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UNION AND GLUEING OF A FAMILY OF COHEN-MACAULAY PARTIALLY ORDERED SETS

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Summary. By means of simple exact sequences in commutative algebra, we can derive some effective criteria for Cohen-Macaulay property of finite partially ordered sets.

Introduction

Given a finite partially ordered set (poset for short), for example,

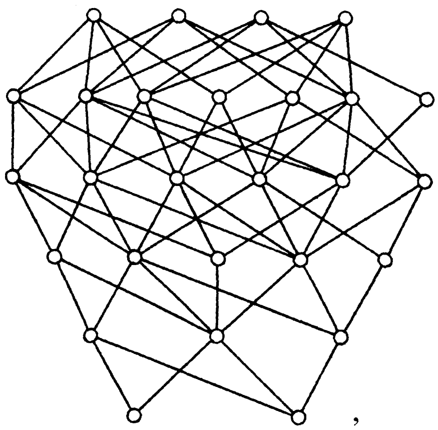

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Figure 1

are there any effective criteria for Cohen-Macaulayness of it? This question is our main motivation to organize this paper. Why is this question so important? Because we have a conjecture that every "integral lattice" (or poset) is Cohen-Macaulay, which is proposed in [15, § 2, d)]. This conjecture is quite open except one partial affirmative answer obtained in [17, § 2, Corollary].

Historically, the notion of Cohen-Macaulay posets originated in

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Baclawski [1] and [2]. Baclawski's definition is purely topological and is influenced by Folkman's result [11, Theorem 4.1], which in current terminology says that a geometric lattice is Cohen-Macaulay. On the other hand, Stanley [26] and Reisner [24] independently gave the ring-theoretical definition of Cohen-Macaulay posets (or complexes) under an influence of a commemorable work of Hochster [19].

The foundation in a theory of Cohen-Macaulay posets is a pioneering work of Reisner [24], in which it is proved that one can define Cohen-Macaulay posets by using either topology or ring theory. Also, the proof by Stanley [26] of the upper bound conjecture for spheres is one of the most dramatic applications of commutative algebra to combinatorics.

In this paper, we will apply local cohomology theory in commutative algebra to a theory of Cohen-Macaulay posets and complexes. Since our tool is based on only depth sensitivity and long exact sequences of local cohomology modules, the method is quite simple and rather formal.

This paper is divided into four sections. In Section 1, after recalling some basic definitions and terminology from commutative algebra and combinatorics, we shall analyze two exact sequences concerning with Stanley-Reisner rings of simplicial complexes, which will play essential roles throughout all of this paper.

In Section 2, based on the exact sequences considered in Section 1, from a purely ring-theoretical viewpoint, we shall give quite simple and elementary alternative proofs to well-known and important combinatorial results, the rank selection theorem and Cohen-Macaulayness of shellable complexes and G-complexes.

On the other hand, Section 3 is a purely combinatorial content, which is also associated with graph theory. In order to apply the argument for simplicial complexes, contained in Section 1, to finite partially ordered sets in Section 4, we shall define explicitly an "intersection", a "union" and the "glueing" of a family of partially ordered sets.

Finally, in Section 4, we shall state our criteria for Cohen-Macaulay property of finite partially ordered sets and, as an example of applications, we shall treat the partially ordered set Π_n of partitions of the integer n ordered by refinement. Also, we shall consider a question that what conditions of partially ordered sets ensure Buchsbaum posets to be Cohen-Macaulay.

It seems likely impossible that we can apply our criteria to *all* finite partially ordered sets. Nevertheless, we hope that our criteria will turn out to be useful in a theory of Cohen-Macaulay posets in our further work, see [34].

§ 1. $0 \to \bigoplus_{i=1}^{n} k[\mathrm{star}_\Delta(\sigma_i)] \to k[\Delta] \to k[\Delta'] \to 0$

In this section we shall consider mainly an exact sequence stated in the above title.

(1.1) To begin with, we will recall some fundamental definitions and results on Stanley-Reisner rings $k[\Delta]$.

Let V be a finite set, called the *vertex set*, and Δ a *simplicial complex* on V. Thus Δ is a family of subsets of V satisfying (i) $\{v\} \in V$ for all $v \in V$ and (ii) $\sigma \in \Delta$, $\tau \subset \sigma$ imply $\tau \in \Delta$.

