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ON THE GAUSS MAP OF MINIMAL SURFACES WITH FINITE TOTAL CURVATURE

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We prove that if a nonflat complete regular minimal surface immersed in R^n is of finite total curvature, then its Gauss map can omit at most $(n-1)(n+2)/2$ hyperplanes in general position in $P^{n-1}(\mathbb{C})$.

1. Introduction

There have been several results devoted to studying the "value distribution" properties of the Gauss map of a nonflat complete regular minimal surface. In [**3**], Fujimoto proved that the Gauss map of a nonflat complete regular minimal surface immersed in R^3 can omit at most four points of the sphere. With the additional condition of finite total curvature, Osserman [**6**] showed that the Gauss map can omit at most three points of the sphere. For the Gauss map of a nonflat complete regular minimal surface immersed in R^n, the author [**7**] has shown that it can omit at most $n(n+1)/2$ hyperplanes in general position in $P^{n-1}(\mathbb{C})$, while the "nondegenerate of the Gauss map" case is due to Fujimoto (see [**4**]). The purpose of this paper is to improve on a theorem of Chern and Osserman [**2**]. We shall show that the nondegenerate condition in the Theorem 4 of [**2**] can be removed by using the powerful tool called "Ṅochka weights". The theorem will be stated in Section 2.

2. Facts concerning minimal surfaces in R^n and the statement of results

We shall recall some basic facts concerning minimal surfaces immersed in R^n. For further details, we refer to Chern and Osserman [**2**].

Let S_0 be a Riemann surface and let $\alpha_1, \ldots, \alpha_n$ be analytic differentials on S_0, which we assume to be not all identically zero. Suppose that in terms of a local parameter ζ, we have $\alpha_k = \phi_k d\zeta$, $1 \leqslant k \leqslant n$. Then under the condition

$$\sum_{1 \leqslant k \leqslant n} \phi_k^2 \equiv 0 \tag{2.1}$$

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the surface $x(p): S_0 \to R^n$, defined by

$$x_k = \operatorname{Re} \int \alpha_k \tag{2.2}$$

is called a *generalised minimal surface* assuming the integrals to have single-valued real parts. If furthermore

$$\sum_{1 \leqslant k \leqslant n} |\phi_k|^2 \neq 0 \tag{2.3}$$

then the surface is called *a regular minimal surface.*

If we set $\zeta = \xi_1 + i\xi_2$, then we have

$$\phi_k = \frac{\partial x_k}{\partial \xi_1} - i\frac{\partial x_k}{\partial \xi_2} \tag{2.4}$$

and if we denote by

$$g_{ij} = \frac{\partial x}{\partial \xi_i} \cdot \frac{\partial x}{\partial \xi_j} \qquad 1 \leqslant i,\ j \leqslant 2$$

the coefficients of the first fundamental form of the surface (2.2), then condition (2.1) becomes

$$g_{11} - g_{22} - 2ig_{12} = 0 \tag{2.6}$$

meaning that ξ_1, ξ_2 are isothermal parameters. This condition may also be written in the form

$$g_{ij} = \lambda^2 \delta_{ij} \qquad \lambda = \lambda(\zeta) \tag{2.7}$$

where

$$\lambda^2 = \left|\frac{\partial x}{\partial \xi_1}\right|^2 = \left|\frac{\partial x}{\partial \xi_2}\right|^2 = \frac{1}{2} \sum_{1 \leqslant k \leqslant n} |\phi_k|^2 . \tag{2.8}$$

The *Gaussian curvature* is given by

$$k = -\frac{\Delta \log \lambda}{\lambda^2} = -\frac{4\,|\phi \wedge \phi'|^2}{|\phi|^6} \tag{2.9}$$

where

$$\phi = (\phi_1, \dots, \phi_n), \quad |\phi|^2 = \sum_{1 \leqslant k \leqslant n} |\phi_k|^2 \tag{2.10}$$

and

$$|\phi \wedge \phi'|^2 = \sum_{1 \leqslant k \leqslant n} |\phi_j \phi_k' - \phi_j' \phi_k|^2 . \tag{2.11}$$

Let D be a domain in the ζ-plane, and denote by S the corresponding part of the surface (2.2). Then the area of S is

$$A(S) = \iint_D \lambda^2 d\xi_1 d\xi_2 = \frac{1}{2} \iint_D |\phi|^2 \, d\xi_1 d\xi_2 \tag{2.12}$$

and *the total curvature* of S is

$$C(S) = \iint_D k dA = \iint_D k\lambda^2 d\xi_1 d\xi_2 = -2 \iint_D \frac{|\phi \wedge \phi'|^2}{|\phi|^4} d\xi_1 d\xi_2. \tag{2.13}$$

