^{+}(\chi_{(a,b)} w^{-1})(a)\big)^{1/p},\end{aligned}$$

and the lemma is proved.

LEMMA 6. *Let* $1 < p < \infty$, $1 < q \le p$ *and* $w \in A_{p,q}{}^{+}$. *Then, there are constants* $C > 0$ *and* $\beta > 0$ *such that*

$$\|\chi_{E_y} w^{-1}\|_{p',q';w} \le C y \big(w(E_{\beta y})\big)^{1/p'} \tag{3.14}$$

for every bounded interval (a, b) *and for every* $y > 0$ *with* $y > M_w{}^{+}(w^{-1}\chi_{(a,b)})(a)$, *where* $E_y = \{x \in (a, b) \mid w^{-1}(x) > y\}$.

PROOF. Let $y > M_w{}^{+}(w^{-1}\chi_{(a,b)})(a)$. By duality,

$$\|\chi_{E_y} w^{-1}\|_{p',q';w} \le C \int_{\{x \in (a,b) \mid w^{-1}(x) > y\}} f w^{-1} w, \tag{3.15}$$

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where $f \geq 0$ and $\|f\|_{p,q;w} \leq 1$. Let $O_y = \{x \in (a,b) \mid M_w{}^{+}(w^{-1}\chi_{(a,b)})(x) > y\}$. Since E_y is contained in O_y, from (3.15) follows

$$\|\chi_{E_y} w^{-1}\|_{p',q';w} \leq C \int_{O_y} fw^{-1}w. \tag{3.16}$$

Let $\{I_j\}$ be the sequence of the connected components of O_y. By the choice of y, the number a cannot be the infimum of any I_j. Every $I_j = (a_j, b_j)$ satisfies

$$y = \frac{b_j - a_j}{w(a_j, b_j)} \leq \frac{b_j - x}{w(x, b_j)}$$

for every $x \in (a_j, b_j)$. Let us fix $I_j = (a_j, b_j)$. Then, Hölder's inequality, Lemma 5 and the fact that $a_j \notin O_y$ give

$$\begin{aligned} \int_{a_j}^{b_j} fw^{-1}w &\leq C\|f\chi_{(a_j,b_j)}\|_{p,q;w} \, \|\chi_{(a_j,b_j)}w^{-1}\|_{p',q';w} \\ &\leq C\|f\chi_{(a_j,b_j)}\|_{p,q;w}(b_j - a_j)^{1/p'}\big(M_w^{+}(\chi_{(a_j,b_j)}w^{-1})(a_j)\big)^{1/p} \\ &\leq C\|f\chi_{(a_j,b_j)}\|_{p,q;w}(b_j - a_j)^{1/p'}y^{1/p}. \end{aligned} \tag{3.17}$$

If we have taken into account that $(b_j - a_j)\big(w(a_j, b_j)\big)^{-1} = y$, then last term of (3.17) equals

$$Cy\|f\chi_{(a_j,b_j)}\|_{p,q;w}\big(w(a_j, b_j)\big)^{1/p'}.$$

Therefore we have shown

$$\int_{a_j}^{b_j} fw^{-1}w \leq Cy\|f\chi_{(a_j,b_j)}\|_{p,q;w}\big(w(a_j, b_j)\big)^{1/p'}. \tag{3.18}$$

From (3.16) and (3.18) we can write

$$\|\chi_{E_y}w^{-1}\|_{p',q';w} \leq Cy\sum_j \|f\chi_{(a_j,b_j)}\|_{p,q;w}\big(w(a_j, b_j)\big)^{1/p'}. \tag{3.19}$$