Let k be a field and $A = k[v; v \in V]$ the polynomial ring in $\sharp(V)$-variables over k, where $\sharp(V)$ means the cardinality of V as a set. By abuse of notation, we are regarding the vertices v as indeterminates over k. Define I_Δ to be the ideal of A generated by all square-free monomials $v_{i_1} v_{i_2} \cdots v_{i_r}$ such that $\{v_{i_1}, v_{i_2}, \cdots, v_{i_r}\} \notin \Delta$, and $k[\Delta] := A/I_\Delta$. The k-algebra $k[\Delta]$ is called the *Stanley-Reisner ring* on Δ in commemoration of Stanley [26] and Reisner [24]. From now on, we will consider A and $k[\Delta]$ as graded rings over k with the standard grading, i.e., the degree of each $v \in V$ is one.

An element $\sigma \in \Delta$ is called a *face* and its dimension $\dim \sigma$ is $\sharp(\sigma)$. A maximal face, with respect to inclusion, is also called a *facet*. The dimension of Δ, denoted by $\dim \Delta$, is $\max\{\dim \sigma; \sigma \in \Delta\}$. Note that $\dim \sigma$ and $\dim \Delta$ in this paper are one more than those in [2] or [10]. Also, Δ is called *pure* if $\dim \sigma = \dim \Delta$ for all facets $\sigma \in \Delta$.

We will denote by $\dim k[\Delta]$ (resp. depth $k[\Delta]$) the dimension (resp. depth) of $k[\Delta]$ as a graded k-algebra. It can be checked, see Stanley [30, p. 63], that $\dim k[\Delta]$ coincides with $\dim \Delta$. In the following, we often regard $k[\Delta]$ as a graded A-module, in this case, of course, $\dim k[\Delta]$ (resp. depth $k[\Delta]$) coincides with $\dim_A k[\Delta]$ (resp. $\mathrm{depth}_A k[\Delta]$), the dimension (resp. depth) of $k[\Delta]$ as a graded A-module. Note that depth $k[\Delta]$ is always positive since $k[\Delta]$ is a reduced k-algebra.

A simplicial complex Δ is called *Cohen-Macaulay* over a field k if $k[\Delta]$ is a Cohen-Macaulay ring. Express the ideal I_Δ as an intersection of prime ideals in A, and we see that every Cohen-Macaulay complex Δ is pure, see Reisner [24, Lemma 9].

(1.2) Let $\varDelta$ be a simplicial complex on the vertex set V. For $\sigma\in\varDelta$, define

$$\mathrm{link}_{\varDelta}(\sigma)=\{\tau\in\varDelta;\ \sigma\cap\tau=\varnothing,\ \sigma\cup\tau\in\varDelta\}\,,$$
$$\mathrm{star}_{\varDelta}(\sigma)=\{\tau\in\varDelta;\ \sigma\cup\tau\in\varDelta\}\,.$$

Then we have

$$k[\mathrm{star}_{\varDelta}(\sigma)]=k[\mathrm{link}_{\varDelta}(\sigma)][v;\ v\in\sigma]\,.$$

By Hochster [20, (5.6)], both $\mathrm{link}_{\varDelta}(\sigma)$ and $\mathrm{star}_{\varDelta}(\sigma)$ are Cohen-Macaulay if $\varDelta$ is Cohen-Macaulay.

Now consider the following graded A-module homomorphism $\varPhi_\sigma$ defined by

$$\begin{array}{ccc}\varPhi_\sigma\colon\ A(-\sharp(\sigma)) & \xrightarrow{\ \sigma\ } & k[\varDelta]\\ \Psi & & \Psi\\ f & \longmapsto & f\prod_{v\in\sigma} v\,,\end{array}$$

where $\sigma\in\varDelta$. Refer to Goto-Watanabe [14, P. 181] for the definition of $A(-\sharp(\sigma))$. Since there is no confusion, for simplicity, we will write A instead of $A(-\sharp(\sigma))$. What is the kernel of $\varPhi_\sigma$? Let $f=\prod_i v_i^{n_i}$ be a monomial in A, then $\varPhi_\sigma(f)=0$ if and only if $\mathrm{Supp}\,(f):=\{v_i;\ n_i>0\}\notin\mathrm{star}_{\varDelta}(\sigma)$. Hence we obtain an injection

$$\varPhi_\sigma\colon\ k[\mathrm{star}_{\varDelta}(\sigma)]\xrightarrow{\ \sigma\ }k[\varDelta]$$

as graded A-modules.

