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Inversion of the Radon Transform on the Free Nilpotent Lie Group of Step Two

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Abstract. Let $F_{2n,2}$ be the free nilpotent Lie group of step two on $2n$ generators, and let $\mathbf{P}$ denote the affine automorphism group of $F_{2n,2}$. In this article the theory of continuous wavelet transform on $F_{2n,2}$ associated with $\mathbf{P}$ is developed, and then a type of radial wavelet is constructed. Secondly, the Radon transform on $F_{2n,2}$ is studied, and two equivalent characterizations of the range for Radon transform are given. Several kinds of inversion Radon transform formulae are established. One is obtained from the Euclidean Fourier transform; the others are from the group Fourier transform. By using wavelet transforms we deduce an inversion formula of the Radon transform, which does not require the smoothness of functions if the wavelet satisfies the differentiability property. In particular, if $n = 1$, $F_{2,2}$ is the 3-dimensional Heisenberg group H^1, the inversion formula of the Radon transform is valid, which is associated with the sub-Laplacian on $F_{2,2}$. This result cannot be extended to the case $n \geq 2$.

1 Introduction

In the past decades, the research of Radon transforms has made considerable progress due to its wide applications in partial differential equations, X-ray technology, radio astronomy and so on (see [10, 18]). We first recall some preliminaries of the Radon transforms on the Euclidean space and of the Heisenberg group, respectively.

The case of Radon transform on the Euclidean space The Radon transform on $\mathbb{R}^n$ is defined by

$$Rf(\tau) = \int_\tau f(x)dm_\tau(x), \tag{1.1}$$

where τ may be parameterized by $(\theta, s) \in \Sigma_{n-1} \times \mathbb{R}$ so that $\tau = \{x \in \mathbb{R}^n : \langle x, \theta\rangle = s\}$ and m_τ is the Euclidean measure on τ. Let δ be the Dirac delta function, then (1.1) can be rewritten in the form

$$Rf(\theta, s) = \int_{\mathbb{R}^n} f(x)\delta\big(s - \langle x, \theta\rangle\big)\, dx.$$

Since Radon obtained the inversion of the Radon transform in 1917, many different inverse methods, such as Fourier inversion and convolution back-projection inversion, have been developed. The wavelet is a useful tool that has been introduced to

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the inversion formula of Radon transforms in recent years. The first result in this area is due to Holschneider who considered the Radon transform on the two-dimensional plane (see [12]). Rubin extended to the cases of k-dimensional Radon transform [19].

The case of Radon transform on the Heisenberg group The Heisenberg group, denoted by H^n, is the most well-known example of a two-step nilpotent Lie group with the underlying manifold $\mathbb{C}^n \times \mathbb{R}$ and the multiplication law (see [6, 22, 25])

$$(z,t)(z',t') = \Big(z + z', t + t' - \frac{1}{2}\operatorname{Im} z\bar{z'}\Big).$$

The Radon transform on the Heisenberg group is called the Heisenberg Radon transform, which is defined by

$$Rf(z,t) = \int_{\mathbb{C}^n} f\big((z,t)(w,0)\big)\, dw = \int_{\mathbb{C}^n} f\Big(w, t - \frac{1}{2}\operatorname{Im}(z\cdot\bar{w})\Big)\, dw.$$

The hyperplane $\{(w, t - \frac{1}{2}\operatorname{Im}(z\cdot\bar{w})) : w \in \mathbb{C}^n\}$ can be written as

$$\Big\{(w,s) : s = t + \frac{1}{2}\operatorname{Re}\big((iz)\cdot\bar{w}\big)\Big\},$$

so it is transversal to the last variable. Let $\delta_2(t)$ be the Dirac delta function in the t variable, then one has another version of the Heisenberg Radon transform:

$$Rf(z,t) = f * \delta_2(t).$$

In [7] Geller and Stein studied this equation and proved that the operator $(\partial/\partial t)^n R$ can be extended to the space $L^2(H^n)$ of square integrable functions on H^n. Moreover,

$$\|(\partial/\partial t)^n Rf\|_2^2 = (4\pi)^{2n}\|f\|_2^2.$$

Strichartz [24] observed that this formula yields the inversion formula:

$$R^{-1} = (4\pi)^{-2n}(\partial/\partial t)^n R(\partial/\partial t)^n.$$

In fact, the Heisenberg Radon transform is just a special case of the transversal Radon transform on Euclidean space shown by [20] by Rubin. Also, by such relation an inversion formula for the Heisenberg Radon transform was obtained. Through a different method, called the wavelet transform defined in [14], another inversion is established by He in [8]. Related works on the more general context of nilpotent Lie groups and Siegel type Lie groups were studied by Felix [4], Nessibi and Trimèche [16], and Peng and Zhang [17].

For the free nilpotent Lie group of step two on $2n$ generators denoted by $F_{2n,2}$, the related harmonic analysis and Radon transform were studied by Strichartz [24]. Inspired by this work, the best constant for the Friedrichs–Knapp–Stein inequality on $F_{2n,2}$ was discussed by Domokos and Franciullo [3]. For more general research on free nilpotent Lie groups of step two, we refer the reader to [1, 5].

We devote this paper to studying the inversion Radon transform on $F_{2n,2}$. In Section 2 we review and present some properties of the group Fourier transform associated with the Schrödinger representations. Moreover, we introduce a unitary representation of the affine automorphism group of $F_{2n,2}$. The direct sum decomposition of $L^2(F_{2n,2})$ under such unitary representation allows us to define the continuous wavelet transform and obtain the Calderón reproducing formula in Section 3. Also, we give an example for the radial wavelet by which we can simplify the Calderón reproducing formula. In Section 4 we study the inversion of the Radon transform on the free nilpotent Lie group of step two. We introduce a Semyanistri–Lizorkin type space, on which the Radon transform is a bijection. The inversion of Radon transform are obtained in different ways. One is from the Euclidean Fourier transform; another is from the group Fourier transform. We show that these descriptions are equivalent. In this case the Plancherel measure is no longer a monomial but a complicated homogeneous polynomial, so the proof for the equivalence of two descriptions is different from that in [9]. In addition, if $n = 1$ we can deduce an inverse Radon transform related with the sub-Laplacian operator. This result cannot be extended to the case $n \geq 2$. In the last section we make use of the wavelet transform to achieve a new inversion formula for the Radon transform on $F_{2n,2}$. The differentiability on f can be neglected if the wavelet function is smooth. Of course, an inversion formula of the Radon transform related to the sub-Laplacian operator is valid only if $n = 1$.

2 Free Nilpotent Lie Group of Step Two

Let $\mathcal{G}_n$ be the free nilpotent Lie algebra of two step (or the free two-step nilpotent Lie algebra) with n generators. The definition used for the universal property can be found in [13]. Roughly speaking, $\mathcal{G}_n$ is a two-step nilpotent Lie algebra with n generators $X_1, \dots, X_n$, such that the vectors $X_1, \dots, X_n$, and $X_{jk} = [X_j, X_k] (j < k)$ form a basis of the vector space $\mathcal{G}_n$. Let $\mathcal{V}$ and $\mathcal{Z}$ be the vector spaces generated by the families of vectors $X_1, \dots, X_n$, and $X_{jk} (1 \leq j < k \leq n)$, respectively; these families become the canonical bases of $\mathcal{V}$ and $\mathcal{Z}$. Thus $\mathcal{G}_n = \mathcal{V} \oplus \mathcal{Z}$ is a stratified algebra, and $\mathcal{Z}$ is the center of the Lie algebra $\mathcal{G}_n$. The simply connected nilpotent Lie group that corresponds to $\mathcal{G}_n$ is called the free two-step nilpotent Lie group denoted by F_n.