Finally, if we apply Hölder's inequality with exponents p and p' to the last sum and Lemmas 1 and 4, we obtain

$$\begin{aligned} \|\chi_{E_y}w^{-1}\|_{p',q';w} &\leq Cy\Big(\sum_j \|f\chi_{(a_j,b_j)}\|_{p,q;w}^p\Big)^{1/p}\Big(\sum_j w(a_j, b_j)\Big)^{1/p'} \\ &\leq Cy\|f\|_{p,q;w}\Big(\sum_j w(a_j, b_j)\Big)^{1/p'} \\ &\leq Cy\Big(\sum_j w\big(\{x \in (a_j, b_j) \mid w^{-1}(x) > \beta y\}\big)\Big)^{1/p'} \\ &\leq Cy\Big(w\big(\{x \in (a,b) \mid w^{-1}(x) > \beta y\}\big)\Big)^{1/p'}. \end{aligned} \tag{3.20}$$

LEMMA 7. *Let $1 < q \leq p < \infty$ and $w \in A_{p,q}{}^{+}$. There are $C > 1$ and $\delta > 0$ such that*

$$\|\chi_{(a,b)}w^{-1}\|_{r',s';w}^{s'} \leq C\big(M_w^{+}(w^{-1}\chi_{(a,b)})(a)\big)^{\delta}\|\chi_{(a,b)}w^{-1}\|_{p',q';w}^{q'} \tag{3.21}$$

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for every bounded interval (a,b), *where* $r' = p' + p'\delta/q'$ *and* $s' = q' + \delta$.

PROOF. Let $y > 0, \delta > 0$ and $E_y = \{x \in (a,b) \mid w^{-1}(x) > y\}$. An easy computation shows

$$\delta \int_0^\infty y^{\delta-1} \|\chi_{E_y} w^{-1}\|_{p',q';w}^{q'} \, dy = \|\chi_{(a,b)} w^{-1}\|_{r',s';w}^{s'}. \tag{3.22}$$

On the other hand, the preceding lemma and $q'/p' = s'/r'$ give

$$\begin{aligned} &\int_{M_w^+(w^{-1}\chi_{(a,b)})(a)}^\infty y^{\delta-1} \|\chi_{E_y} w^{-1}\|_{p',q';w}^{q'} \\ &\leq C \int_{M_w^+(w^{-1}\chi_{(a,b)})(a)}^\infty y^{s'-1} \Big(w\big(\{x \in (a,b) \mid w^{-1}(x) > \beta y\}\big)\Big)^{q'/p'} \, dy \\ &\leq \frac{C}{s'} s' \int_0^\infty y^{s'-1} \Big(w\big(\{x \in (a,b) \mid w^{-1}(x) > y\}\big)\Big)^{q'/p'} = \frac{C}{s'} \|\chi_{(a,b)} w^{-1}\|_{r',s';w}^{s'}. \end{aligned} \tag{3.23}$$

It follows from (3.23) that

(3.24)
$$\int_0^\infty y^{\delta-1} \|\chi_{E_y} w^{-1}\|_{p',q';w}^{q'} \leq \frac{C}{s'} \|\chi_{(a,b)} w^{-1}\|_{r's';w}^{s'} + \frac{1}{\delta} \big(M_w^+(\chi_{(a,b)} w^{-1})(a)\big)^\delta \|\chi_{(a,b)} w^{-1}\|_{p',q';w}^{q'}.$$

Now (3.22) and (3.24) give

$$\frac{1}{\delta} \|\chi_{(a,b)} w^{-1}\|_{r',s';w}^{s'} \leq \frac{1}{\delta} \big(M_w^+(\chi_{(a,b)} w^{-1})(a)\big)^\delta \|\chi_{(a,b)} w^{-1}\|_{p',q';w}^{q'} + \frac{C}{s'} \|\chi_{(a,b)} w^{-1}\|_{r',s';w}^{s'},$$

i.e.,

$$\|\chi_{(a,b)} w^{-1}\|_{r',s';w}^{s'} \leq \frac{1}{\delta} \left(\frac{1}{\frac{1}{\delta} - \frac{C}{s'}} \right) \big(M_w^+(\chi_{(a,b)} w^{-1})(a)\big)^\delta \|\chi_{(a,b)} w^{-1}\|_{p',q';w}^{q'},$$

which is (3.21) taking δ small enough.