The *generalised Gauss map* is the map

$$G\colon S_0 \to Q_{n-2} \subset P^{n-1}(\mathbb{C}) \tag{2.14}$$

with homogeneous coordinates $\overline{\phi_1(\zeta)}, \dots, \overline{\phi_n(\zeta)}$, where

$$Q_{n-1} = \{[z_1 : \dots : z_n] \subset P^{n-1}(\mathbb{C}) \mid z_1^2 + \dots + z_n^2 = 0\}.$$

We have the following theorem:

Main Theorem. *Let S be a nonflat complete regular minimal surface in R^n defined by a map (2.2) on a Riemann surface S_0, with the Gauss map defined by (2.14). Suppose that S is of finite total curvature. Then $G_0(S_0)$ can fail to intersect at most $(n-1)(n+2)/2$ hyperplanes in general position in $P^{n-1}(\mathbb{C})$.*

3. Plüker formulas for algebraic curve and Nochka weights for hyperplanes in subgeneral position

(A) Plüker formulas.

We state the Plüker formulas for an algebraic curve in a complex projective space; see [8], pp.41–65. Let W be a compact Riemann surface of genus g and let $f\colon W \to P^m(\mathbb{C})$ be a nondegenerate algebraic curve (that is, $f(W)$ is not contained in any hyperplane in $P^m(\mathbb{C})$). For a suitable choice of homogeneous coordinates $\zeta_0, \dots, \zeta_m$ in $P^m(\mathbb{C})$, the equations of the curve can be put locally into the normal form

$$\begin{aligned} \zeta_0 &= t^{\zeta_0} + \dots \\ &\cdots\cdots\cdots \\ \zeta_m &= t^{\delta_m} + \dots \end{aligned} \tag{3.1}$$

where

$$0 = \delta_0 < \delta_1 < \dots < \delta_m, \tag{3.2}$$

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and where t is a local parameter on W. The integers

$$\nu_k = \delta_{k+1} - \delta_k - 1 \qquad 0 \leqslant k \leqslant m-1 \tag{3.3}$$

are called the *stationary indices of order* k at the point $t = 0$. The stationary point, that is, points with non-zero stationary index, are isolated and hence are finite in number. We will denote by σ_k the sum of all stationary indices of order k. Let d_k, $0 \leqslant k \leqslant m-1$, be the *order of rank* k of the algebraic curve; geometrically this is the order of the associated curve of rank k, that is, the curve formed by the osculating space of dimension k. Then Plüker formulas are

$$\sigma_k = 2d_k - d_{k+1} - d_{k-1} + 2(g-1), \qquad 1 \leqslant k \leqslant m-1 \tag{3.4}$$

with the convention $d_{-1} = d_m = 0$. From (3.4), it follows that

$$\sum_{1 \leqslant h \leqslant m} (m+1-h)\sigma_{h-1} = (m+1)d_0 + m(m+1)(g-1). \tag{3.5}$$

(B) Nochka weights.

We consider q hyperplanes H_j $(1 \leqslant j \leqslant q)$ in $P^m(\mathbb{C})$ which are given by

$$H_j : (A_j, W) = 0 \qquad (1 \leqslant j \leqslant q)$$

for $A_j \in C^{m+1} - \{0\}$, where $q > N \geqslant m$ and (A, W) means $a_0 w_0 + \ldots + a_n w_n$ for a vector $A = (a_0, \ldots, a_m)$ and homogeneous coordinates $W = [w_0, \ldots, w_n]$.

According to Nochka [5] and Chen [1], we give the following definition.

Definition 3.1: We say that hyperplane $H_1, \ldots, H_q$ are in N-subgeneral position if, for every $1 \leqslant j_0 < \ldots < j_N \leqslant q$, $A_{j_0}, \ldots, A_{j_N}$ generate C^{m+1}. If $N = m$, then we say that $H_1, \ldots, H_q$ are in general position.

It is easy to check that if $H_1, \ldots, H_q$ are hyperplanes in general position in $P^{n-1}(\mathbb{C})$, and we embed $P^k(\mathbb{C})$ as the subspace of $P^{n-1}(\mathbb{C})$ for $1 \leqslant k < n-1$, then $H_i \cap P^k(\mathbb{C})$ $(1 \leqslant i \leqslant q)$ are in $(n-1)$-subgeneral position in $P^k(\mathbb{C})$.

Nochka [5] and Chen [1] have given the following lemmas to prove Cartan's conjecture.

Lemma 3.2. *Let $H_1, \ldots, H_q$ be hyperplanes in $P^m(\mathbb{C})$ located in N-subgeneral position, where $q > 2N - m + 1$. Then there exists some constants $\omega(1), \ldots, \omega(q)$ and θ satisfying the following conditions:*

(i) $0 < \omega(j)\theta \leqslant 1 \qquad (1 \leqslant j \leqslant q)$

(ii) $\theta\left(\sum_{j=1}^{q} \omega(j) - m - 1\right) = q - 2N + m - 1$

(iii) $(N+1)/(m+1) \leqslant \theta \leqslant (2N-m+1)/(m+1)$.

