THE INVERSIVE DISTANCE BETWEEN TWO CIRCLES

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Dedicated to H. S. M. Coxeter, geometer

1. Introduction. H. S. M. Coxeter **(3)** has recently studied the correspondence between two geometries the isomorphism of which was well known, but to which he was able to add some remarkable consequences. The two geometries are the *inversive geometry* of a plane E (the Euclidean plane completed with a single point at infinity or, what is the same thing, the plane of complex numbers to which ∞ is added) on the one hand, and the *hyperbolic geometry* of three-dimensional space S.

Each concept and each theorem of one geometry may be translated into the other one. Points of E correspond to points at infinity (or points on the absolute quadric Ω) of S; a circle of E corresponds to a plane of S; two non-intersecting circles of E correspond to two ultraparallel planes of S. One of the interesting concepts introduced by Coxeter is the *inversive distance* d of two non-intersecting circles, which is the translation of the hyperbolic distance of two ultraparallel planes. In terms of elementary geometry it comes to this: if the line of centres of the two circles intersects them in A, A' and B, B' (Fig. 1), then

(1) $$th^2\frac{d}{2} = (AA', BB') = \frac{A'B'.AB}{A'B.AB'},$$

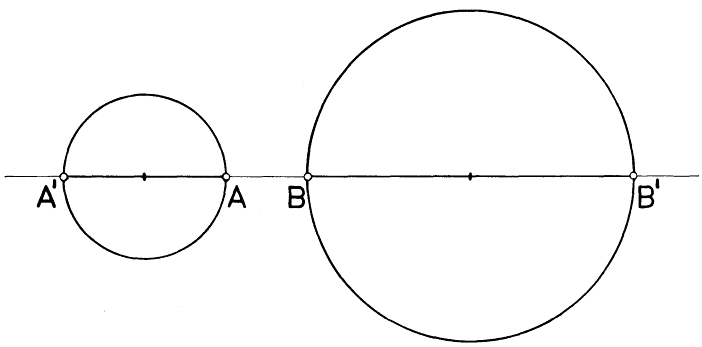

FIGURE 1

the cross-ratio on the right-hand side being invariant for inversive transformations. If the two circles are inverted into concentric circles (which is always possible) d is seen to be the logarithm of the ratio of their radii.

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2. Casey's invariants for two point pairs. As Coxeter remarks, the inversive distance "seems to have been sadly neglected." As far as we know, this is true, but it is perhaps worth while to mention that J. Casey, a century ago, in the *Proceedings of the Royal Irish Academy* for 1866 was not far from the idea when he showed, in an elementary way, that the two cross-ratios

$$\frac{A'B'.AB}{A'A.BB'} \quad \text{and} \quad \frac{A'B.AB'}{A'A.BB'} \tag{2}$$

are invariant for inversion. He made use of them to give an extension of Ptolemy's theorem, dealing with four circles all touched by the same circle, and as an application he proved Feuerbach's theorem in a very elegant way. Casey's invariants are equal to the square of a common tangent of the circles, divided by the product of their radii. It must be noted that these invariants are properly related not to the configuration of two circles but to that of two cycles (or oriented circles) and therefore are concepts of the geometry of Laguerre. We want only to say here that the cross-ratio (1), which is the key to the definition of inversive distance, is equal to the ratio of the invariants (2). Casey's considerations are reproduced in his classical treatise **(1)** and in later books on the subject **(2; 6)**.

3. The Jacobian of two point pairs. The object of this note is to formulate a problem on the inversive distance and to give a solution by means of the isomorphism of the two geometries.

In the plane E two pairs of points A_1, B_1 and A_2, B_2 are given. Consider a circle c_1 through A_1, B_1 and a circle c_2 through A_2, B_2 such that c_1 and c_2 do not intersect.

What can be said about the range of values of the inversive distance $d(c_1, c_2)$ if c_1 and c_2 vary? Obviously d does not have a minimum, for as c_1 and c_2 approach two mutually tangent circles d tends to zero. Is there a maximum for d and, if so, how should c_1 and c_2 be chosen to realize it?

A_1 and B_1 correspond to two points on Ω and therefore the pair $A_1 B_1$ corresponds to a line l_1 of S, $A_2 B_2$ to a line l_2, the circles c_1 and c_2 to ultraparallel planes p_1 and p_2 through l_1 and l_2 respectively and their inversive distance to the distance between these planes. It is well known **(4)** that the latter has a *maximum* which is attained by the planes through l_1 and l_2 respectively which are both orthogonal to the common perpendicular of l_1 and l_2; the value of this maximum is the length of the perpendicular.