The information provided by Lemmas 5 and 7 can be summarized as:

LEMMA 8. *Let* $1 < q \leq p < \infty$ *and* $w \in A_{p,q}{}^+$. *There are constants* $C > 0$ *and* $\delta > 0$ *such that*

$$\|\chi_{(a,b)} w^{-1}\|_{r',s';w} \leq C(b-a)^{1/r'} \big(M_w^+(\chi_{(a,b)} w^{-1})(a)\big)^{1/r} \tag{3.25}$$

for every bounded interval (a,b), *where* $r' = p' + p'\delta/q'$ *and* $s' = q' + \delta$.

PROOF. The Lemmas 5 and 7 give immediately

$$\|\chi_{(a,b)} w^{-1}\|_{r',s';w}^{s'} \leq C(b-a)^{q'/p'} \big(M_w^+(\chi_{(a,b)} w^{-1})(a)\big)^{\delta + \frac{q'}{p}} \tag{3.26}$$

and (3.25) follows from (3.26) taking into account that $q'/p' = s'/r'$ and $\delta + q'/p = s'/r$.

LEMMA 9. *Let* $1 < q \leq p < \infty$ *and* $w \in A_{p,q}{}^+$. *Let* δ, r *and* s *be the real numbers associated to* w *by Lemma 8. Then* $w \in A_{r,s}{}^+$.

PROOF. Let $a, b, c \in \mathbf{R}$ with $a < b < c$ and let us consider the finite decreasing sequence $x_0 = b > x_1 > \cdots > x_N \geq a = x_{N+1}$ such that

$$\|\chi_{(x_k,c)} w^{-1}\|_{r',s';w} = 2^k \|\chi_{(b,c)} w^{-1}\|_{r',s';w} \tag{3.27}$$

if $k = 0, 1, \ldots, N$ and

$$\|\chi_{(a,c)} w^{-1}\|_{r',s';w} \le 2^{N+1} \|\chi_{(b,c)} w^{-1}\|_{r',s';w}.$$

The inequality (3.27), Lemma 8 and Hölder's inequality give

$$
\begin{aligned}
\int_a^b w \frac{\|\chi_{(b,c)} w^{-1}\|^r_{r',s';w}}{(c-a)^{r/r'}} &= \sum_{k=0}^{N} 2^{-kr} \int_{x_{k+1}}^{x_k} w \frac{\|\chi_{(x_k,c)} w^{-1}\|^r_{r',s';w}}{(c-a)^{r/r'}} \\
&\le \sum_{k=0}^{N} 2^{-kr} \int_{x_{k+1}}^{x_k} w(y) \frac{\|\chi_{(y,c)} w^{-1}\|^r_{r',s';w}}{(c-y)^{r/r'}}\, dy \\
&\le C \sum_{k=0}^{N} 2^{-kr} \int_{x_{k+1}}^{x_k} w(y) M_w^+(\chi_{(y,c)} w^{-1})(y)\, dy \\
&\le C \sum_{k=0}^{N} 2^{-kr} \int_{x_{k+1}}^{x_k} w(y) M_w^+(\chi_{(x_{k+1},c)} w^{-1})(y)\, dy \\
&\le C \sum_{k=0}^{N} 2^{-kr} \|\chi_{(x_{k+1},x_k)}\|_{r,s;w} \|M_w^+(\chi_{(x_{k+1},c)} w^{-1})\|_{r',s';w} \\
&\le C \|\chi_{(a,b)}\|_{r,s;w} \sum_{k=0}^{N} 2^{-kr} \|M_w^+(\chi_{(x_{k+1},c)} w^{-1})\|_{r',s';w}.
\end{aligned}
\tag{3.28}
$$

The boundedness of ${M_w}^+$ in $L_{r',s'}(w)$ and the definition of the finite sequence $\{x_k\}$ allow us to dominate the last term of (3.28) by

$$
\begin{aligned}
C\|\chi_{(a,b)}\|_{r,s;w} &\sum_{k=0}^{N} 2^{-kr} \|\chi_{(x_{k+1},c)} w^{-1}\|_{r',s';w} \\
&\le C\|\chi_{(a,b)}\|_{r,s;w} \sum_{k=0}^{N} 2^{-kr+k+1} \|\chi_{(b,c)} w^{-1}\|_{r',s';w} \\
&\le C\|\chi_{(a,b)}\|_{r,s;w} \|\chi_{(b,c)} w^{-1}\|_{r',s';w}.
\end{aligned}
$$

