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# ON THE LOWEST ZERO OF THE DEDEKIND ZETA FUNCTION

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**Abstract**

Let $\zeta_K(s)$ denote the Dedekind zeta-function associated to a number field $K$. We give an effective upper bound for the height of the first nontrivial zero other than $1/2$ of $\zeta_K(s)$ under the generalised Riemann hypothesis. This is a refinement of the earlier bound obtained by Sami ['Majoration du premier zéro de la fonction zêta de Dedekind', *Acta Arith.* **99**(1) (2000), 61–65].

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## 1. Introduction

Let $K/\mathbb{Q}$ be a number field. The Dedekind zeta-function associated with $K$ is defined on $\mathrm{Re}(s) > 1$ by

$$\zeta_K(s) := \sum_{\mathfrak{a}} \frac{1}{N\mathfrak{a}^s}.$$

Here, $\mathfrak{a}$ runs over all nonzero integral ideals of $K$. This function has an analytic continuation to $\mathbb{C}$ except for a simple pole at $s = 1$. The zeros of $\zeta_K(s)$ in the critical strip $0 < \mathrm{Re}(s) < 1$ are called the nontrivial zeros. One of the central problems in analytic number theory is to study the order and magnitude of these nontrivial zeros. The generalised Riemann hypothesis (GRH) says that all the nontrivial zeros of $\zeta_K(s)$ lie on the vertical line $\mathrm{Re}(s) = \frac{1}{2}$. Under GRH, one can consider the height of a zero, that is, its distance from the point $s = \frac{1}{2}$. Define

$$\tau(K) := \min\{t > 0 : \zeta_K(1/2 + it) = 0\},$$

the lowest height of a nontrivial zero of $\zeta_K(s)$ other than $\frac{1}{2}$. It is possible that $\zeta_K(\frac{1}{2}) = 0$, as shown by Armitage [1] in 1971. However, it is believed that as we vary over number fields, $\zeta_K(\frac{1}{2})$ vanishes very rarely. Indeed, Soundararajan [13] showed that for a large proportion (87.5%) of quadratic number fields, $\zeta_K(\frac{1}{2}) \neq 0$.

One of the natural questions is to obtain upper and lower bounds on $\tau(K)$. The importance of studying $\tau(K)$ is evident from its connection to the discriminant of the number field, as highlighted in the survey paper by Odlyzko [9]. The low-lying zeros of $\zeta_K(s)$ also have consequences for Lehmer's conjecture on heights of algebraic numbers (see [4]). In 1979, Hoffstein [5] showed that for number fields $K$ with sufficiently large degree,

$$\tau(K) \leq 0.87.$$

For a number field $K$, denote by $n_K$ the degree $[K : \mathbb{Q}]$ and by $d_K$ the discriminant $\mathrm{disc}(K/\mathbb{Q})$. Let $\alpha_K$ be the log root discriminant of $K$ defined by

$$\alpha_K := \frac{\log |d_K|}{n_K}.$$

In 1985, Neugebauer [8] showed the existence of a nontrivial zero of $\zeta_K(s)$ in the rectangle

$$\mathfrak{R} = \{\sigma + it \mid 1/2 \leq \sigma \leq 1, |t - T| \leq 10\},$$

for every $T \geq 50$. Later in 1988, Neugebauer [7] derived an explicit upper bound, namely either $\zeta_K(1/2) = 0$ or

$$\tau(K) \leq \min\left\{60, \frac{64\pi^2}{\log\left(\frac{1}{4}\log(82 + 27\alpha_K)\right)}\right\}. \tag{1.1}$$

Tollis [14] conjectured that

$$\tau(K) \ll \frac{1}{\log |d_K|}, \tag{1.2}$$

where the implied constant is absolute. Although this remains open, Sami [12] showed that under GRH,

$$\tau(K) \ll_{n_K} \frac{1}{\log\log(|d_K|)}.$$

Thus, the lowest zero of the Dedekind zeta function converges to $\frac{1}{2}$ as we vary over number fields with a fixed degree. In [6], an ineffective upper bound of a similar nature has been obtained for newforms of weight $k$ on $\Gamma_0(N)$.