For the proof, see [**5**] or Chen [**1**, Theorem 0.3]. We call the constants $\omega(j)$ $(1 \leqslant j \leqslant q)$ and θ in the above lemma the Nochka weights and the Nochka constant for $H_1, \ldots, H_q$ respectively.

The following lemma is crucial to the proof of the main theorem using Nochka weights.

LEMMA 3.3. *Let $H_1, \ldots, H_q$ be hyperplanes in $P^m(\mathbb{C})$ located in N-subgeneral position, where $q > 2N - m + 1$. Let $\omega(j)$ $(1 \leqslant j \leqslant q)$ be their Nochka weights. Take $A \subset \{1, 2, \ldots, q\}$, with $0 < \#A = N + 1$. Let E_j $(1 \leqslant j \leqslant q)$ be real numbers with $E_j \geqslant 1$. Then there exists a subindex set $\{j_0, \ldots j_m\} \subset A$ such that*

$$\prod_{j \in A} (E_j)^{\omega(j)} \leqslant \prod_{i=0}^{m} E_{j_i}$$

and the hyperplanes $H_{j_0}, \ldots, H_{j_m}$ are in general position in $P^m(\mathbb{C})$.

For the proof, see Chen [**1**, Theorem 1.2].

4. Some theorems proved by Chern and Osserman

We recall some theorems proved by Chern and Osserman [**2**].

DEFINITION 4.1: The Gauss map (2.14) is called algebraic if the surface S_0 is conformally equivalent to a region D on a compact Riemann surface W, and if, when the differentials α_k are considered as analytic differentials on D, the ratios α_k/α_m extend to meromorphic functions on W, whenever $\alpha_m \not\equiv 0$.

THEOREM A. *Let S be a minimal surface defined by (2.2) on a Riemann surface S_0. If S is a complete regular minimal surface, then the following four statements are equivalent:*

(a) *S has finite total curvature;*
(b) *there exists an integer N such that the image of S_0 under the Gauss map intersects at most N times all hyperplanes which do not contain it;*
(c) *the Gauss map of S_0 is algebraic;*
(d) *the surface S_0 is conformally equivalent to a compact surface W punctured at a finite number of points $p_1, \ldots, p_r$, and the differentials α_k are either regular or have a pole at each p_j.*

For the proof, see [**2**], Theorem 1.

THEOREM B. *Let S be a complete regular minimal surface with Euler characteristic χ and r boundary components. Then*

$$C(S) \leqslant 2\pi(\chi - r). \tag{4.1}$$

For the proof, see [**2**], Theorem 2.

THEOREM C. *The total curvature of a complete regular minimal surface is either $-\infty$ or $-2\pi N$ where N is the integer in statement (b) of Theorem A.*

For the proof, see [**2**] Corollary.

5. PROOF OF THE MAIN THEOREM

PROOF OF THE MAIN THEOREM: Under the hypotheses of the main theorem, it follows from Theorem A that S_0 is conformally equivalent to a compact Riemann surface W punctured at the points p_j, $1 \leqslant j \leqslant r$, and that the Gauss map G extends to an antiholomorphic map of W into $P^{n-1}(\mathbb{C})$. Take a number m, $1 \leqslant m \leqslant n-1$ such that $G(W)$ is contained in $P^m(\mathbb{C})$ but none of lower dimension. Then G is a non-degenerate algebraic curve in $P^m(\mathbb{C})$. Let π_i, $1 \leqslant i \leqslant q$, be hyperplanes in general position in $P^{n-1}(\mathbb{C})$ which do not intersect $G(S_0)$. Let $H_i = \pi_i \cap P^m(\mathbb{C})$. Then H_i, $1 \leqslant i \leqslant q$, are hyperplanes in $P^m(\mathbb{C})$ in $(n-1)$-subgeneral position, and do not intersect $G(S_0)$. Then H_i intersects $G(W)$ at certain of the points p_j, with a multiplicity which we denote by μ_{ij}. We have

$$\sum_{1 \leqslant j \leqslant r} \mu_{ij} = d_0 \tag{5.1}$$

where d_0 is the order of the algebraic curve $G(W)$.

Because the hyperplanes are in $(n-1)$-subgeneral position in $P^m(\mathbb{C})$, at most $n-1$ of the hyperplanes H_i can intersect $G(W)$ at p_j. It follows that there exists a subset $A \subset \{1, 2, \dots, q\}$, $\#A = n$ such that

$$\sum_{1 \leqslant i \leqslant q} \omega(i)\mu_{ij} \leqslant \sum_{i \in A} \omega(i)\mu_{ij} \tag{5.2}$$

where $\omega(i)$, $1 \leqslant i \leqslant q$, are Nochka weights for hyperplanes H_i, $1 \leqslant i \leqslant q$.