We have shown

$$w(a,b) \frac{\|\chi_{(b,c)} w^{-1}\|^r_{r',s';w}}{(c-a)^{r/r'}} \le C\big(w(a,b)\big)^{1/r} \|\chi_{(b,c)} w^{-1}\|_{r',s';w},$$

i.e.,

$$\big(w(a,b)\big)^{1/r} \|\chi_{(b,c)} w^{-1}\|_{r',s';w} \le C(c-a),$$

which is the expression of ${A_{r,s}}^+$.

We are ready to prove Theorem 2.

The statement iv) implies i) by Theorem 1 (see Remark 1) and the implication iii) $\Rightarrow$ iv) is obvious. To prove ii) $\Rightarrow$ iii) we use an argument of interpolation: if $w \in {A_p}^+$, by Lemma 9, there is $\varepsilon > 0$ such that $w \in {A_{p-\varepsilon}}^+$; by Theorem 1 this means that M^+ applies $L_{p-\varepsilon}(w)$ in $L_{p-\varepsilon,\infty}(w)$ and Marcinkiewicz's theorem gives iii). Finally, to prove i) $\Rightarrow$ ii) let us assume that $w \in {A_{p,q}}^+$. We can reduce the problem to the case $q \le p$, since ${A_{p,q}}^+$ implies ${A_{p,s}}^+$ for every $s \le q$. By Lemma 9, there exist r and s with $r < p$, $s \le r$ and $w \in {A_{r,s}}^+$. Then, by Theorem 1, M^+ is bounded from $L_{r,s}(w\,dx)$ to $L_{r,\infty}(w\,dx)$ and interpolation gives that M^+ is bounded in $L_p(w\,dx)$, which implies immediately that $w \in {A_p}^+$.

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4. **Weights for the ergodic maximal operator and pointwise convergence of the averages of functions in $L_{p,q}$.** Let (X, M, μ) be a σ-finite measure space and let $T: X \longrightarrow X$ be an invertible measure preserving transformation with measurable inverse. Let $M_T{}^+$ be the ergodic maximal operator defined by

$$M_T^+ f(x) = \sup_{m \geq 0} |T_{0,m} f(x)|,$$

where

$$T_{0,m} f(x) = \frac{1}{m+1} \sum_{i=0}^{m} f(T^i x).$$

If u is a nonnegative measurable function defined on X, u^x will denote the function on $\mathbf{Z}$ defined by $u^x(i) = u(T^i x)$.

DEFINITION 2. A pair (u, v) of positive measurable functions on X satisfies the condition $A_{p,q}{}^+(T)$ (or belongs to the class $A_{p,q}{}^+(T)$), $1 < p < \infty$ and $1 \leq q \leq \infty$ or $p = q = 1$, if there exists $C > 0$ such that

$$\|\chi_{[0,k]}\|_{p,q;u^x} \cdot \|\chi_{[k,m]}(v^x)^{-1}\|_{p',q';v^x} \leq C(m+1)$$

for all $k, m \in \mathbf{N}$ with $0 \leq k \leq m$ and a.e. $x \in X$.

THEOREM 3. *Let $1 \leq q \leq p < \infty$ and u, v be positive measurable functions. The following statements are equivalent:*

i) $\|M_T{}^+ f\|_{p,\infty;u} \leq C\|f\|_{p,q;v}$

ii) $\sup_{m \geq 0} \|T_{0,m} f\|_{p,\infty;u} \leq C\|f\|_{p,q;v}$

iii) The pair (u, v) satisfies $A_{p,q}{}^+(T)$.

PROOF. The proof is essentially the same as the proof of Theorem 1 in [7]. The implication i) $\Rightarrow$ ii) is clear and iii) $\Rightarrow$ i) follows by transference from the discrete version of Theorem 1. The proof ii) $\Rightarrow$ iii) is the larger one and we only sketch it. We shall need the following lemma which is slightly different from the lemma which appears in [7].