Let $\Lambda^2(\mathbb{R}^{2n})$ be the set of $2n \times 2n$ skew-symmetric real matrices. For $y, y' \in \Lambda^2(\mathbb{R}^{2n})$, the inner product of y and y' is given by

$$y \cdot y' = \frac{1}{2} \sum_{j,k} y_{jk} y'_{jk}. \tag{2.1}$$

Here the factor $1/2$ is present to compensate for the redundancy involved in using both y_{jk} and y_{kj} as distinct coordinates, although $y_{jk} = -y_{kj}$. For $x, x' \in \mathbb{R}^{2n}$, we can write $x \wedge x'$ for the skew-symmetric matrix by the law

$$(x \wedge x')_{jk} = x_j x'_k - x_k x'_j. \tag{2.2}$$

In this paper we consider the free nilpotent Lie group of step two on $2n$ generators, that is, $F_{2n,2} = \mathbb{R}^{2n} \oplus \Lambda^2(\mathbb{R}^{2n})$, which equipped with the group law

$$(x, y)(x', y') = \Big(x + x', y + y' - \frac{1}{2} x \wedge x' \Big).$$

It is easy to see that the dimension of $F_{2n,2} = \mathbb{R}^{2n} \oplus \Lambda^2(\mathbb{R}^{2n})$ is $2n+(2n^2-n) = 2n^2+n$. The homogeneous dimension is $2n + 2(2n^2 - n) = 4n^2$. The vector fields

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{k=1}^{2n} x_k \frac{\partial}{\partial y_{jk}}, \quad j = 1, \cdots, 2n,$$
$$Y_{jk} = \frac{\partial}{\partial y_{jk}}, \quad 1 \le j < k \le 2n$$

form a basis for the Lie algebra of left-invariant fields on $F_{2n,2}$, and the sub-Laplacian operator is

$$\begin{aligned} \mathcal{L} &= -\sum_{j=1}^{2n} X_j^2 \\ &= -\Big(\sum_{j=1}^{2n} \frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^{2n} \sum_{k=1}^{2n} x_k \frac{\partial}{\partial x_j} \frac{\partial}{\partial y_{jk}} + \frac{1}{4} \sum_{j=1}^{2n} \Big(\sum_{k=1}^{2n} x_k \frac{\partial}{\partial y_{jk}} \Big)^2 \Big). \end{aligned} \tag{2.3}$$

We note that for $n = 1$, the group $F_{2,2}$ is the 3-dimensional Heisenberg group H^1, but for $n > 1$ there are important differences between $F_{2n,2}$ and H^n, especially arising from the size of the vertical subspace of the Lie algebra spanned by the commutators. From this point of view $F_{2n,2}$ can be considered maximal and H^n minimal.

3 Fourier Analysis on $F_{2n,2}$

The group Fourier transform on $F_{2n,2}$ is similar to that of the Heisenberg group, which is related to the so-called Schrödinger representation. Note that for any $\eta \in \Lambda^2(\mathbb{R}^{2n})$, one has the canonical form

$$\eta = \sum_{j=1}^{n} \lambda_j a_j \wedge b_j,$$

where $\lambda_j \ge 0$ and $\{a_1, \cdots, a_n, b_1, \cdots, b_n\}$ is an orthonormal basis of $\mathbb{R}^{2n}$. It can be verified that $|\det \eta|^{1/2} = \prod_{j=1}^{n} \lambda_j$.

Proposition 3.1

(i) *For $x, x', w, w' \in \mathbb{R}^{2n}$, we have*

$$(x \wedge x') \cdot (w \wedge w') = (x \cdot w)(x' \cdot w') - (x \cdot w')(x' \cdot w). \tag{3.1}$$

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(ii) *For the skew-symmetric matrix* $\eta = \sum_{j=1}^{n} \lambda_j a_j \wedge b_j$, *we have*

$$(x \wedge x') \cdot \eta = \sum_{j=1}^{n} \lambda_j \big((x \cdot a_j)(x' \cdot b_j) - (x \cdot b_j)(x' \cdot a_j) \big) . \tag{3.2}$$

Proof By (2.1) and (2.2) we have

$$\begin{aligned}(x \wedge x') \cdot (w \wedge w') &= \frac{1}{2} \sum_{j,k} (x_j x'_k - x_k x'_j)(w_j w'_k - w_k w'_j) \\ &= \frac{1}{2} \sum_{j,k} (x_j x'_k w_j w'_k - x_j x'_k w_k w'_j - x_k x'_j w_j w'_k + x_k x'_j w_k w'_j) \\ &= (x \cdot w)(x' \cdot w') - (x \cdot w')(x' \cdot w).\end{aligned}$$

Formula (3.2) can be easily obtained by substituting the canonical form of η into (3.1). ∎

Now let $\eta \in \Lambda^2(\mathbb{R}^{2n})$, $\det \eta \neq 0$. The Schrödinger representation π_η is defined on $L^2(\mathbb{R}^n)$ by

$$\begin{aligned}\pi_\eta(x, y)\varphi(\xi) = \exp & \Big(2\pi i y \cdot \eta + 2\pi i \sum_{j=1}^{n} \lambda_j \xi_j (x \cdot a_j) + \pi i \sum_{j=1}^{n} \lambda_j (x \cdot a_j)(x \cdot b_j) \Big) \\ & \times \varphi\Big(\xi + \sum_{j=1}^{n} (x \cdot b_j) e_j \Big),\end{aligned}$$

where $\{e_1, \ldots, e_n\}$ is a standard basis of $\mathbb{R}^n$. By Proposition 3.1 and the group law of $F_{2n,2}$, we then obtain

$$\pi_\eta(x, y)\pi_\eta(x', y') = \pi_\eta\big((x, y)(x', y') \big) ,$$

which indicates that π_η is unitary. In addition, analogues to the proof of [25, Theorem 1.2.1], we deduce that π_η is irreducible.

Suppose that $f \in L^1(F_{2n,2})$. The Fourier transform of f is an operator-valued function acting on $L^2(\mathbb{R}^n)$ by

$$\widehat{f}(\eta)\varphi(\xi) = \int_{F_{2n,2}} f(x, y)\pi_\eta(x, y)\varphi(\xi) dx dy. \tag{3.3}$$

If we write $\pi_\eta(x) = \pi_\eta(x, 0)$ and

$$f^\eta(x) = \mathscr{F}_2 f(x, \eta) = \int_{\Lambda^2} f(x, y) \exp(2\pi i y \cdot \eta) dy,$$

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then we can rewrite (3.3) as

$$\widehat{f}(\eta) = \int_{\mathbb{R}^{2n}} f^{\eta}(x)\pi_{\eta}(x)dx = \mathcal{W}_{\eta}(f^{\eta}),$$

where $\mathcal{W}_{\eta}$ is the Weyl transform. By the same argument of the theory of Weyl transforms on the Heisenberg group we have

$$\|f^{\eta}\|^2_{L^2(\mathbb{R}^{2n})} = |\det \eta|^{1/2}\|\widehat{f}(\eta)\|^2_{HS}. \tag{3.4}$$

From this identity we obtain the Plancherel formula

$$\|f\|^2_{L^2(F_{2n,2})} = \int_{\Lambda^2} \|\widehat{f}(\eta)\|^2_{HS}|\det \eta|^{1/2} d\eta \tag{3.5}$$

and the Parseval identity

$$\langle f, g\rangle_{L^2(F_{2n,2})} = \int_{\Lambda^2} \operatorname{tr}(\widehat{g}(\eta)^*\widehat{f}(\eta))|\det \eta|^{1/2} d\eta,$$

where $\widehat{g}(\eta)^*$ is the adjoint of $\widehat{g}(\eta)$, and tr is the trace of an operator. Moreover, we have the inversion Fourier transform

$$f(x, y) = \int_{\Lambda^2} \operatorname{tr}\big(\widehat{f}(\eta)\pi^*_{\eta}(x, y)\big)\,|\det \eta|^{1/2} d\eta.$$

Let $f * g$ denote the convolution for f and g on $F_{2n,2}$ by

$$f * g(x, y) = \int_{F_{2n,2}} f(x', y')g\big((-x', -y')(x, y)\big)\,dx'dy'.$$

From the definition of the group Fourier transform we have

$$(f * g)\widehat{\ }(\eta) = \widehat{f}(\eta)\widehat{g}(\eta). \tag{3.6}$$

Let $O(2n)$ be the orthogonal group consisting of all $2n \times 2n$ orthogonal matrices. For $u \in O(2n)$, we see that $u(x \wedge x')u^{-1} = ux \wedge ux'$. Then the action

$$(x, y) \mapsto (ux, uyu^{-1})$$

is an automorphism of $F_{2n,2}$. The following proposition is useful here.

