Proceedings of the Edinburgh Mathematical Society (1997) 40, 243–246 ©

A REMARK ON THE *C*-NORMALITY OF MAXIMAL SUBGROUPS OF FINITE GROUPS

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(Received 4th April 1995)

A subgroup H is called c-normal in a group G if there exists a normal subgroup N of G such that $HN = G$ and $H \cap N \leq H_G$, where $H_G =: Core(H) = \cap_{g \in G} H^g$ is the maximal normal subgroup of G which is contained in H. We use a result on primitive groups and the c-normality of maximal subgroups of a finite group G to obtain results about the influence of the set of maximal subgroups on the structure of G.

1991 *Mathematics subject classification:* Primary 20D10.

1. Introduction

The relationship between the properties of maximal subgroups of a finite group G and the structure of G have been studied by many people. In fact, the knowledge of the maximal subgroups of a finite group often yields a wealth of information about the group itself. The normality of subgroups in a finite group plays an important role in the study of finite groups. It is well known that a finite group G is nilpotent if and only if every maximal subgroup of G is normal in G. As for the class of supersolvable groups, B. Huppert's well known theorem shows that a finite group G is supersolvable if and only if every maximal subgroup of G has prime index in G. In term of normality, we have that G is supersolvable if and only if every maximal subgroup of G is weakly normal in G. [**6**, Theorem 1.8.7] In [**5**], we have shown that G is solvable, if and only if M is c-normal in G for every maximal subgroup M of G, if and only if there exists a solvable c-normal maximal subgroup M of G. In this paper, we extend the theorem to a π-separable group by using Lafuente's result on primitive groups [**3**].

All the groups in this paper are finite, p denotes a prime, π denotes a set of primes and π' the complementary set of primes. $M < \cdot G$ means M is a maximal subgroup of G. For a subgroup H of G, H_G denotes $Core(H) = \cap_{g \in G} H^g$. We denote by $[N]M$ the semi-direct product of N by M.

Definition 1.1. Let G be a group. We call a subgroup H *c-normal* in G if there exists a normal subgroup N of G such that $HN = G$ and $H \cap N \leq H_G$.

* Project supported in part by National Natural Science Foundation of China and STFG.

It is clear that a normal subgroup of G is a c-normal subgroup of G but the converse is not true. For example, $S_3 = [C_3]C_2$, $C_2 \ntrianglelefteq S_3$ but C_2 is c-normal in S_3.

Definition 1.2. A finite group G is called *π-separable* if every composition factor of G is either a π-group or a π'-group.

A finite group G is called *π-solvable* if every composition factor of G is either a p-group with $p \in \pi$ or a π'-group.

Definition 1.3. Let G be a finite group. We say G is a *special primitive group* if we have the following:

(1) G is a primitive group. That is, there exists a maximal subgroup M of G with $M_G = 1$.

(2) G has a unique nonabelian minimal normal subgroup N and G has one maximal subgroup U such that $U \cap N = 1$.

The structure of the special primitive groups were characterized in [**1**], [**2**] and [**3**]. We will use the following lemma of [**3**].

Lemma 1.4. ([**3**, p. 2032 Corollary]). *Let G be a special primitive group. Let N be the minimal normal subgroup of G. Let U be the maximal subgroup of G with $U \cap N = 1$. Let K be a minimal normal subgroup of U. Let S (resp. T) be a simple factor of N (resp. K). Then S is isomorphic to a section of T.*

2. Preliminaries

Property 2.1. *Let G be a group. Then*

(1) *G is π-separable if and only if every chief factor of G is either a π-group or a π'-group.*

(2) *G is π-solvable if and only if every chief factor of G is either a p-group with $p \in \pi$ or a π'-group.*

(3) *G is p-solvable if and only if G is p-separable.*

(4) *Let $N \lhd G$. Then G is π-separable (resp. π-solvable) if and only if both N and G/N are π-separable (resp. π-solvable).*

(5) *G is π-solvable if and only if G is p-solvable for every $p \in \pi$.*

Proof. (1)–(4) follow directly from the definition and induction. Now we prove (5). If G is p-solvable for every $p \in \pi$, then by definition G is π-solvable. Conversely, assume G is π-solvable. Let N be a minimal normal subgroup of G and $p \in \pi$. If N is a p'-group, then, by induction, both G/N and N are p-solvable and so is G. If N is not a p'-group, then N is a p-group by the definition of π-separable group. The same argument shows that G is p-solvable.