LEMMA 10. *Let s, $k \in \mathbf{N}$ with $s \leq k$ and let B be a measurable set. For every $x \in B$ and $n \in \mathbf{Z}$, let $H_n{}^x = \{i \in [s,k] \mid v^{-1}(T^i x) > 3^n\}$. Let $\mathbf{A}$ be the collection of all the decreasing sequences in $\mathbf{Z} \cup \{-\infty\}$ with no more than 2^{k-s+1} different terms and with at least one term in $\mathbf{Z}$. If $\alpha = \{a_n\} \in \mathbf{A}$, let A_α be the following set:*

$$A_\alpha = \{x \in B \mid H_n{}^x = \emptyset \text{ if } a_n = -\infty \text{ and } 2^{a_n} < \sum_{i \in H_n{}^x} v(T^i x) \leq 2^{a_n+1} \text{ if } a_n \neq -\infty\}.$$

Then $\{A_\alpha\}_{\alpha \in \mathbf{A}}$ is countable, its elements are pairwise disjoint and $B = \bigcup_{\alpha \in \mathbf{A}} A_\alpha$.

PROOF OF THE LEMMA. It is clear that $\mathbf{A}$ is countable and that $\alpha \neq \beta$ in $\mathbf{A}$ implies $A_\alpha \cap A_\beta = \emptyset$. To see that $B = \bigcup_{\alpha \in \mathbf{A}} A_\alpha$, let $x \in B$ and, for every $n \in \mathbf{Z}$ with $H_n{}^x \neq \emptyset$, let a_n be the only integer such that

$$2^{a_n} < \sum_{i \in H_n{}^x} v(T^i x) \leq 2^{a_n+1}.$$

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If ${H_n}^x = \emptyset$, let $a_n = -\infty$. Then, the sequence $\alpha = \{a_n\}$ is decreasing (since ${H_{n-1}}^x \supset {H_n}^x$) with no more than 2^{k-s+1} different terms (since there are no more than 2^{k-s+1} different ${H_n}^x$) and $x \in A_\alpha$.

ii) $\Rightarrow$ iii). Let $r, k \in \mathbf{N}$ with $r \leq k$ and let $\{B_i\}$ be the sequence of measurable sets associated to X and k by Lemma 2.10 in [4]. Let us fix B_i and suppose $s(i) = k$. By Lemma 10, with $s = r$, $B_i = \bigcup_{\alpha \in \mathrm{A}} A_\alpha$. Let us fix A_α and consider, for each $(n_0, n_1, \ldots, n_k) \in \mathbf{Z}^{k+1}$, the set

$$H_{n_0,n_1,\ldots,n_k} = \{x \in A_\alpha \mid 2^{n_i} < v(T^i x) \leq 2^{n_i+1}, i = 0, 1, \ldots, k\}.$$

It is clear that the sets $H_{n_0,n_1,\ldots n_k}$ are measurable, their union is A_α and they are pairwise disjoint. Let us fix $H_{n_0,n_1,\ldots,n_k}$ and let A be a measurable subset of $H_{n_0,n_1,\ldots,n_k}$. Let $R = \bigcup_{0 \leq j \leq k} T^j A$, $R_1 = \bigcup_{0 \leq j \leq r} T^j A$ and $R_2 = \bigcup_{r \leq j \leq k} T^j A$. First we prove the inequality

$$\|\chi_{R_1}\|_{p,q;u} \|\chi_{R_2} v^{-1}\|_{p',q';v} \leq C\mu(R) \tag{4.1}$$

with C independent of k, r and A. To prove (4.1), we see first that

$$\|\chi_{R_2} v^{-1}\|_{p',q';v} \leq C\mu(A)^{1/p'} \|\chi_{[r,k]} w^{-1}\|_{p',q';w}, \tag{4.2}$$

where w is defined over $\mathbf{Z}$ by $w(j) = 2^{n_j}\chi_{[r,k]}$ and the (p', q')-norm of the right hand side is a norm in the integers. Then we use an argument of duality: there exists $w' \geq 0$ with $\|w'\|_{p,q;w} = 1$ such that

$$C\|\chi_{[r,k]} w^{-1}\|_{p',q';w} \leq \sum_{j=r}^{k} w'(j). \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$\|\chi_{R_2} v^{-1}\|_{p',q';v} \leq C\mu(A)^{1/p'} \sum_{j=r}^{k} w'(j). \tag{4.4}$$