Proposition 3.2

(i) *Let $\tilde{f}(x, y) = \overline{f}(-x, -y)$, then $\widehat{\tilde{f}}(\eta) = \widehat{f}(\eta)^*$.*
(ii) *For $\rho > 0$, let $f_{\rho}(x, y) = \rho^{-2n^2} f\big(\frac{x}{\sqrt{\rho}}, \frac{y}{\rho}\big)$, then $\widehat{f_{\rho}}(\eta) = \widehat{f}(\rho\eta)$.*
(iii) *For $u \in O(2n)$, let $f_u(x, y) = f(u^{-1}x, u^{-1}yu)$, then $\widehat{f_u}(\eta) = \widehat{f}(u^{-1}\eta u)$.*

(iv) *For $u \in O(2n)$, $\rho > 0$, let*

$$f_{\rho,u}(x,y) = \rho^{-2n^2} f\Big(\frac{u^{-1}x}{\sqrt{\rho}}, \frac{u^{-1}yu}{\rho}\Big),$$

then $\widehat{f}_{\rho,u}(\eta) = \widehat{f}(\rho u^{-1}\eta u)$.

Let $MF_{2n,2}$ denote the semi-direct product of $F_{2n,2}$ and $O(2n)$. Then $MF_{2n,2}$ forms a motion group endowed with the group law

$$(x,y,u)(x',y',u') = (x+ux', y+uy'u^{-1} - \frac{1}{2}x \wedge ux', uu').$$

Strichartz in [24] used the "polar coordinate" to obtain the decomposition of $L^2(F_{2n,2})$ into an integral of irreducible representations of $MF_{2n,2}$. Now in the following, we will decompose $L^2(F_{2n,2})$ into the direct sum of the irreducible invariant closed subspaces under the unitary representations of the affine automorphism group.

For the free nilpotent Lie group $F_{2n,2}$, we define the translation, dilation, and rotation operators respectively by

$$\begin{aligned} T_{(x,y)}\colon (x',y') &\mapsto \Big(x+x', y+y' - \frac{1}{2}x\wedge x'\Big), \\ T_\rho\colon (x,y) &\mapsto (\sqrt{\rho}x, \rho y), \quad \rho > 0, \\ T_u\colon (x,y) &\mapsto (ux, uyu^{-1}), \quad u \in O(2n). \end{aligned}$$

Write (x,y,ρ,u) for $T_{(x,y)}T_\rho T_u$, and let

$$\mathbf{P} = \big\{ (x,y,\rho,u) : (x,y) \in F_{2n,2}, \rho > 0, u \in O(2n) \big\}.$$

Then an element $(x,y,\rho,u) \in \mathbf{P}$ acts on $F_{2n,2}$ by

$$(x,y,\rho,u)(x',y') = \Big(x + \sqrt{\rho}ux', y + \rho uy'u^{-1} - \frac{1}{2}x \wedge \sqrt{\rho}ux'\Big).$$

Now the group law of $\mathbf{P}$ is given by

$$(x,y,\rho,u)(x',y',\rho',u') = \Big(x + \sqrt{\rho}ux', y + \rho uy'u^{-1} - \frac{1}{2}x \wedge \sqrt{\rho}ux', \rho\rho', uu'\Big).$$

It is easy to see that $\mathbf{P}$ is a locally compact non-unimodular group with the left and right Haar measures denoted by $dm_l(x,y,\rho,u) = dxdyd\rho du/\rho^{2n^2+1}$ and $dm_r(x,y,\rho,u) = dxdyd\rho du/\rho$, respectively. We now consider the unitary representation U of $\mathbf{P}$ on $L^2(F_{2n,2})$ defined by

$$\begin{aligned} U(x,y,\rho,u)f(x',y') &= \rho^{-n^2} f\big(T_u^{-1}T_\rho^{-1}T_{(x,y)}^{-1}(x',y')\big) \\ &= \rho^{-n^2} f\bigg(\frac{u^{-1}(x'-x)}{\sqrt{\rho}}, \frac{u^{-1}(y'-y+\frac{1}{2}x\wedge x')u}{\rho}\bigg). \end{aligned}$$

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Then by Proposition 3.2 we calculate that

$$\big(U(x,y,\rho,u)f\big)\widehat{\ }(\eta) = \rho^{n^2}\pi_\eta(x,y)\widehat{f_u}(\rho\eta). \tag{3.7}$$

Let $\mathbf{Z}_+$ be the set of all nonnegative integers, and let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbf{Z}_+^n$. We choose the orthonormal basis in $L^2(\mathbb{R}^n)$ to be the Hermite functions (see [6])

$$h_\alpha(\xi) = \prod_{j=1}^{n} \frac{2^{1/4}}{\sqrt{\alpha_j!}} \left(\frac{-1}{2\sqrt{\pi}}\right)^{\alpha_j} e^{\pi\xi_j^2} \frac{d^{\alpha_j}}{d\xi_j^{\alpha_j}} (e^{-2\pi\xi_j^2}).$$

From [24, (6.8)] we find that the entry functions $\langle \pi_\eta(x,y)h_\alpha,\ h_\beta\rangle$ as $\alpha, \beta \in \mathbf{Z}_+^n$ give a family of eigenfunctions for the sub-Laplacian defined by (2.3), namely

$$\mathcal{L}\langle \pi_\eta(x,y)h_\alpha,\ h_\beta\rangle = \left(2\pi\sum_{j=1}^{n}\lambda_j(1+2\alpha_i)\right)\langle \pi_\eta(x,y)h_\alpha,\ h_\beta\rangle.$$

From this formula we then have

$$\widehat{\mathcal{L}f}(\eta)h_\alpha = \left(2\pi\sum_{j=1}^{n}\lambda_j(1+2\alpha_i)\right)\widehat{f}(\eta)h_\alpha. \tag{3.8}$$

Notice that this case is similar to that of the Heisenberg group if and only if $n = 1$.

Now we consider the space $\mathcal{H}_l$ spanned by $\{h_\alpha : |\alpha| = l\}$. It is clear that the dimension of $\mathcal{H}_l$ is $d_l = (n+l-1)!/((n-1)!l!)$, which just equals the dimension of $\mathscr{P}_l$ discussed in [23, p. 138]. Moreover, we have $L^2(\mathbb{R}^n) = \bigoplus_{l=0}^{\infty}\mathcal{H}_l$. Let $\mathcal{P}_l$ denote the orthogonal projection operator from $L^2(\mathbb{R}^n)$ to $\mathcal{H}_l$. We define the operator P_l on $L^2(F_{2n,2})$ in terms of the Fourier transform by

$$\widehat{P_l f}(\eta) = \widehat{f}(\eta)\mathcal{P}_l$$

and the subspace H_l of $L^2(F_{2n,2})$ by

$$H_l = \{f \in L^2(F_{2n,2}) : \widehat{f}(\eta) = \widehat{f}(\eta)\mathcal{P}_l\}.$$

By an argument analogous to [14, Theorem 1] we have the following theorem.

Theorem 3.3 *Let H_l be defined as above. Then H_l is an irreducible invariant closed subspace of $L^2(F_{2n,2})$ under the unitary representation U of* $\mathbf{P}$. *Moreover, we have the orthogonal direct sum decomposition $L^2(F_{2n,2}) = \bigoplus_{l=0}^{\infty} H_l$.*

4 Continuous Wavelet Transforms

In this section we will develop the theory of continuous wavelet transforms on $F_{2n,2}$ and then obtain the Calderón reproducing formula. In addition, we shall give an example for radial wavelets on this group.

