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A note on flabby sheaves

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It is shown that any sheaf of $\tilde{R}$-modules, all of whose stalks are injective, is necessarily a flabby sheaf. This generalizes the result of Grothendieck that the sheaf $\tilde{M}$ determined by an injective module M over a commutative noetherian ring with 1 is flabby.

In the following, R shall be a commutative ring with 1. Let $X = \mathrm{Spec}(R)$ and $\tilde{R}$ be the structure sheaf of R over X. Campbell [1] gives an elementary proof of a result of Grothendieck in [3] (Corollary 2.7). The following generalizes Grothendieck's result, which we shall state as a corollary, and yet the proof avoids any use of local cohomology. We use the same notation and terminology as Macdonald [4].

DEFINITION. We say the $\tilde{R}$-module $\underline{F}$ is flabby (flasque) if for every open set $U \subseteq X$, the restriction map $\underline{F}(X) \to \underline{F}(U)$ is onto.

PROPOSITION. *Let $\underline{F}$ be an $\tilde{R}$-module (not necessarily quasi-coherent) such that $\underline{F}_x$ is an injective R_{p_x}-module for all $x \in X$. Then $\underline{F}$ is flabby.*

Proof. If U is an open set in X, let us write $\underline{F}_U = (\underline{F}|U)^X$, where $\underline{F}|U$ is the restriction of $\underline{F}$ to U and $(\ )^X$ is prolongation by zero. There is an exact sequence of sheaves

$$0 \to \underline{F}_U \xrightarrow{f} \underline{F} ,$$

where f is the injection map. Let $U \subseteq X$ be open and suppose

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$\sigma \in \underline{F}(U)$. Let $E(\underline{F}_U)$ be the $\tilde{R}$-module defined as follows: If $V \subseteq X$ is open, then $E(\underline{F}_U)(V) = \prod_{x \in V} (\underline{F}_U)_x$, and with the obvious restriction maps. Now $E(\underline{F}_U)$ is an injective $\tilde{R}$-module since each $E(\underline{F}_U)$ is an injective R_{p_x}-module [2; proof of 7.1.1]. Consider the sheaf map

$$\alpha : \underline{F}_U \to E(\underline{F}_U)$$

which, for every open set $V \subseteq X$, is given by

$$\alpha(V) : \underline{F}_U(V) \to E(\underline{F}_U)(V) ,$$

the homomorphism sending $\gamma \to (\gamma(x))_{x \in V}$, where $\gamma(x)$ is the germ of γ at x . Since $E(\underline{F}_U)$ is injective, there exists a sheaf map $\hat{\alpha} : \underline{F} \to E(\underline{F}_U)$ such that $\hat{\alpha} f = \alpha$. That is,

$$\begin{array}{ccc} 0 \to & \underline{F}_U & \xrightarrow{f} \underline{F} \\ & \alpha \downarrow & \swarrow \hat{\alpha} \\ & E(\underline{F}_U) & \end{array}$$

commutes. In particular, $\hat{\alpha}(X)f(X) = \alpha(X)$. Note that $\underline{F}_U(X) = \underline{F}(U)$. Thus, if $\sigma \in \underline{F}(U)$, then $\sigma \in \underline{F}_U(X)$. Hence $\hat{\alpha}(X)(\tau) = \alpha(X)(\sigma)$ where $\tau = f(X)(\sigma) \in \underline{F}(X)$. Since f is the injection map, this means $(\tau(x))_{x \in U} = (\sigma(x))_{x \in U}$. Therefore $\tau | U = \sigma$, so that $\underline{F}$ is flabby.

COROLLARY (Grothendieck). *If R is noetherian and M is an injective R-module, then $\tilde{M}$ is flabby.*

Proof. Since R is noetherian and M is an injective R-module, M_{p_x} is an injective R_{p_x}-module for all $x \in X$. Now $\tilde{M}_x = M_{p_x}$ for all $x \in X$, so by the above proposition, $\tilde{M}$ is flabby.

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