Let f be the function defined on X by

$$f(x) = \sum_{j=r}^{k} w'(j)\chi_{T^j A}(x).$$

The function f satisfies $\|f\|_{p,q;v} \leq C\mu(A)^{1/p}$ and

$$R_1 \subset \left\{x \in X \mid |T_{0,k} f(x)| > C\frac{\sum_{j=r}^{k} w'(j)}{k+1}\right\},$$

so that our hypothesis about ${M_T}^+$ yields

$$u(R_1) \leq C\frac{(k+1)^p}{\left(\sum_{j=r}^{k} w'(j)\right)^p}\mu(A). \tag{4.5}$$

This inequality together with (4.4) give:

$$u(R_1) \leq C\frac{(k+1)^p}{\|\chi_{R_2} v^{-1}\|_{p',q';v}^p}\mu(A)^p. \tag{4.6}$$

Raising to $1/p$ and taking into account that $(k+1)\mu(A) = \mu(R)$, we obtain (4.1).

The inequality (4.1) can be written as follows:

(4.7)

$$\Big(\int_A \sum_{j=0}^{r} u(T^j x)\,d\mu\Big)^{1/p}\Big(q'\int_0^\infty\Big(\int_A \sum_{\{j\in[r,k]\,|\,v^{-1}(T^j x)>y\}} v(T^j x)\Big)^{q'/p'} y^{q'-1}\,dy\Big)^{1/q'} \le C(k+1)\mu(A).$$

From (4.7) we shall obtain

(4.8)

$$\Big(\int_A \sum_{j=0}^{r} u(T^j x)\,d\mu\Big)^{p'/p}\int_A\Big(\int_0^\infty\Big(\sum_{\{j\in[r,k]\,|\,v^{-1}(T^j x)>y\}} v(T^j x)\Big)^{q'/p'} y^{q'-1}\,dy\Big)^{p'/q'}\,d\mu \le C(k+1)^{p'}\mu(A)^{p'},$$

and then, since A is an arbitrary measurable subset of $H_{n_0,n_1,\ldots,n_k}$, the union of the H's is A_α and the union of the A_α's is B_i, we shall get

$$\Big(\sum_{j=0}^{r} u(T^j x)\Big)^{1/p}\Big(\int_0^\infty q'\Big(\sum_{\{j\in[r,k]\,|\,v^{-1}(T^j x)>y\}} v(T^j x)\Big)^{q'/p'} y^{q'-1}\,dy\Big)^{1/q'} \le C(k+1)$$

for almost every $x \in B_i$, *i.e.*,

$$\|\chi_{[0,r]}\|_{p,q;u^x}\|\chi_{[r,k]}(v^x)^{-1}\|_{p',q';v^x} \le C(k+1)$$

for almost every $x \in B_i$.

Let us consider the second factor on the left-hand side of (4.8) and let us dominate it by the corresponding in (4.7):