Suppose $\phi \in H_l$, not identically zero. If there exists a constant C_ϕ such that for all $\eta \in \Lambda^2(\mathbb{R}^{2n})$,

$$C_\phi = \frac{1}{d_l}\int_0^\infty \|\widehat{\phi}(\rho\eta)\|_{HS}^2 \frac{d\rho}{\rho} < \infty,$$

then we say that ϕ is an admissible wavelet, and write $\phi \in AW_l$. Now suppose that $\phi \in AW_l$, $f \in H_l$. The continuous wavelet transform W_ϕ of f with respect to ϕ is defined by

$$W_\phi f(x,y,\rho,u) = \big\langle f, U(x,y,\rho,u)\phi\big\rangle_{L^2(F_{2n,2})}.$$

Theorem 4.1 *Suppose that $\phi \in AW_l$, $f, g \in H_l$. Then we have*

$$\langle W_\phi f, W_\phi g\rangle_{L^2(\mathbf{P},dm_l)} = C_\phi \langle f, g\rangle_{L^2(F_{2n,2})}.$$

In particular, we obtain

$$\|W_\phi f\|_{L^2(\mathbf{P},dm_l)} = C_\phi^{1/2}\|f\|_{L^2(F_{2n,2})}. \tag{4.1}$$

Proof Let $\phi \in AW_l$, $f \in H_l$. It follows from (3.7) that

$$\int_{F_{2n,2}} \big\langle f, U(x,y,\rho,u)\phi\big\rangle_{L^2(F_{2n,2})} \pi_\eta(x,y)\,dx\,dy = \rho^{n^2}\widehat{f}(\eta)\widehat{\phi_u}(\rho\eta)^*.$$

By the Parseval formula, we have

$$\begin{aligned}
&\int_{\mathbf{P}} W_\phi f(x,y,\rho,u)\overline{W_\phi g}(x,y,\rho,u)\,dm_l(x,y,\rho,u)\\
&= \int_{O(2n)}\int_0^\infty \Big(\int_{\Lambda^2} \operatorname{tr}\big(\widehat{g}(\eta)^*\widehat{f}(\eta)\widehat{\phi}_u(\rho\eta)^*\widehat{\phi}_u(\rho\eta)\big)\,|\det\eta|^{1/2}d\eta\Big)\frac{d\rho du}{\rho}\\
&= \int_0^\infty \Big\{\int_{\Lambda^2}\sum_{|\alpha|=l}\Big\langle\Big(\int_{O(2n)}\widehat{\phi}_u(\rho\eta)^*\widehat{\phi}_u(\rho\eta)du\Big)h_\alpha, \widehat{f}(\eta)^*\widehat{g}(\eta)h_\alpha\Big\rangle\,\big|\det\eta\big|^{1/2}d\eta\Big\}\frac{d\rho}{\rho}.
\end{aligned}$$

Writing

$$\int_{O(2n)}\widehat{\phi_u}(\rho\eta)^*\widehat{\phi_u}(\rho\eta)du = \int_{O(2n)}\widehat{\widetilde{\phi_u} * \phi_u}(\rho\eta)du = \widehat{\psi}(\rho\eta),$$

where

$$\psi(x,y) = \int_{O(2n)}\widetilde{\phi_u} * \phi_u(x,y)du,$$

it is clear that for all $u \in O(2n)$, $\psi_u(x,y) = \psi(x,y)$. Moreover, we have $\psi \in H_l$ since $\phi \in H_l$. It follows that $\widehat{\psi}(\eta) = \widehat{\psi}(\eta)\mathcal{P}_l$. Then by Schur's lemma (see [2, p .78]), we have

$$\int_{O(2n)}\widehat{\phi_u}(\rho\eta)^*\widehat{\phi_u}(\rho\eta)du = \frac{1}{d_l}\operatorname{tr}\big(\widehat{\phi}(\rho\eta)^*\widehat{\phi}(\rho\eta)\big)\mathcal{P}_l = \frac{1}{d_l}\|\widehat{\phi}(\rho\eta)\|_{HS}^2\mathcal{P}_l.$$

Therefore,

$$\langle W_\phi f, W_\phi g\rangle_{L^2(\mathbf{P},dm_l)} = \int_{\Lambda^2}\int_0^\infty \frac{1}{d_l}\|\widehat{\phi}(\rho\eta)\|_{HS}^2\frac{d\rho}{\rho}\operatorname{tr}\big(\widehat{g}(\eta)^*\widehat{f}(\eta)\big)\,\big|\det\eta\big|^{1/2}d\eta$$

$$= C_\phi\langle f, g\rangle_{L^2(F_{2n,2})}.$$

Setting $f = g$ then gives (4.1). ∎

Now let $\mathscr{S}(F_{2n,2})$ be the Schwartz space that coincides with the Schwartz space on $\mathbb{R}^{2n}\times\mathbb{R}^{n(2n-1)}$. Then we have the following theorem.

Theorem 4.2 *Let $\phi \in AW_l$ and $f \in H_l$. Then the Calderón reproducing formula*

$$f(x', y') = \frac{1}{C_\phi}\int_{\mathbf{P}} W_\phi f(x, y, \rho, u)U(x, y, \rho, u)\phi(x', y')dm_l(x, y, \rho, u) \tag{4.2}$$

holds in the L^2-sense. Moreover, if $f \in H_l \cap \mathscr{S}(F_{2n,2})$, then formula (4.2) is valid pointwise.

From Theorem 4.2 we see that the reproducing formula holds for functions in H_l, since the wavelets are restricted to the subspace H_l. We claim that these restrictions can be removed. Now suppose that $\phi \in L^2(F_{2n,2})$; we write $\phi = \Sigma_{l\in\mathbf{Z}_+}\phi_l$, where $\phi_l \in H_l$. If there exists a non-zero constant C_ϕ, which is independent of l such that for all $\eta \in \Lambda^2(\mathbb{R}^{2n})$,

$$\frac{1}{d_l}\int_0^\infty \|\widehat{\phi_l}(\rho\eta)\|_{HS}^2\frac{d\rho}{\rho} = C_\phi < \infty, \tag{4.3}$$

then we write $\phi \in AW$. The continuous wavelet transform for $f \in L^2(F_{2n,2})$ with respect to ϕ is defined by

$$W_\phi f(x, y, \rho, u) = \big\langle f, U(x, y, \rho, u)\phi\big\rangle_{L^2(F_{2n,2})}.$$

Theorem 4.3 *Suppose that $\phi \in AW$ and $f \in L^2(F_{2n,2})$. Then we have*

$$\|W_\phi f\|_{L^2(\mathbf{P},dm_l)}^2 = C_\phi\|f\|_{L^2(F_{2n,2})}^2.$$

Moreover, the Calderón reproducing formula

$$f(x', y') = \frac{1}{C_\phi}\int_{\mathbf{P}} W_\phi f(x, y, \rho, u)U(x, y, \rho, u)\phi(x', y')dm_l(x, y, \rho, u) \tag{4.4}$$

holds in the L^2-sense. If $f \in \mathscr{S}(F_{2n,2})$, then formula (4.4) holds pointwise.

Proof For $f \in L^2(F_{2n,2})$, by Theorem 3.3 we can write $f = \Sigma f_l$, where $f_l \in H_l$. From Theorem 4.1 together with (4.3) we have

$$\|W_\phi f\|_{L^2(\mathbf{P},dm_l)}^2 = \sum_{l=1}^\infty C_\phi\|f_l\|_{L^2(F_{2n,2})}^2 = C_\phi\|f\|_{L^2(F_{2n,2})}^2.$$

Then by the same argument as Theorem 4.2, we obtain (4.4). ∎

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Now we are going to construct a type of radial wavelets satisfying condition (4.3). The Calderón reproducing formula can be simplified by using the radial wavelets. We will use c to denote the positive constant, which is not necessarily the same at each occurrence.

Suppose that $f \in L^2(F_{2n,2})$. Similar to [24, (6.17)], we can write the Plancherel formula (3.5) in the integration of the "polar coordinate" form

$$\|f\|^2_{L^2(F_{2n,2})} = c\int_{O(n)}\int_S \big\|\widehat{f}(\Sigma\lambda_j a_j \wedge b_j)\big\|^2_{HS}\prod_{j=1}^n \lambda_j \times \prod_{j<k}(\lambda_j^2-\lambda_k^2)^2 d\lambda_1\cdots d\lambda_n d(a,b), \tag{4.5}$$

where $d(a,b)$ represents the Haar measure on $O(n)$, S denotes the simplicial cone $0<\lambda_1<\cdots<\lambda_n$ in $\mathbb{R}^n$, and the constant c depends on the normalization of Haar measure.