Let $\tau_0 := \tau(\mathbb{Q})\,(= 14.1347\ldots)$ be the lowest zero of the Riemann zeta-function $\zeta(s)$. Recall the famous Dedekind conjecture, which states that $\zeta_K(s)/\zeta(s)$ is entire. Therefore, one expects $\zeta_K(1/2 + i\tau_0) = 0$ for all number fields $K$. Explicit upper bounds for the height of the lowest zero (under GRH) for automorphic $L$-functions were studied in [3], and Bllaca [2] examined the $L$-functions in the Selberg class. The goal of this paper is to give a simple and effective version of Sami's upper bound [12] on the first zero of the Dedekind zeta function under GRH. We obtain the following effective upper bound for the lowest zero of $\zeta_K(s)$.

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THEOREM 1.1. *Let $K$ be a number field such that the log root discriminant $\alpha_K > 6.6958$ and $\zeta_K(1/2) \neq 0$. Then, under GRH, either $\tau(K) \geq \tau_0$ or*

$$\tau(K) \leq \frac{\pi}{\sqrt{2}\log(\frac{\alpha_K - 1.2874}{5.4084})}.$$

REMARK 1.2. One can improve this bound using Hoffstein's result [5, page 194], which states that $\tau(K) \leq 0.87$ for all number fields with sufficiently large degree. Indeed, the method of our proof shows that for number fields $K$ with sufficiently large degree, if $\alpha_K > 6.4435$, then under GRH,

$$\tau(K) \leq \frac{\pi}{\sqrt{2}\log(\frac{\alpha_K - 1.2874}{5.1561})}.$$

Further, it follows from Hoffstein's result that $\tau(K) \leq 0.37$ except for finitely many number fields with $\alpha_K \leq 6.6958$. Therefore,

$$\tau(K) \leq \min\left\{0.37, \frac{\pi}{\sqrt{2}\log(\frac{\alpha_K - 1.2874}{5.1561})}\right\}$$

for all but finitely many number fields.

Assuming GRH, Sami's bound was improved by Carneiro *et al.* [3, Theorem 7], where they showed that as $\alpha_K \to \infty$,

$$\tau(K) \leq \frac{\pi}{2\log\alpha_K} + O\Big(\frac{\log\log\alpha_K}{(\log\alpha_K)^2}\Big). \tag{1.3}$$

Note that Theorem 1.1 yields

$$\tau(K) \leq \frac{\pi}{\sqrt{2}\log\alpha_K} + O\Big(\frac{1}{(\log\alpha_K)^2}\Big).$$

So, Theorem 1.1 is weaker than (1.3) asymptotically. However, it holds for all number fields $K$ with $\alpha_K \geq 6.6958$ without any error term.

Next, we address the case where $\zeta_K(s)$ vanishes at $s = 1/2$.

THEOREM 1.3. *Suppose $K$ is a number field with $\alpha_K > 12.1048$ and $\zeta_K(1/2) = 0$. Let*

$$A := \frac{\pi^2}{34.4}\frac{\log\log|d_K|}{\alpha_K}(\alpha_K - 1.2874) \quad \text{and} \quad B := 2\log\Big(\frac{\alpha_K - 1.2874}{10.8168}\Big).$$

*Then, under GRH, either $\tau(K) \geq \tau_0$ or*

$$\tau(K) \leq \frac{\sqrt{2}\pi}{\min\{A, B\}}.$$

From Tollis's conjecture (1.2), it is clear that over any family of number fields $\{K_i\}$, the height of the lowest zero $\tau(K)$ tends to 0. However, in Theorems 1.1 and 1.3 (also in [12]), we show this for families of number fields $\{K_i\}$, where the root discriminant tends to infinity. This property is also discussed in [15, Proposition 5.2]. Also note

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that the bound in Theorem 1.3 is weaker than that in Theorem 1.1. This is perhaps indicative of the 'zero repulsion' effect due to the existing zero at $\frac{1}{2}$.