$$\begin{aligned}
&\int_A\Big(\int_0^\infty\Big(\sum_{\{j\in[r,k]\,|\,v^{-1}(T^j x)>y\}} v(T^j x)\Big)^{q'/p'} y^{q'-1}\,dy\Big)^{p'/q'}\,d\mu\\
&\quad\le \int_A\Big(\sum_{n=-\infty}^{\infty}\int_{3^n}^{3^{n+1}}\Big(\sum_{\{j\in[r,k]\,|\,v^{-1}(T^j x)>3^n\}} v(T^j x)\Big)^{q'/p'} y^{q'-1}\,dy\Big)^{p'/q'}\,d\mu\\
&\quad= C\int_A\Big(\sum_{n=-\infty}^{+\infty}\int_{3^{n-1}}^{3^n}\Big(\sum_{\{j\in[r,k]\,|\,v^{-1}(T^j x)>3^n\}} v(T^j x)\Big)^{q'/p'} y^{q'-1}\,dy\Big)^{p'/q'}\,d\mu\\
&\quad\le C\int_A\Big(\sum_{n=-\infty}^{+\infty}\int_{3^{n-1}}^{3^n} 2^{(a_n+1)q'/p'} y^{q'-1}\,dy\Big)^{p'/q'}\,d\mu\\
&\quad= C\Big(\sum_{n=-\infty}^{+\infty}\int_{3^{n-1}}^{3^n}\Big(\int_A 2^{a_n}\,d\mu\Big)^{q'/p'} y^{q'-1}\,dy\Big)^{p'/q'}\\
&\quad\le C\Big(\sum_{n=-\infty}^{+\infty}\int_{3^{n-1}}^{3^n}\Big(\int_A \sum_{\{j\in[r,k]\,|\,v^{-1}(T^j x)>3^n\}} v(T^j x)\,d\mu\Big)^{q'/p'} y^{q'-1}\,dy\Big)^{p'/q'}\\
&\quad\le C\Big(\int_0^\infty\Big(\int_A \sum_{\{j\in[r,k]\,|\,v^{-1}(T^j x)>y\}} v(T^j x)\,d\mu\Big)^{q'/p'} y^{q'-1}\,dy\Big)^{p'/q'}.
\end{aligned}$$

This proves (4.8).

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Let us consider now the case $s(i) < k$. Once we have fixed B_i with $s(i) < k$, we apply Lemma 10 for $s = 0$. We have $B_i = \bigcup_\alpha A_\alpha$, where these A_α's are defined from the $H_n{}^x$'s associated to $s = 0$. Fix A_α and consider, for every $(n_0, n_1, \ldots, n_k) \in \mathbf{Z}^{k+1}$, the set

$$H_{n_0,n_1,\ldots,n_k} = \{x \in A_\alpha \mid 2^{n_i} < v(T^i x) \leq 2^{n_i+1}, i = 0, 1, \ldots, k\}.$$

Let us fix $H_{n_0,n_1,\ldots n_k}$ and let A be a measurable subset of $H_{n_0,n_1,\ldots,n_k}$. Let $R = \bigcup_{0\leq j\leq s(i)} T^j A = \bigcup_{0\leq j\leq k} T^j A$. First, we prove

$$\|\chi_R\|_{p,q;u}\, \|\chi_R v^{-1}\|_{p',q';v} \leq C\mu(R) \tag{4.9}$$

with C independent of k and R, as in the previous case. The inequality (4.9) can be written as

(4.10)

$$\Big(\int_A \sum_{j=0}^{s(i)} u(T^j x)\, d\mu\Big)^{1/p} \Big(q' \int_0^\infty \Big(\int_A \sum_{\{j\in[0,s(i)] \mid v^{-1}(T^j x)>y\}} v(T^j x)\Big)^{q'/p'} y^{q'-1}\, dy\Big)^{1/q'} \leq C\big(s(i)+1\big)\mu(A)$$

and the Lemma 2.10 in [4] allows us to prove that (4.10) is also valid replacing $s(i)$ by k.

Finally, in the same way as in the case $s(i) = k$, from (4.10) and using the definition of $H_n{}^x$ we deduce:

(4.11)

$$\Big(\int_A \sum_{j=0}^{k} u(T^j x)\, d\mu\Big)^{p'/p} \int_A \Big(\int_0^\infty \Big(\sum_{\{j\in[0,k] \mid v^{-1}(T^j x)>y\}} v(T^j x)\Big)^{q'/p'} y^{q'-1}\, dy\Big)^{p'/q'} d\mu \leq C(k+1)^{p'} \mu(A)^{p'},$$

and since (4.11) holds for every measurable subset A of $H_{n_0,n_1,\ldots,n_k}$, the union of the $H_{n_0,n_1,\ldots,n_k}$'s is A_α and the union of the A_α's is B_i, it follows

$$\Big(\sum_{j=0}^{k} u(T^j x)\Big)^{1/p} \Big(\int_0^\infty q' \Big(\sum_{\{j\in[0,k] \mid v^{-1}(T^j x)>y\}} v(T^j x)\Big)^{q'/p'} y^{q'-1}\, dy\Big)^{1/q'} \leq C(k+1)$$

a.e. $x \in B_i$, *i.e.*,

$$\|\chi_{[0,k]}\|_{p,q;u^x}\, \|\chi_{[0,k]}(v^x)^{-1}\|_{p',q';v^x} \leq C(k+1)$$

a.e. $x \in B_i$. This inequality implies clearly

$$\|\chi_{[0,r]}\|_{p,q;u^x}\, \|\chi_{[r,k]}(v^x)^{-1}\|_{p',q';v^x} \leq C(k+1). \tag{4.12}$$

Therefore, either $s(i) = k$ or $s(i) < k$, we have proved that (4.12) holds a.e. $x \in B_i$. Since $X = \bigcup_i B_i$, iii) is proved.

REMARK 3. Observe that i) $\Rightarrow$ ii) and ii) $\Rightarrow$ iii) also hold for $1 < p < \infty$ and $1 < q < \infty$.

In the single weight case we have the following theorem:

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THEOREM 4. *Let $1 < p < \infty$, $1 < q < \infty$ and w be a positive measurable function. The following statements are equivalent:*

i) $\|M_T{}^+f\|_{p,\infty;w} \leq C\|f\|_{p,q;w}$

ii) $\sup_{n\geq 0} \|T_{0,n}f\|_{p,\infty;w} \leq C\|f\|_{p,q;w}$

iii) $\|M_T{}^+f\|_{p,q;w} \leq C\|f\|_{p,q;w}$

iv) $\sup_{n\geq 0} \|T_{0,n}f\|_{p,q;w} \leq C\|f\|_{p,q;w}$

v) $w \in A_{p,q}{}^+(T)$

vi) $w \in A_p{}^+(T)$.

PROOF. The implications i) $\Rightarrow$ ii), iii) $\Rightarrow$ iv) and v) $\Rightarrow$ ii) are clear. To prove the equivalence between v) and vi) it suffices to write in the integers from the proof of Theorem 2. The implication ii) $\Rightarrow$ v) may be proved as in Theorem 3 (see Remark 3). Finally, vi) $\Rightarrow$ i) and vi) $\Rightarrow$ iii) follow from Marcinkiewickz's interpolation theorem and from the fact that $w \in A_p{}^+(T)$ implies $w \in A_{p-\varepsilon}{}^+(T)$ for any $\varepsilon > 0$ with $p - \varepsilon > 1$.

When we work in a finite measure space and T is only a null-preserving transformation, the equivalence of the weak and strong type inequalities for the maximal operator reduces to the case in which the measure is equivalent to an invariant measure. Moreover, the uniform boundedness of the ergodic averages implies a.e. convergence:

THEOREM 5. *Let (X, M, ν) be a finite measure space and let T be a null-preserving invertible transformation over X. Let $1 < p < \infty$ and $1 < q < \infty$. The following statements are equivalent:*

i) $\|M_T{}^+f\|_{p,\infty} \leq C\|f\|_{p,q}$

ii) $\sup_{n\geq 0} \|T_{0,n}f\|_{p,\infty} \leq C\|f\|_{p,q}$

iii) $\|M_T{}^+f\|_{p,q} \leq C\|f\|_{p,q}$

iv) $\sup_{n\geq 0} \|T_{0,n}f\|_{p,q} \leq C\|f\|_{p,q}$.

Moreover, if one of the above conditions holds, then the sequence $\{T_{0,n}f\}$ converges almost everywhere for every $f \in L_{p,q}$.

PROOF. It works as in Theorem 3 of [7].

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Análisis Matemático
Facultad de Ciencias
Universidad de Málaga
29071-Málaga
Spain