Let h satisfy

$$\int_S \big|h\big(\Pi_{j=1}^n\lambda_j\big)\big|^2\prod_{j=1}^n\lambda_j\prod_{j<k}(\lambda_j^2-\lambda_k^2)^2 d\lambda_1\cdots d\lambda_n<\infty \tag{4.6}$$

and

$$C_h=\int_0^\infty \big|h(\rho)\big|^2\frac{d\rho}{\rho}<\infty. \tag{4.7}$$

We set

$$\widehat{\phi}_h(\eta)=\sum_{l=0}^\infty h\big(d_l^{2n}|\det\eta|^{1/2}\big)\,\mathcal{P}_l. \tag{4.8}$$

We claim that $\phi_h\in AW$. Noticing that $\sum_{l=0}^\infty d_l^{-(4n^2-1)}<\infty$, together with (4.5) and (4.6), we have

$$\begin{aligned}\|\phi_h\|^2_{L^2(F_{2n,2})} &= c\int_S\sum_{l=0}^\infty d_l\big|h\big(d_l^{2n}\Pi_{j=1}^n\lambda_j\big)\big|^2\prod_{j=1}^n\lambda_j\prod_{j<k}(\lambda_j^2-\lambda_k^2)^2 d\lambda_1\cdots d\lambda_n\\ &= c\sum_{l=0}^\infty d_l^{-(4n^2-1)}\int_S\big|h\big(\Pi_{j=1}^n\lambda_j\big)\big|^2\prod_{j=1}^n\lambda_j\prod_{j<k}(\lambda_j^2-\lambda_k^2)^2 d\lambda_1\cdots d\lambda_n\\ &= c\int_S\big|h\big(\Pi_{j=1}^n\lambda_j\big)\big|^2\prod_{j=1}^n\lambda_j\prod_{j<k}(\lambda_j^2-\lambda_k^2)^2 d\lambda_1\cdots d\lambda_n<\infty.\end{aligned}$$

Now write $\phi_h=\sum_{l=0}^\infty\phi_{h,l}$, where $\phi_{h,l}\in H_l$. Then

$$\widehat{\phi}_{h,l}(\eta)=h\big(d_l^{2n}|\det\eta|^{1/2}\big)\,\mathcal{P}_l.$$

Thus we have

$$
\begin{aligned}
C_{\phi_h} &= \frac{1}{d_l}\int_0^\infty \|\widehat{\phi}_{h,l}(\rho\eta)\|_{HS}^2 \frac{d\rho}{\rho} = \int_0^\infty \big| h(d_l^{2n}\rho^n |\det\eta|^{1/2})\big|^2 \frac{d\rho}{\rho} \\
&= \int_0^\infty \big| h(\rho')\big|^2 \frac{d\rho'}{\rho'} = C_h.
\end{aligned}
$$

Note that the radial wavelet defined above is independent of the action of rotation operators, that is, $(\phi_h)_u(x,y) = \phi_h(x,y)$. For brevity we write $W_{\phi_h} f(x,y,\rho)$ and $U(x,y,\rho)$ instead of $W_{\phi_h} f(x,y,\rho,1)$ and $U(x,y,\rho,1)$, respectively.

Theorem 4.4 *Suppose the radial wavelet ϕ_h defined above and $f \in L^2(F_{2n,2})$. Then we have*

$$
\int_{F_{2n,2}}\int_0^\infty |W_{\phi_h} f(x,y,\rho)|^2 \frac{dx\,dy\,d\rho}{\rho^{2n^2+1}} = C_h\|f\|^2_{L^2(F_{2n,2})}.
$$

Moreover, the Calderón reproducing formula

$$
f(x',y') = \frac{1}{C_h}\int_{F_{2n,2}}\int_0^\infty W_{\phi_h} f(x,y,\rho)U(x,y,\rho)\phi(x',y')\frac{dx\,dy\,d\rho}{\rho^{2n^2+1}} \tag{4.9}
$$

holds in the L^2-sense. If $f \in \mathscr{S}(F_{2n,2})$, then formula (4.9) holds pointwise.

In the following we shall give a concrete example of radial wavelets on $F_{2n,2}$.

Example 4.5 Let $\lambda = (\lambda_1,\dots,\lambda_n) \in S$ and $\eta = \Sigma\lambda_j a_j \wedge b_j$. We define

$$
h(|\det\eta|^{1/2}) = \begin{cases} |\det\eta|^{1/2}\exp\{-|\det\eta|^{1/2}\}, & \lambda_1 > 2,\\ 0, & \text{others.}\end{cases} \tag{4.10}
$$

Then ϕ_h defined by (4.8) belongs to AW.

In order to prove that ϕ_h defined by (4.8) belongs to AW, we need to show that $h(|\det\eta|^{1/2})$ satisfies (4.6) and (4.7). First we claim that if $2 < \lambda_1 < \cdots < \lambda_n$, then $\Pi_{j=1}^n\lambda_j \geq \Sigma_{j=1}^n\lambda_j$. In fact, we set $\lambda_k = 2+\alpha_k, \alpha_k \geq 0, k = 1,2,\dots,n$. If $n = 2$, we can verify the inequality directly: $\lambda_1\lambda_2 = (2+\alpha_1)(2+\alpha_2) \geq 4+(\alpha_1+\alpha_2) = \lambda_1+\lambda_2$. By the induction we have the desired result. Substituting (4.10) into (4.6), we have

$$
\begin{aligned}
&\int_S \big| h\big(\Pi_{j=1}^n\lambda_j\big)\big|^2 \prod_{j=1}^n \lambda_j \prod_{j<k}(\lambda_j^2-\lambda_k^2)^2 d\lambda_1\cdots d\lambda_n \\
&= \int_S \Big| \prod_{j=1}^n \lambda_j \exp\{-\Pi_{j=1}^n\lambda_j\}\Big|^2 \prod_{j=1}^n \lambda_j \prod_{j<k}(\lambda_j^2-\lambda_k^2)^2 d\lambda_1\cdots d\lambda_n \\
&\leq \int_S \exp\{-2\Pi_{j=1}^n\lambda_j\}\prod_{j=1}^n\lambda_j^3\prod_{j=1}^n\lambda_j^{4(j-1)} d\lambda_1\cdots d\lambda_n \\
&\leq c\int_S \exp\{-2\Sigma_{j=1}^n\lambda_j\}\prod_{j=1}^n\lambda_j^{4j-1} d\lambda_1\cdots d\lambda_n \\
&\leq c\int_{\mathbb{R}^n} \exp\{-2\Sigma_{j=1}^n\lambda_j\}\prod_{j=1}^n\lambda_j^{4j-1} d\lambda_1\cdots d\lambda_n = c\prod_{j=1}^n\Gamma(4j) < \infty.
\end{aligned}
$$

On the other hand, we can see that

$$C_h = c\int_0^\infty \rho^{2n}\det\eta\exp\{-2\rho^n|\det\eta|^{1/2}\}\frac{d\rho}{\rho} = c\Gamma(1) < \infty.$$

This implies that $\phi_h \in AW$.

Liu and Peng [14] constructed admissible wavelets on the Heisenberg group, and gave the explicit expressions of wavelet functions by using group Fourier transforms. In present case, the explicit expression can be calculated in a similar way. We omit the details.

5 The Radon Transform on $F_{2n,2}$

The Radon transform R on $F_{2n,2}$ is defined by

$$Rf(x,y) = \int_{\mathbb{R}^{2n}} f\big((x,y)(x',0)\big)\,dx' = \int_{\mathbb{R}^{2n}} f\big(x', y - \frac{1}{2}x\wedge x'\big)\,dx'. \tag{5.1}$$

The hyperplane $\{(x', y - \frac{1}{2}x\wedge x') : x' \in \mathbb{R}^{2n}\}$ can be written as

$$\Big\{(x',y') : y'_{jk} = y_{jk} + \frac{1}{2}(x_j x'_k - x_k x'_j), 1 \le j < k \le 2n\Big\}.$$

Clearly, when $n = 1$, (5.1) is just the Radon transform on the Heisenberg group H^1.

Strichartz [24] obtained two inversion formulas for the inversion Radon transform by means of the Euclidean Fourier transform. However, he did not indicate on which spaces these formula hold. One such formula is

$$R^{-1} = \mathscr{F}^{-1} J \mathscr{F}_2,$$

where $\mathscr{F}$ is the Euclidean Fourier transform for all variables and J is an operator defined by

$$Jf(x,y) = f(-2y^{-1}x, y).$$

Another is

$$R^{-1} = (-1)^n(4\pi)^{-2n}\,\mathrm{Pf}(\partial/\partial y) R\,\mathrm{Pf}(\partial/\partial y), \tag{5.2}$$

where Pf denotes the Pfaffian, which will be discussed later. He also pointed out that even if $f \in \mathscr{S}(F_{2n,2})$, the Radon transform $R(f)$ may not rapidly decrease at infinity. In this section we will find a subspace of $\mathscr{S}(F_{2n,2})$ on which the Radon transform is a bijection.