## 2. Preliminaries

In this section, we state and prove some results which will be useful in the proof of the main theorems. We first recall Weil's explicit formula. Let $F$ be a real-valued even function such that:

(i) $F$ is continuously differentiable on $\mathbb{R}$ except at a finite number of points $a_i$ where $F(x)$ and its derivative $F'(x)$ have only discontinuities of the first kind for which $F$ satisfies the mean condition, that is,

$$F(a_i) = \tfrac{1}{2}(F(a_i + 0) + F(a_i - 0));$$

(ii) there exists $b > 0$ such that $F(x)$ and $F'(x)$ are $O(e^{-(1/2+b)|x|})$ as $x \to \infty$.

Then, the Mellin transform of $F$, given by

$$\Phi(s) := \int_{-\infty}^{\infty} F(x)e^{(s-1/2)x}\,dx,$$

is holomorphic in any strip $-a \le \sigma \le 1 + a$, where $0 < a < b$, $a < 1$. The following explicit formula is due to Weil [10] (formulated by Poitou).

THEOREM 2.1 (Weil). *Let $F$ satisfy conditions (i) and (ii) above with $F(0) = 1$. Then, the sum $\sum \Phi(\rho)$ taken over the nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta_K(s)$ with $|\gamma| < T$ has a limit when $T$ tends to infinity given by the formula*

$$\sum_{\rho} \Phi(\rho) = \Phi(0) + \Phi(1) - 2\sum_{\mathfrak{p}}\sum_{m=1}^{\infty} \frac{\log(N(\mathfrak{p}))}{N(\mathfrak{p})^{m/2}} F(m\log(N(\mathfrak{p}))) + \log(|d_K|)$$
$$- n_K[\log(2\pi) + \gamma + 2\log(2)] - r_1 J(F) + n_K I(F), \qquad (2.1)$$

*where*

$$J(F) = \int_0^{\infty} \frac{F(x)}{2\cosh(x/2)}\,dx, \quad I(F) = \int_0^{\infty} \frac{1 - F(x)}{2\sinh(x/2)}\,dx$$

*and $\gamma = 0.57721566\ldots$ denotes the Euler–Mascheroni constant. Here, $\mathfrak{p}$ runs over all the prime ideals of $K$, $N(\mathfrak{p})$ denotes the ideal norm of $\mathfrak{p}$ and $r_1$ denotes the number of real embeddings of $K$.*

Observe that

$$\Phi(0) + \Phi(1) = 4\int_0^{\infty} F(x)\cosh(x/2)\,dx.$$

For a function $F \in L^1(\mathbb{R})$, the Fourier transform of $F$ is given by

$$\widehat{F}(t) := \int_{-\infty}^{\infty} F(x)e^{2\pi i t x}\,dx.$$

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Under GRH, we have $\Phi(\rho) = \widehat{F}(t)$, where $\rho = 1/2 + it$. Set $F_T(x) := F(x/T)$, then $\widehat{F_T}(u) = T\widehat{F}(Tu)$. We now recall the following lemma proved in [12].

LEMMA 2.2 (Sami). *Let $F$ be a compactly supported even function defined on $\mathbb{R}$ by*

$$F(x) = \begin{cases} (1-|x|)\cos(\pi x) + \dfrac{3}{\pi}\sin(\pi|x|) & \text{if } 0 \le |x| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

*Then, $F$ satisfies the growth conditions of the explicit formula and*

$$\widehat{F}(u) = 2\Big(2 - \frac{u^2}{\pi^2}\Big)\Big[\frac{2\pi}{\pi^2 - u^2}\cos(u/2)\Big]^2.$$

We also need the following straightforward lemma (proved by contradiction).

LEMMA 2.3. *Let $a, b, c$ be three positive real constants satisfying $c > 2b$. If $T > 0$ and $aT + be^{T/2} \ge c$, then*

$$T \ge \min\Big(\frac{c}{2a}, 2\log\Big(\frac{c}{2b}\Big)\Big).$$

## 3. Proof of the main theorems

The proof of our theorems follows a similar method to [12]. We start with the following lemma.