Applying Lemma 3.3 with $E_i = e^{\mu_{ij}}$, it follows that there exists a subindex set $\{i_0, \dots, i_m\} \subset A$ such that

$$\prod_{i \in A} (e^{\mu_{ij}})^{\omega(i)} \leqslant \prod_{k=0}^{m} e^{\mu_{i_k j}} \tag{5.3}$$

and the $H_{i_0 j}, \dots, H_{i_m j}$ are hyperplanes in general position in $P^m(\mathbb{C})$. Hence

$$\sum_{i \in A} \omega(i)\mu_{ij} \leqslant \sum_{k=0}^{m} \mu_{i_k j} \tag{5.4}$$

Since $H_{i_0 j}, \dots, H_{i_m j}$ are in general position in $P^m(\mathbb{C})$, we can adopt Chern and Osserman's argument [**2**, p.30].

Let (3.1) be the equations of $G(W)$ at p_j, whose parameter value is $t = 0$. At p_j the maximum possible value of $\mu_{i_k j}$ is $\delta_{m-1}(p_j)$, and that for the unique hyperplane $\zeta_{m-1} = 0$. A second hyperplane can intersect $G(W)$ at p_j with multiplicity at most $\delta_{m-2}(p_j)$, and a third, if in general position with respect to the first two, at most $\delta_{m-3}(p_j)$, etc. It follows that at most m of the hyperplanes $H_{i_k j}$, $0 \leqslant k \leqslant m$, can intersect $G(W)$ at p_j and we have

$$\sum_{0 \leqslant k \leqslant m} \mu_{i_k j} \leqslant \delta_1(p_j) + \ldots + \delta_m(p_j) \tag{5.5}$$

Combining this with (5.4) and (5.2), we have

$$\sum_{1 \leqslant i \leqslant q} \omega(i)\mu_{ij} \leqslant \delta_1(p_j) + \ldots + \delta_m(p_j). \tag{5.6}$$

By (3.3) the right side is equal to

$$\sum_{1 \leqslant h \leqslant m} (m+1-h)\nu_{h-1}(p_j) + \frac{1}{2}m(m+1).$$

Combining this with (3.5), (5.1), we get

$$\sum_{1 \leqslant i \leqslant q} \omega(i)d_0 \leqslant \sum_{1 \leqslant i \leqslant q} \sum_{1 \leqslant j \leqslant r} \omega(i)\mu_{ij} \leqslant (m+1)d_0 + \frac{1}{2}m(m+1)\{2(g-1)+r\}. \tag{5.7}$$

On the other hand, by Theorem B and Theorem C

$$d_0 \geqslant 2(r+g-1). \tag{5.8}$$

Eliminating g in the inequalities (5.7) and (5.8), we get

$$\frac{1}{2}m(m+1)r \leqslant \left\{ \frac{1}{2}(m+1)(m+2) - \sum_{1 \leqslant i \leqslant q} \omega(i) \right\} d_0 \tag{5.9}$$

So

$$\frac{1}{2}m(m+1)r\theta \leqslant \left\{ \frac{1}{2}(m+1)(m+2)\theta - \sum_{1 \leqslant i \leqslant q} \theta\omega(i) \right\} d_0$$

where θ is the Nochka constant. By Lemma 3.2 (ii)

$$\sum_{1 \leqslant i \leqslant q} \theta\omega(i) = q - 2(n-1) + (m-1) + \theta(m+1).$$

We have

(5.11)
$$\frac{1}{2}m(m+1)r\theta \leqslant \{\frac{1}{2}(m+1)(m+2)\theta - q + 2(n-1) - m + 1 - \theta(m+1)\}d_0$$
$$\leqslant \left\{\frac{1}{2}m(m+1)\theta - 1 + 2(n-1) - m + 1\right\} d_0$$

by Lemma 3.2 (iii) $\theta \leqslant (2(n-1) - m + 1)/(m+1)$ (5.11) becomes

$$\frac{1}{2}m(m+1)r\theta \leqslant \left\{\left(\frac{1}{2}m+1\right)(2(n-1) - m + 1) - q\right\} d_0 \tag{5.12}$$
$$\leqslant \left\{\frac{1}{2}(2n-m-1)(m+2) - q\right\} d_0.$$

For $1 \leqslant m \leqslant n-1$, $(2n-m-1)(m+2)/2 \leqslant n(n+1)/2$ therefore

$$\frac{1}{2}m(m+1)r\theta \leqslant \left\{\frac{1}{2}n(n+1) - q\right\} d_0. \tag{5.13}$$

Since the left-hand side is strictly positive, this gives

$$q < \frac{1}{2}n(n+1)$$

which proves the theorem. □

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