In this way the problem is solved in the isomorphic geometry S; the translation backwards into E will give us the answer we require. We shall bring the data (the two pairs A_1, B_1 and A_2, B_2) into such a form that the translation may be read easily. Accordingly, we remark that by a suitable inversion of E the two pairs can be transformed into the vertices of a parallelogram. This well-known theorem, which is given by Johnson **(5)**, for example, may be proved as follows. Two point pairs (A_1, B_1) and (A_2, B_2) in E have their *Jacobian*, that is the unique pair (J_1, J_2) which is harmonic with both. Consider an inversion with J_2 as its centre and denote the transforms by a prime. J'_2 is the point at infinity and, as the harmonic property is invariant for an inversion, J'_1 is the mid-point of A'_1, B'_1 and of A'_2, B'_2; hence $A'_1 A'_2 B'_1 B'_2$ is a parallelogram, and thus a representative of the general configuration of four points. It is given in Figure 2, with the primes omitted and $J'_1 = O$. If the four points are concyclic, the vertices of the parallelogram are concylic, which means that they are either on a line through J'_1 or they are the vertices of a rectangle.

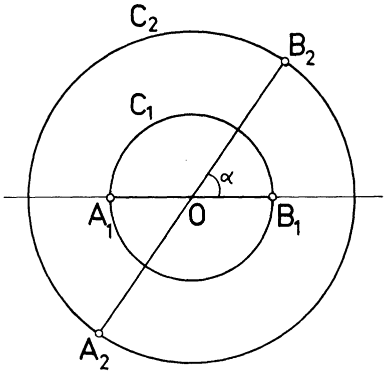

FIGURE 2

4. Maximizing the inversive distance between circles through two point pairs. The images in S of the two pairs of points are l_1 and l_2, which are in general *skew* lines. Let h be their common perpendicular. We know that a line that intersects a line l orthogonally corresponds to a point pair in E that is harmonic with the pair corresponding to l. The conclusion is: The image of h is the Jacobian of A_1, B_1 and A_2, B_2, i.e. in our case the pair $(0, \infty)$. Hence a plane through h corresponds to a straight line through O and a plane orthogonal to h corresponds to a circle having its centre at O. It follows from this that the two circles asked for are the circles $O(B_1)$ and $O(B_2)$. Therefore Figure 2 is a portrait of the configuration of two lines l_1 and l_2 of S, giving all the particulars. The distance between the lines is $|\log A_1 B_1/A_2 B_2|$; moreover, the angle α between $A_1 B_1$ and $A_2 B_2$ is equal to the angle between l_1 and l_2, for it is the angle between the planes $l_1 h$ and $l_2 h$.

There are two special cases. If the parallelogram is a rectangle, c_1 and c_2 coincide, the distance is zero, l_1 and l_2 are intersecting lines; if $A_1 B_1$, $A_2 B_2$ are on a line through O, we have $\alpha = 0$, l_1 and l_2 are ultraparallel lines, and the distance is still the length of the common perpendicular.

We may, of course, describe Figure 2 by means of inversive concepts without making use of the chosen special position of the points. The circles $A_1 B_1 A_2$ and $A_1 B_1 B_2$, being intersecting circles, have two *mid-circles*, which are easily seen to be the straight line $A_1 B_1$ and the circle c_1. Of these two mid-circles, only the former separates the points A_2 and B_2.

In the same way $A_2 B_2$ and c_2 are the mid-circles of $A_2 B_2 A_1$ and $A_2 B_2 B_1$. Therefore we have reached the following conclusion. Let $A_1 B_1$ and $A_2 B_2$ be arbitrary point pairs of E, c_1 and c'_1 the mid-circles of $A_1 B_1 A_2$ and $A_1 B_1 B_2$ (c_1 *not* separating A_2 and B_2), c_2 and c'_2 the mid-circles of $A_2 B_2 A_1$ and $A_2 B_2 B_1$ (c_2 *not* separating A_1 and B_1): then the maximum inversive distance of any circle through $A_1 B_1$ and any (non-intersecting) circle through $A_2 B_2$ is the inversive distance of c_1 and c_2; it may be called "the inversive distance of the pairs $A_1 B_1$ and $A_2 B_2$." The angle "between $A_1 B_1$ and $A_2 B_2$" is the angle between c'_1 and c'_2; the points of intersection of c'_1 and c'_2 stand for the common perpendicular of $A_1 B_1$ and $A_2 B_2$.

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