Let $y^s = \Pi_{1\le j<k\le 2n} y_{jk}^{s_{jk}}$, where $s \in \mathbf{Z}_+^{2n^2-n}$, we define the subspace $\mathscr{S}_*(F_{2n,2})$ of $\mathscr{S}(F_{2n,2})$ by

$$\mathscr{S}_*(F_{2n,2}) = \Big\{ f \in \mathscr{S}(F_{2n,2}) : \int_{\Lambda^2} f(x,y) y^s dy = 0 \text{ for all } x \in \mathbb{R}^{2n}, s \in \mathbf{Z}_+^{2n^2-n}\Big\}.$$

Write

$$\partial_y^s f(x,0) = \frac{\partial^{|s|}}{\Pi_{1\le j<k\le 2n}\partial y_{jk}^{s_{jk}}} f(x,y)|_{y=0}$$

and define another subspace $\mathscr{S}^*(F_{2n,2})$ by

$$\mathscr{S}^*(F_{2n,2}) = \left\{ f \in \mathscr{S}(F_{2n,2}) : \partial_y^s f(x,0) = 0 \text{ for all } x \in \mathbb{R}^{2n}, s \in \mathbf{Z}_+^{2n^2-n} \right\}.$$

By argument analogous to [20, Proposition 5.1], we also find that $f \in \mathscr{S}_*(F_{2n,2})$ if and only if $\mathscr{F}_2(f) \in \mathscr{S}^*(F_{2n,2})$, and $\mathscr{F}_2$ is an isomorphism from $\mathscr{S}_*(F_{2n,2})$ onto $\mathscr{S}^*(F_{2n,2})$. The spaces $\mathscr{S}^*(F_{2n,2})$ and $\mathscr{S}_*(F_{2n,2})$ are regarded as Semyanistyi–Lizonkin type spaces that have many applications (see [16, 18, 21]).

Let $f \in \mathscr{S}_*(F_{2n,2})$. Similar to [20, Theorem 2], we conclude that the Radon transform R is a bijection on $\mathscr{S}_*(F_{2n,2})$. We also have the following theorem.

Theorem 5.1 *Let $f \in \mathscr{S}_*(F_{2n,2})$. Then $R^{-1}f = \mathscr{F}^{-1}J\mathscr{F}_2 f$ holds pointwise.*

Now we give another proof of the formula (5.2) by using of the group Fourier transform on $F_{2n,2}$. The next theorem shows the relation between the Radon transform and the group Fourier transform.

Theorem 5.2 *Suppose $f \in L^2(F_{2n,2})$. Then we have*

$$\widehat{Rf}(\eta)h_\alpha(\xi) = (-1)^{|\alpha|}|\det\eta|^{-\frac{1}{2}}\widehat{f}(\eta)h_\alpha(\xi).$$

Proof Note that the Radon transform of f can be written as

$$Rf(x,y) = f * \delta_2(x,y),$$

where δ_2 is the Direct delta function with respect to the second variable. By (3.6) we have

$$\widehat{Rf}(\eta)h_\alpha(\xi) = \widehat{f}(\eta)\widehat{\delta_2}(\eta)h_\alpha(\xi).$$

Here,

$$\begin{aligned}
\widehat{\delta_2}(\eta)h_\alpha(\xi) = &\int_{\mathbb{R}^{2n}} h_\alpha\Big(\xi + \sum_{j=1}^n (x\cdot b_j)e_j\Big) \\
&\times \exp\Big(2\pi i\Big(\sum_{j=1}^n \lambda_j\xi_j(x\cdot a_j) + \frac{1}{2}\sum_{j=1}^n \lambda_j(x\cdot a_j)(x\cdot b_j)\Big)\Big)\,dx \\
= &\int_{\mathbb{R}^{2n}} h_\alpha\Big(\xi + \sum_{j=1}^n (x\cdot b_j)e_j\Big) \\
&\times \exp\Big(2\pi i\Big(\xi + \sum_{j=1}^n (x\cdot b_j)e_j\Big)\cdot\frac{1}{2}\sum_{j=1}^n \lambda_j(x\cdot a_j)e_j\Big) \\
&\times \exp\Big(2\pi i\Big(\frac{1}{2}\sum_{j=1}^n \lambda_j(x\cdot a_j)e_j\cdot\xi\Big)\Big)\,dx.
\end{aligned}$$

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In view of the Euclidean Fourier transform and its inversion, we get

$$\widehat{\delta_2}(\eta)h_\alpha(\xi) = \prod_{j=1}^{n} \lambda_j h_\alpha(-\xi).$$

By the definition of the Hermite function we have $h_\alpha(-\xi) = (-1)^{|\alpha|}h_\alpha(\xi)$. As desired, we finish the proof of this theorem. ■

Now for $y \in \Lambda^2(\mathbb{R}^{2n})$, the Pfaffian of y is defined by

$$\mathrm{Pf}(y) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) y_{\sigma(1)\sigma(2)} y_{\sigma(3)\sigma(4)} \cdots y_{\sigma(n-1)\sigma(n)},$$

where S_{2n} is the set of all permutations on $2n$ letters, and $\mathrm{sgn}(\sigma)$ is the signature of σ. Indeed, the determinant of y can be written as the square of a polynomial in the entries of y, that is $\det y = \mathrm{Pf}(y)^2$. Thus the determinant of a real skew-symmetric matrix is always non-negative. Suppose that A is an $2n \times 2n$ matrix, then one has

$$\mathrm{Pf}(A^t yA) = \mathrm{Pf}(AyA^t) = \det A \ \mathrm{Pf}(y), \tag{5.3}$$

where A^t denotes the transpose of A. Moveover, for $\rho > 0$,

$$\mathrm{Pf}(\rho y) = \rho^n \,\mathrm{Pf}(y). \tag{5.4}$$

For more properties of the skew-symmetric matrix and the Pfaffian, we refer the reader to [11, 15].

Now we write
$$\frac{\partial}{\partial y} = \left(\frac{\partial}{\partial y_{jk}} \right).$$

A direct computation shows that

$$\mathscr{F}_2\big((2\pi)^{-1} i\partial/\partial y_{jk} f\big)(x,\eta) = \eta_{jk} \mathscr{F}_2(f)(x,\eta).$$

It follows that

$$\mathscr{F}_2\Big((2\pi)^{n-2n^2}\, \mathrm{Pf}\big(i\partial/\partial y\big)\, f\Big)(x,\eta) = \mathrm{Pf}(\eta)\mathscr{F}_2(f)(x,\eta).$$

Set $L = (2\pi)^{n-2n^2}\, \mathrm{Pf}\big(i\partial/\partial y\big)$, then

$$\widehat{Lf}\,(\eta) = \mathrm{Pf}(\eta)\widehat{f}(\eta). \tag{5.5}$$

Notice that if $f \in \mathscr{S}(F_{2n,2})$, then $L^k f \in L^2(F_{2n,2})$ for all $k \in \mathbf{Z}_+$. This means that

$$\int_{\Lambda^2} \|\widehat{f}(\eta)\|_{HS}^2 \big|\det \eta\big|^{k+\frac{1}{2}} d\eta < \infty \quad \text{for all } k \in \mathbf{Z}_+. \tag{5.6}$$

Now let R^k be the k-th Radon transform. By Theorem 5.2 we have

$$\|\widehat{R^k f}(\eta)\|_{HS}^2 \big|\det\eta\big|^k = \|\widehat{f}(\eta)\|_{HS}^2. \tag{5.7}$$

By (5.6) it is natural to define the space $\mathscr{S}_R(F_{2n,2})$ by

$$\mathscr{S}_R(F_{2n,2}) = \Big\{ f \in \mathscr{S}(F_{2n,2}) : \int_{\Lambda^2} \|\widehat{f}(\eta)\|_{HS}^2 \big|\det\eta\big|^{-k+\frac{1}{2}} d\eta < \infty \quad \text{for all } k \in \mathbf{Z}_+ \Big\}.$$

Obviously, the Radon transform R on $\mathscr{S}_R(F_{2n,2})$ is a bijection. Moreover, by Theorem 5.2 and (5.5) we have

$$\big((LR)^2 f\big)\widehat{\ }(\eta)h_\alpha(\xi) = \widehat{f}(\eta)h_\alpha(\xi),$$

from which we deduce that $R^{-1}f = LRLf$ holds pointwise in $\mathscr{S}_R(F_{2n,2})$.