Lemma 2.2. *Let G be a group. Then*

(1) *If H is c-normal in G, $H \leq K \leq G$, then H is c-normal in K.*

(2) *Let $K \lhd G$ and $K \leq H$. Then H is c-normal in G if and only if H/K is c-normal in G/K.*

Proof. (1) Suppose that H is c-normal in G. Then there exists a normal subgroup N of G such that $HN = G$ and $H \cap N \leq H_G$. Now $K = K \cap G = H(K \cap N)$, $K \cap N$ is normal in K, and $H \cap (K \cap N) \leq H_G \cap K \leq H_K$. Hence H is c-normal in G.

(2) Suppose that H/K is c-normal in G/K. Then there exists $N/K \lhd G/K$ such that $G/K = (H/K)(N/K)$ with $(H/K) \cap (N/K) \leq (H/K)_{G/K}$. It is easy to see that $G = HN$ and $H \cap N \leq H_G$. The converse is the same.

3. Theorems

In order to minimize the number of restricted maximal subgroups, we localize our condition on the c-normality of one maximal subgroup. It was shown in [5] that a finite group G is solvable if and only if there exists a solvable c-normal maximal subgroup M of G. Since a nonabelian simple group has no c-normal maximal subgroup, (refer to Lemma 2.2), we cannot extend the "only if" part of this theorem to π-separable groups. (Simply choose G, a nonabelian simple group, $\pi = \pi(G)$, as a counterexample.) However, we will extend the "if" part of this theorem to π-separable groups by proving the following:

Theorem 3.1. *Let G be a finite group. Then G is π-separable if there exists a π-separable c-normal maximal subgroup M of G.*

Proof. Assume the theorem is false and let G be a minimal counterexample. Let M be a c-normal solvable maximal subgroup of G. Then G must satisfy the following:

(a) M is corefree.

If $M_G \neq 1$, then M/M_G is a π-separable c-normal maximal subgroup of G/M_G by Lemma 2.2. Since M_G is also π-separable, by minimal choice of G, we have that G/M_G is π-separable and therefore G is π-separable, a contradiction.

(b) There exists a minimal normal subgroup N of G such that $G = [N]M$.

Since M is c-normal in G, there exists a normal subgroup N of G such that $G = NM$ and $M \cap N \leq M_G = 1$. Therefore N must be minimal normal in G since $M < \cdot G$.

(c) N is the unique minimal normal subgroup of G.

In fact, if K is another minimal normal subgroup of G, then $G = [N]M = KM$. Note that both G/N and G/K are π-separable. We have that $G \cong G/(N \cap K) \lesssim (G/N) \times (G/K)$ and hence G is π-separable, which is absurd.

https://doi.org/10.1017/S0013091500023683 Published online by Cambridge University Press

(d) N is either a π-subgroup or a π'-subgroup.

If N is abelian, then N is a p-subgroup for a $p \in \pi(G)$, so (d) holds. Assume that N is nonabelian. By (a)–(c), G is a special primitive group. Then $N = N_1 \times \cdots \times N_k$ is a product of isomorphic nonabelian simple groups. Let T be a minimal normal subgroup of M. Since M is π-separable, T is either a π-subgroup or a π'-subgroup. Without loss of generality, we assume that T is a π-subgroup. By Lemma 1.4, we have that N_1 is isomorphic to a section of T, which is a π-subgroup. Hence N is a π-subgroup.

Now both N and G/N are π-separable which implies that G is π-separable, contrary to our choice.

This completes our proof.

Corollary 3.2. *Let G be a finite group. Then G is π-solvable (resp. solvable) if there exists a π-solvable (resp. solvable) c-normal maximal subgroup M of G.*

Proof. Use Theorem 3.1 and Property 2.1. Note that, for any group H, H is π-solvable if and only if H is p-separable for every $p \in \pi$ and H is solvable if and only if G is p-separable for every $p \in \pi(H)$.

Acknowledgement. This work was done when the author worked in the University of Minnesota. He is grateful to the hospitality of the School of Mathematics. The author is also grateful to the referee for his/her comments.

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