DERIVATIONS RELATED TO A SEPARABLE ELEMENT OF AN ALGEBRA

JAMES MARTIN and W. G. LEAVITT

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Let R be a C-algebra over a commutative ring C of zero characteristic. An element $a \in R$ will be called *separable* if there exists $p \in C[x]$ for which $p(a) = 0$ and such that $p'(a)$ is invertible, where p' is the formal derivative of p. Call A the C-algebra generated by a_R, and a_L, the right and left multiplications by a, and write D_a for the inner derivation defined by a. It will be shown that when a is separable there exists $\phi \in A$ such that $[p'(a)]^{-1}\phi D_a$ is idempotent. As a consequence it follows that the additive group of R may be decomposed into a direct sum of Ker D_a and Im D_a. Another result is that for an arbitrary C-derivation δ there exists $d \in \text{Im } D_a$ such that $a\delta = aD_d$. Thus Ker D_a (and also Im D_a) is a δ-subgroup of R^+ if and only if $a\delta = 0$.

THEOREM 1. *Let a be separable over C with $p(a) = 0$ and $h = p'(a)$ invertible. There exists ϕ in the C-algebra A generated by a_R and a_L such that $h^{-1}\phi D_a$ is idempotent.*

PROOF. Suppose p has degree n, and we write

$$\theta = \sum_{k=1}^{n} \frac{p^{(k)}(a_L)}{k!} D_a^{k-1}, \tag{1}$$

where $p^{(k)}$ is the formal k^{th} derivative of p. Now $p(a_L) = 0$ and $a_R = a_L + D_a$. Then since A is a commutative algebra, it is easy to see that

$$\theta D_a = \sum_{0}^{n} \frac{1}{k!} p^{(k)}(a_L) D_a^k = p(a_R) = 0.$$

Relation (1) can be written in form $h^{-1}\theta = 1 - h^{-1}\phi D_a$ for some $\phi \in A$. Let $\alpha = h^{-1}\phi D_a$ then since h^{-1}, ϕ, and θ commute, we have $(1-\alpha)\alpha = h^{-2}\phi\theta D_a = 0$. Thus α is idempotent.

COROLLARY 1. $R = \text{Ker } D_a \oplus \text{Im } D_a$.

PROOF. Since α is idempotent it is well known that $R = \text{Ker } \alpha \oplus \text{Im } \alpha$. But $\text{Im } \alpha \subseteq \text{Im } D_a$ and since the elements of Ker D_a commute with a, we have $\text{Ker } D_a \subseteq \text{Ker } \alpha$. Also $\theta D_a = 0$ so (1) implies $0 = D_a - \alpha D_a = D_a - D_a \alpha$. Thus $\text{Ker } \alpha \subseteq \text{Ker } D_a$ and $\text{Im } D_a \subseteq \text{Im } \alpha$.

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Also note that Ker θ = Im D_a and Im θ = Ker D_a.

LEMMA 1. *Let* $p(x) = \sum_0^n c_j x^j$, *then* (1) *may be written alternately as*

$$\theta = \sum_{j=1}^{n} c_j \sum_{i=0}^{j-1} a_L^i a_R^{j-i-1}. \tag{2}$$

PROOF. Substituting $a_R = a_L + D_a$ in (2) yields

$$\theta = \sum_{j=1}^{n} c_j \sum_{i=0}^{j-1} \sum_{k=0}^{j-i-1} \binom{j-i-1}{k} a_L^{j-k-1} D_a^k.$$

Invert the last two summations and sum the binomial coefficients over i to obtain $\binom{j}{k+1}$. Then changing limits on k and inverting the summations leads immediately to the result (1).

THEOREM 2. *If a is separable over C and δ is an arbitrary C-derivation in R, then there exists a unique $d \in Im\ D_a$ such that $a\delta = aD_d$.*

PROOF. Since $p(a) = 0$ it follows from Lemma 1 that $0 = p(a)\delta = a\delta\theta$. Thus $a\delta(1-\alpha) = 0$. Since $\alpha = h^{-1}\phi D_a$ is idempotent this implies

$$a\delta = a\delta(h^{-1}\phi D_a)^2 = aD_d$$

where

$$d = -a\delta h^{-2}\phi^2 D_a \in \text{Im } D_a.$$

To establish uniqueness suppose $a\delta = aD_c$ for some $c \in \text{Im } D_a$. Then

$$0 = aD_c - aD_d = aD_{c-d} \text{ so } c-d \in \text{Ker } D_a.$$

But also $c-d \in \text{Im } D_a$ so by Corollary 1 we have $c-d = 0$.