Now for $\mu \in \mathbb{R}$, by (5.5) we have

$$\widehat{L^\mu f}(\eta) = \mathrm{Pf}(\eta)^\mu \widehat{f}(\eta). \tag{5.8}$$

It is clear that $L^\mu R^k$ is well defined for all $\mu \in \mathbb{R}$ and $k \in \mathbf{Z}_+$ on $\mathscr{S}_R(F_{2n,2})$. Then we have an extension of (5.2).

Theorem 5.3 *Suppose $f \in \mathscr{S}_R(F_{2n,2})$. Then for any $k \in \mathbf{Z}_+$, we have*

$$\|L^k R^k f\|_{L^2(F_{2n,2})} = \|f\|_{L^2(F_{2n,2})}. \tag{5.9}$$

Moreover, for $\mu, \nu \in \mathbb{R}$ and $\mu + \nu = k + 1 \in 2\mathbf{Z}_+$, then

$$R^{-1} f = L^\mu R^k L^\nu f \tag{5.10}$$

holds pointwise.

Proof Formula (5.9) follows from (5.7) and (5.8). Formula (5.10) can be easily derived from (5.9). ∎

The following theorem will indicate that our two treatments about the Radon transform are essentially equivalent.

Theorem 5.4 $\mathscr{S}_R(F_{2n,2}) = \mathscr{S}_*(F_{2n,2})$.

Proof Let $\eta = (\eta_{jk}) \in \Lambda^2(\mathbb{R}^{2n})$. We say that $\eta \to 0$ if all the $\eta_{jk} \to 0$, and the symbol (η_{jk}^0) represents that all the elements except η_{jk} $(j \neq k)$ are equal to zero. From (3.4) we know that

$$\|\widehat{f}(\eta)\|_{HS}^2 = |\det\eta|^{-1/2}\|f^\eta\|_{L^2(\mathbb{R}^{2n})}^2.$$

If $f \in \mathscr{S}_R(F_{2n,2})$, then for all $k \in \mathbf{Z}_+$,

$$\lim_{\det\eta \to 0} |\det\eta|^{-k} \int_{\mathbb{R}^{2n}} |\mathscr{F}_2 f(x,\eta)|^2 dx = 0.$$

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In particular, we have

$$\lim_{\eta\to 0} |\det \eta|^{-k} \int_{\mathbb{R}^{2n}} |\mathscr{F}_2 f(x,\eta)|^2 dx = 0,$$

which implies that

$$\int_{\mathbb{R}^{2n}} |\mathscr{F}_2 f(x,0)|^2 dx = \lim_{\eta\to 0} \int_{\mathbb{R}^{2n}} |\mathscr{F}_2 f(x,\eta)|^2 dx = 0.$$

Therefore we deduce

$$\int_{\mathbb{R}^{2n}} |\partial_{\eta_{jk}} \mathscr{F}_2 f(x,0)|^2 dx = \lim_{\eta_{jk}\to 0} \eta_{jk}^{-2} \int_{\mathbb{R}^{2n}} \left| \mathscr{F}_2 f\big(x, (\eta^0_{jk})\big) \right|^2 dx = 0.$$

By induction, we obtain that for all $s \in \mathbf{Z}_+^{2n^2-n}$,

$$\int_{\mathbb{R}^{2n}} |\partial^s_\eta \mathscr{F}_2 f(x,0)|^2 dx = 0.$$

This proves that $\mathscr{F}_2 f \in \mathscr{S}^*(F_{2n,2})$, and equivalently $f \in \mathscr{S}_*(F_{2n,2})$.

Conversely, suppose that $f \in \mathscr{S}_*(F_{2n,2})$, *i.e.*, $\mathscr{F}_2 f \in \mathscr{S}^*(F_{2n,2})$. Then we get for all $x \in \mathbb{R}^{2n}$,

$$\lim_{\eta\to 0} \int_{\mathbb{R}^{2n}} |\mathscr{F}_2 f(x,\eta)|^2 dx = \int_{\mathbb{R}^{2n}} |\mathscr{F}_2 f(x,0)|^2 dx = 0.$$

Then we have

$$\lim_{\eta\to 0} |\det \eta|^{-1} \int_{\mathbb{R}^{2n}} |\mathscr{F}_2 f(x,\eta)|^2 dx = \\ \lim_{\eta\to 0} \int_{\mathbb{R}^{2n}} \left| \sum_{1\le j<k\le 2n} \frac{\eta_{jk}}{|\det \eta|^{1/2}} \partial_{\eta_{jk}} \mathscr{F}_2 f\big(x, (\theta\eta_{jk})\big) \right|^2 dx = 0,$$

where $0 < \theta < 1$. Inductively, for all $k \in \mathbf{Z}_+$,

$$\lim_{\eta\to 0} |\det \eta|^{-k} \int_{\mathbb{R}^{2n}} |\mathscr{F}_2 f(x,\eta)|^2 dx = 0.$$

Hence $f \in \mathscr{S}_R(F_{2n,2})$. This completes the proof. ∎

Now we are going to find another inversion formula for the Radon transform on the group $F_{2,2}$, which is homogeneous to the Heisenberg group H^1. First, Theorem 5.2 tells us that for $f \in L^2(F_{2,2})$, we have

$$\widehat{Rf}(\eta) h_l(\xi) = (-1)^l \lambda^{-1} \widehat{f}(\eta) h_l(\xi), \tag{5.11}$$

where $h_l(\xi)$ denotes a one dimensional Hermite function. On the other hand, in Theorem 3.3 we have proved that $L^2(F_{2,2}) = \bigoplus_{l=0}^{\infty} H_l$. Then given a function $f \in H_l$, we have by (3.8) that

$$\widehat{\mathcal{L} f}(\eta) h_l = 2\pi\lambda(1+2l) \widehat{f}(\eta) h_l. \tag{5.12}$$

Theorem 5.5 *Let $f \in H_l \cap \mathscr{S}_R(F_{2,2})$. Then for any $k \in \mathbf{Z}_+$ we have*

$$\|\mathcal{L}^k R^k(f)\|^2_{L^2(F_{2,2})} = C_{l,k}\|f\|^2_{L^2(F_{2,2})},$$

where $C_{l,k} = (2\pi(1+2l))^{2k}$. Moreover, for $\mu, \nu \in \mathbb{R}$ and $\mu + \nu = k + 1 \in 2\mathbf{Z}_+$, then

$$R^{-1}(f) = C^{-1}_{l,k+1}\, \mathcal{L}^\mu R^k \mathcal{L}^\nu(f).$$

Proof It is easy to see that $L^k R^k(f)$ is well defined for all $k \in \mathbf{Z}_+$, since $f \in H_l \cap \mathscr{S}_R(F_{2,2})$. By (5.11) and (5.12) we get

$$\widehat{\mathcal{L}^k R^k(f)}(\eta)h_l = (-1)^{kl}(2\pi(1+2l))^k \widehat{f}(\eta)h_l.$$

Then

$$\begin{aligned}\|\mathcal{L}^k R^k(f)\|^2_{L^2(F_{2,2})} &= \int_{\Lambda^2} \|(2\pi(1+2l))^k \widehat{f}(\eta)\|^2_{HS} |\det \eta|^{1/2} d\eta \\ &= (2\pi(1+2l))^{2k}\|f\|^2_{L^2(F_{2,2})}.\end{aligned}$$

This indicates that for $\mu + \nu = k + 1 \in 2\mathbf{Z}_+$,

$$R^{-1}(f) = C^{-1}_{l,k+1}\, \mathcal{L}^\mu R^k \mathcal{L}^\nu(f).$$

The desired result is proved. ■

From expression (3.8) of eigenvalues for sub-Laplacian $\mathcal{L}$ we can see that Theorem 5.5 will not hold on the group $F_{2n,2}$ when $n \geq 2$.

6 Inverse Radon Transform by Using Wavelets

Note that the inversion formulas of the Radon transform in Section 5 require the smoothness of functions. In this section we will establish an inversion formula of the Radon transform in L^2-sense by using the inverse wavelet transform. This formula does not require the smoothness of functions. Instead, we will use smooth wavelets.