LEMMA 3.1. *Let $F_T(x) = F(x/T)$ as in the explicit formula (2.1). Then,*

$$\sum_{\mathfrak{p}}\sum_{m=1}^{\infty}\frac{\log(N(\mathfrak{p}))}{N(\mathfrak{p})^{m/2}}F_T(m\log(N(\mathfrak{p}))) \le 1.2571\, n_K(2\,e^{T/2} - 1),$$

*where $\mathfrak{p}$ runs over all prime ideals of $K$.*

PROOF. Let $p$ be a rational prime. Since $\sum_{p|\mathfrak{p}}\log N(\mathfrak{p}) \le n_K \log p$,

$$\sum_{p|\mathfrak{p}}\frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{m/2}} \le n_K\frac{\log p}{p^{m/2}}.$$

From the definition of $F(x)$, it follows that $|F(x)| \le 1.21$. Hence, the above inequality gives

$$\begin{aligned}\sum_{\mathfrak{p},m}\frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{m/2}}F_T(m\log N(\mathfrak{p})) &= \sum_{m,p}\sum_{p|\mathfrak{p}}\frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{m/2}}F_T(m\log N(\mathfrak{p})) \\ &\le 1.21\, n_K\sum_{m\log p\le T}\frac{\log p}{p^{m/2}} \\ &= 1.21\, n_K\sum_{n\le e^T}\frac{\Lambda(n)}{\sqrt{n}}, \qquad (3.1)\end{aligned}$$

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where $\Lambda$ is the von Mangoldt function. Now, recall the Chebyshev function,

$$\Psi(x) := \sum_{n \le x} \Lambda(n).$$

Applying partial summation and using the bound $\Psi(x) \le 1.0389\, x$ by Rosser and Schoenfeld [11],

$$\sum_{n \le e^T} \frac{\Lambda(n)}{\sqrt{n}} = \frac{\Psi(e^T)}{e^{T/2}} + \frac{1}{2} \int_1^{e^T} \frac{\Psi(t)}{t^{3/2}}\, dt \le 1.0389\,(2e^{T/2} - 1). \tag{3.2}$$

From (3.1) and (3.2), the lemma follows. □

Let $T = \sqrt{2}\pi/\tau(K)$ and let $F(x)$ be the function defined in Lemma 2.2. Applying Theorem 2.1 to $F_T(x) = F(x/T)$,

$$\sum_{\rho} \Phi(\rho) = \Phi_T(0) + \Phi_T(1) - 2 \sum_{\mathfrak{p},m} \frac{\log(N(\mathfrak{p}))}{N(\mathfrak{p})^{m/2}} F_T(m \log(N(\mathfrak{p}))) \\ + \log|d_K| - n_K[\log(2\pi) + \gamma + 2\log(2)] - r_1 J(F_T) + n_K I(F_T). \tag{3.3}$$

Since $\tau(K) \le \tau_0$, we have $T \ge 0.314$. For such $T$, the remaining terms on the right-hand side of (3.3) can be bounded by

$$J(F_T) = \int_0^T \frac{F(x/T)}{2\cosh(x/2)}\, dx \le 0.276\, e^{T/2}, \tag{3.4}$$

$$I(F_T) = \int_0^T \frac{1 - F(x/T)}{2\sinh(x/2)}\, dx \ge -0.1034\, e^{T/2}. \tag{3.5}$$

We are now ready to prove our theorems.