THEOREM 3. *If a is separable over C then a C-derivation δ maps* Ker D_a *into itself if and only if $a\delta = 0$. When this is the case δ also maps* Im D_a *into itself.*

PROOF. By Theorem 2 we have $a\delta = aD_d = -dD_a \in \text{Im } D_a$. Thus if δ maps Ker D_a into itself it follows that $a\delta \in \text{Ker } D_a \cap \text{Im } D_a = 0$. Conversely, if $a\delta = 0$ then $ax = xa$ implies $a(x\delta) = (x\delta)a$. It is also easy to see that $a\delta = 0$ implies $D_a\delta = \delta D_a$ so δ also maps Im D_a into itself.

Let γ be the projection of R onto Ker D_a.

COROLLARY 2. *A C-derivation δ satisfies $a\delta = 0$ if and only if $\gamma\delta = \delta\gamma$.*

PROOF. Since $a\delta \in \text{Im } D_a$ and $a\gamma = a$ it is clear that $\gamma\delta = \delta\gamma$ implies $a\delta = 0$. The converse follows immediately from Theorem 3.

REMARK. If we write $a\delta = aD_d = -dD_a$ it is easy to show that $a\delta = 0$ is also equivalent to either $d = 0$ or $D_d = 0$. Two other conditions equivalent to $a\delta = 0$ are: $\gamma D_d = 0$ or $\gamma D_d = D_d\gamma$.

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LEMMA 3. For an arbitrary $c \in \operatorname{Im} D_a$ we have $\gamma D_c \gamma = 0$.

PROOF. Let $x \in \operatorname{Ker} D_a$ then $0 = xD_a = -aD_x$. By Corollary 2 this implies $\gamma D_x = D_x \gamma$. But $c\gamma = 0$ and so for an arbitrary $y \in R$ we have $y\gamma D_c \gamma = -cD_{y\gamma}\gamma = -c\gamma D_{y\gamma} = 0$.

The question of conditions on a C-derivation δ so that $\operatorname{Ker} D_a \delta \subseteq \operatorname{Im} D_a$ is open. The following is a partial answer:

THEOREM 4. *The derivation D_c maps* Ker D_a *into* Im D_a *if and only if* $c = c_1 + c_2$ *where* $c_2 \in \operatorname{Im} D_a$ *and* $c_1 \in$ *centralizer of* Ker D_a in R.

PROOF. Since $D_c = D_{c_1} + D_{c_2}$ the sufficiency follows from Lemma 3 and the fact that D_{c_1} is zero on Ker D_a. Conversely, suppose $\gamma D_c \gamma = 0$ for $c = c_1 + c_2$ where $c_1 \in \operatorname{Ker} D_a$ and $c_2 \in \operatorname{Im} D_a$. Then by Lemma 3 we have $xD_{c_1} \in \operatorname{Im} D_a$ for all $x \in \operatorname{Ker} D_a$. But Ker D_a is a subring of R so $xD_{c_1} \in \operatorname{Ker} D_a \cap \operatorname{Im} D_a = 0$. Thus $xc_1 = c_1 x$ for all $x \in \operatorname{Ker} D_a$.

NOTE. For an arbitrary C-derivation δ it follows from $a\delta \in \operatorname{Im} D_a$, by an easy induction, that $C[a]\delta \subseteq \operatorname{Im} D_a$. By the last theorem δ will not in general map all of Ker D_a into Im D_a. In fact, the following is an example in which a derivation maps a member of Ker D_a into a non-zero member of $C[a]$.

Let C be the rational field and $K = C[a]/(a^2+a+1)$. Write $A = K[x, y]$ where $xy \neq yx$, and $I = (yx - xy - a)$, with $R = A/I$. The element $a \in R$ is separable over R with $p(a) = a^2+a+1 = 0$ and $p'(a)$ invertible. Also $yD_x = a$ for $y \in \operatorname{Ker} D_a = R$. Thus the only question is whether or not R is trivial. However, it is clear that in each coset of A modulo I there is a unique element in form $\sum \alpha_i x^{n_i} y^{m_i}$ where $\alpha_i \in K$. Thus $R \neq 0$.

Remark that this example also shows that conditions on the images of Im D_a by a C-derivation δ (such as $\operatorname{Im} D_a \delta \subseteq \operatorname{Im} D_a$ or $\operatorname{Im} D_a \delta \subseteq \operatorname{Ker} D_a$) do not in general restrict δ.

Norfolk, Nebraska
and
University of Nebraska

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