Let

$$L^2_\sharp(F_{2n,2}) = \left\{ f \in L^2(F_{2n,2}) : \int_{\Lambda^2} \|\widehat{f}(\eta)\|^2_{HS} |\det \eta|^{-1/2} d\eta < \infty \right\},$$

$$L^2_\natural(F_{2n,2}) = \left\{ f \in L^2(F_{2n,2}) : \int_{\Lambda^2} \|\widehat{f}(\eta)\|^2_{HS} |\det \eta|^{3/2} d\eta < \infty \right\}.$$

It is easy to see that if $f \in L^2_\sharp(F_{2n,2})$, then $Rf \in L^2(F_{2n,2})$ and if $f \in L^2_\natural(F_{2n,2})$, then $R^{-1}f \in L^2(F_{2n,2})$. Suppose that $\phi \in AW \bigcap \mathscr{S}_R(F_{2n,2})$ and $f \in L^2_\sharp(F_{2n,2})$. Then $W_{L^\mu R^k L^\nu \phi} Rf$ is well defined for all $\mu + \nu = k + 1 \in 2\mathbf{Z}_+$, and we have the following theorem.

Theorem 6.1 *Let $\mu+\nu=k+1\in 2\mathbf{Z}_+$. Suppose $\phi\in AW\cap\mathscr{S}_R(F_{2n,2})$ and $f\in L^2_\sharp(F_{2n,2})$. Then $W_{L^\mu R^k L^\nu\phi}Rf=\rho^n W_\phi f$.*

Proof The wavelet transform can be written in the form

$$W_\phi f(x,y,\rho,u)=\rho^{n^2} f*\widetilde{\phi}_{\rho,u}(x,y).$$

By a direct computation we have

$$\int_{F_{2n,2}} W_\phi f(x,y,\rho,u)\pi_\eta(x,y)dxdy=\rho^{n^2}\widehat{f}(\eta)\widehat{\phi_{\rho,u}}(\eta)^*.$$

Note that if $u\in O(2n)$, then $u^{-1}=u^t$ and $\det u=\pm 1$. Since $\mu+\nu=k+1$ is an even number, by Proposition 3.2 together with (5.3) and (5.4), we then have

$$\begin{aligned}\big(W_{L^\mu R^k L^\nu\phi}Rf\big)\widehat{\ }(\eta)&=\rho^{n^2}\widehat{Rf}(\eta)(L^\mu R^k L^\nu\phi)_{\rho,u}\widehat{\ }(\eta)^*\\&=\mathrm{Pf}(\rho u^{-1}\eta u)^{k+1}\rho^{n^2}\widehat{Rf}(\eta)\widehat{R^k\phi}\,(\rho u^{-1}\eta u)^*\\&=\rho^n\det(u)^{k+1}|\det(u)|^{-k}\,\rho^{n^2}\widehat{f}(\eta)\widehat{\phi}\,(\rho u^{-1}\eta u)^*\\&=\rho^n\int_{F_{2n,2}}W_\phi f(x,y,\rho,u)\pi_\eta(x,y)dxdy,\end{aligned}$$

from which we deduce the desired result. ∎

By this theorem together with the Calderón reproducing formula (4.4), we have the following theorem.

Theorem 6.2 *Let $\mu+\nu=k+1\in 2\mathbf{Z}_+$. Suppose $\phi\in AW\cap\mathscr{S}_R(F_{2n,2})$ and $f\in H_l\cap L^2_\sharp(F_{2n,2})$. Then*

$$f(x',y')=C_\phi^{-1}\int_{\mathbf{P}}W_{L^\mu R^k L^\nu\phi}Rf(x,y,\rho,u)U(x,y,\rho,u)\phi(x',y')\frac{dxdyd\rho du}{\rho^{2n^2+n+1}}\tag{6.1}$$

in the L^2-sense. Furthermore, if $f\in\mathscr{S}(F_{2n,2})\cap L^2_\sharp(F_{2n,2})$, then (6.1) holds pointwise. Equivalently, if $f\in L^2_\natural(F_{2n,2})$, then

$$\begin{aligned}&R^{-1}f(x',y')=\\&\qquad C_\phi^{-1}\int_{\mathbf{P}}W_{L^\mu R^k L^\nu\phi}f(x,y,\rho,u)U(x,y,\rho,u)\phi(x',y')\frac{dxdyd\rho du}{\rho^{2n^2+n+1}}\end{aligned}\tag{6.2}$$

in the L^2-sense. Furthermore, if $f\in\mathscr{S}(F_{2n,2})\cap L^2_\natural(F_{2n,2})$, then (6.2) holds pointwise.

Suppose that h is given by (4.10) and ϕ_h is defined by (4.8). It is easy to verify that $\phi_h\in\mathscr{S}_R(F_{2n,2})$. Then by Theorem 4.4, we have the following theorem.

Theorem 6.3 *Let $\mu+\nu = k+1 \in 2\mathbf{Z}_+$. Suppose h is given by (4.10) and ϕ_h is defined by (4.8). Then for $f \in L^2_\sharp(F_{2n,2})$,*

$$(6.3)\quad f(x',y') = C_h^{-1}\int_{F_{2n,2}}\int_0^\infty W_{L^\mu R^k L^\nu \phi_h} Rf(x,y,\rho)U(x,y,\rho)\phi_h(x',y')\frac{dx\,dy\,d\rho}{\rho^{2n^2+n+1}}$$

in the L^2-sense. Furthermore, if $f \in \mathscr{S}(F_{2n,2}) \cap L^2_\sharp(F_{2n,2})$, then (6.3) holds pointwise. Equivalently, if $f \in L^2_\natural(F_{2n,2})$, then

$$(6.4)\quad R^{-1}f(x',y') = C_h^{-1}\int_{F_{2n,2}}\int_0^\infty W_{L^\mu R^k L^\nu \phi_h} f(x,y,\rho)U(x,y,\rho)\phi_h(x',y')\frac{dx\,dy\,d\rho}{\rho^{2n^2+n+1}}$$

in the L^2-sense. Furthermore, if $f \in \mathscr{S}(F_{2n,2}) \cap L^2_\natural(F_{2n,2})$, then (6.4) holds pointwise.

In fact, the wavelet transform on the group $F_{2,2}$ can be defined in the obvious way. Then we have the Calderón reproducing formula

$$f(x',y') = \frac{1}{C_\phi}\int_{F_{2,2}}\int_0^\infty W_\phi f(x,y,\rho)U(x,y,\rho)\phi(x',y')\frac{dx\,dy\,d\rho}{\rho^3},$$

which is a special case of the formula (4.2). Moreover, by Theorem 5.5 we have the following inversion of the Radon transform associated with the sub-Laplacian.

Theorem 6.4 *Let $\mu+\nu = k+1 \in 2\mathbf{Z}_+$. Suppose $\phi \in AW_l \cap \mathscr{S}_R(F_{2,2})$ and $f \in H_l \cap L^2_\sharp(F_{2,2})$. Then*

$$(6.5)\quad f(x',y') = C_\phi^{-1}C_{l,k+1}^{-1}\int_{F_{2,2}}\int_0^\infty W_{\mathcal{L}^\mu R^k \mathcal{L}^\nu \phi} Rf(x,y,\rho)U(x,y,\rho)\phi(x',y')\frac{dx\,dy\,d\rho}{\rho^4}$$

in the L^2-sense. Furthermore, if $f \in H_l \cap \mathscr{S}(F_{2,2}) \cap L^2_\sharp(F_{2,2})$, then (6.5) holds pointwise. Equivalently, if $f \in H_l \cap L^2_\natural(F_{2,2})$, then

$$(6.6)\quad R^{-1}f(x',y') = C_\phi^{-1}C_{l,k+1}^{-1}\int_{F_{2,2}}\int_0^\infty W_{\mathcal{L}^\mu R^k \mathcal{L}^\nu \phi} f(x,y,\rho)U(x,y,\rho)\phi(x',y')\frac{dx\,dy\,d\rho}{\rho^4}$$

in the L^2-sense. Furthermore, if $f \in H_l \cap \mathscr{S}(F_{2,2}) \cap L^2_\natural(F_{2,2})$, then (6.6) holds pointwise.

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