Proof of Theorem 1.1. Since $\zeta_K(1/2) \ne 0$, (3.3) gives

$$\log|d_K| + \Phi_T(0) + \Phi_T(1) \le 2 \sum_{\mathfrak{p},m} \frac{\log(N(\mathfrak{p}))}{N(\mathfrak{p})^{m/2}} F_T(m \log(N(\mathfrak{p}))) \\ + n_K[\log(2\pi) + \gamma + 2\log(2)] + r_1 J(F_T) - n_K I(F_T).$$

From Lemma 2.2 along with (3.4) and (3.5),

$$\log|d_K| \le 5.4084\, n_K e^{T/2} + 1.2874\, n_K.$$

Thus, $\alpha_K - 1.2874 \le 5.4084\, e^{T/2}$ and, for $\alpha_K > 6.6958$,

$$T \ge 2\log\Big(\frac{\alpha_K - 1.2874}{5.4084}\Big).$$

Since $T = \sqrt{2}\pi/\tau(K)$, the theorem follows. □

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PROOF OF THEOREM 1.3. Here, $\zeta_K(\frac{1}{2}) = 0$ and therefore (3.3) gives

$$\log|d_K| + \Phi_T(0) + \Phi_T(1) \leq 2\sum_{\mathfrak{p},m} \frac{\log(N(\mathfrak{p}))}{N(\mathfrak{p})^{m/2}} F_T(m\log(N(\mathfrak{p})))+ \\ + n_K[\log(2\pi) + \gamma + 2\log(2)] + r_1 J(F_T) - n_K I(F_T) + \frac{16}{\pi^2} rT,$$

where $r$ is the order of $\zeta_K(s)$ at $1/2$. As before, using Lemma 2.2 along with (3.4) and (3.5),

$$\log|d_K| \leq 5.4084\, n_K e^{T/2} + 1.2874\, n_K + \frac{16}{\pi^2} rT.$$

From [12, Proposition 1], we can bound the order of the zero of $\zeta_K(s)$ at $s = 1/2$ by

$$r \leq \frac{\log|d_K|}{\log\log|d_K|} + \frac{n_K}{2\log\log|d_K|}.$$

Thus,

$$\alpha_K - 1.2874 \leq 5.4084\, e^{T/2} + \left(\frac{17.2}{\pi^2}\frac{\alpha_K}{\log\log|d_K|}\right)T.$$

Using Lemma 2.3 with

$$a = \left(\frac{17.2}{\pi^2}\frac{\alpha_K}{\log\log|d_K|}\right), \quad b = 5.4084, \quad c = \alpha_K - 1.2874,$$

we conclude that

$$\tau(K) \leq \frac{\sqrt{2}\pi}{\min\{A, B\}},$$

where $A, B$ are as in the statement of the theorem. This completes the proof. □

## 4. Computational data and concluding remarks

Let $K = \mathbb{Q}(\beta)$ be a number field and $m_\beta(x)$ be the minimal polynomial of $\beta$. Using SageMath, we can compare the lowest zero and the bounds obtained using Theorem 1.1 (see Table 1).

However, we can also compare Theorem 1.1 with Neugebaur's bound in (1.1). Although the bound in (1.1) is unconditional, it applies only for the cases where $\alpha_K$ is very large ($> 10^{64849}$), whereas Theorem 1.1 applies for all $K$ with $\alpha_K \geq 6.6958$.

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TABLE 1. Comparing the bound in Theorem 1.1 with the height of the first zero.

| $m_\beta(x)$ | $\alpha_K$ | $\tau(K)$ | Bound in Theorem 1.1 |
|---|---|---|---|
| $x^2 + 510510$ | 7.26472993307674 | 0.195366057287247 | 22.2098243056698 |
| $x^2 + 9699690$ | 8.73694942265996 | 0.250485767971509 | 6.93766313396318 |
| $x^2 + 223092870$ | 10.3046965306245 | 0.282126995483731 | 4.34561699877460 |
| $x^2 + 6469693230$ | 11.9883444456178 | 0.223870166465309 | 3.25543786648311 |
| $x^2 + 200560490130$ | 13.7053380478603 | 0.0869456767128933 | 2.67260773966497 |
| $x^3 + 30030$ | 7.97191372931969 | 0.249553262973507 | 10.4864035098435 |
| $x^4 + 30030$ | 9.11875848185292 | 0.0668359001429184 | 6.00093283699129 |

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