Moreover, if $\sigma_1,\sigma_2,\cdots,\sigma_n\in\varDelta$ then we have

$$\begin{array}{ccc}\varPhi_{(\sigma_1,\sigma_2,\cdots,\sigma_n)}\colon\ \bigoplus_{i=1}^{n}k[\mathrm{star}_{\varDelta}(\sigma_i)] & \xrightarrow{(\sigma_1,\sigma_2,\cdots,\sigma_n)} & k[\varDelta]\\ \Psi & & \Psi\\ (f_1,f_2,\cdots,f_n) & \longmapsto & \sum_{i=1}^{n}\{f_i\prod_{v\in\sigma_i}v\}.\end{array}$$

However, this map $\varPhi_{(\sigma_1,\sigma_2,\cdots,\sigma_n)}$ is not necessarily injective and it is easy to see that

LEMMA. *$\varPhi_{(\sigma_1,\sigma_2,\cdots,\sigma_n)}$ is injective if and only if $\sigma_i\cup\sigma_j\notin\varDelta$ for all $i\neq j$.*

What is the cokernel of $\varPhi_{(\sigma_1,\sigma_2,\cdots,\sigma_n)}$? The image of $\varPhi_{(\sigma_1,\sigma_2,\cdots,\sigma_n)}$ is an ideal of $k[\varDelta]$ which is generated by $\sigma_1,\sigma_2,\cdots,\sigma_n$. Hence the cokernel of $\varPhi_{(\sigma_1,\sigma_2,\cdots,\sigma_n)}$ is

$$k[\Delta - \{\tau \in \Delta;\ \tau \supset \sigma_i \text{ for some } i\}].$$

Summarizing the above observations, we have

THEOREM. *Let Δ be a simplicial complex on the vertex set V and $\sigma_1, \sigma_2, \cdots, \sigma_n$ faces of Δ satisfying $\sigma_i \cup \sigma_j \notin \Delta$ for all $i \neq j$. Then we have the exact sequence*

$$(*) \qquad 0 \longrightarrow \bigoplus_{i=1}^{n} k[\mathrm{star}_\Delta(\sigma_i)] \longrightarrow k[\Delta] \longrightarrow k[\Delta'] \longrightarrow 0$$

as graded A-modules, where $A = k[v;\ v \in V]$ and

$$\Delta' = \Delta - \{\tau \in \Delta;\ \tau \supset \sigma_i \text{ for some } i\}.$$

(1.3) Let $A = \bigoplus_{n\geq 0} A_n$ be a noetherian graded ring defined over a field $k = A_0$, $m = A_+ = \bigoplus_{n>0} A_n$, and $M = \bigoplus_{n\in Z} M_n$ a finitely generated graded A-module. We denote by $\underline{H}^i_m(M)$ the i-th local cohomology module, that is to say,

$$\underline{H}^i_m(M) = \varinjlim_{\nu} \underline{\mathrm{Ext}}^i_A(A/m^\nu, M),$$

see Goto-Watanabe [14, P. 187].

Recall some fundamental properties of $\underline{H}^i_m(M)$. First, if $d = \dim_A M$ and $t = \mathrm{depth}_A M$, then $\underline{H}^i_m(M) = 0$ unless $t \leq i \leq d$ and $\underline{H}^d_m(M) \neq 0$ and $\underline{H}^t_m(M) \neq 0$. Secondly, if

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is an exact sequence of graded A-modules, then there exists a long exact sequence

$$\begin{aligned} 0 &\longrightarrow \underline{H}^0_m(L) \longrightarrow \underline{H}^0_m(M) \longrightarrow \underline{H}^0_m(N) \\ &\longrightarrow \underline{H}^1_m(L) \longrightarrow \underline{H}^1_m(M) \longrightarrow \underline{H}^1_m(N) \\ &\longrightarrow \cdots \\ &\longrightarrow \underline{H}^i_m(L) \longrightarrow \underline{H}^i_m(M) \longrightarrow \underline{H}^i_m(N) \\ &\longrightarrow \cdots \end{aligned}$$

of local cohomology modules.

Combining the above local cohomology theory with Theorem (1.2), we obtain the following

COROLLARY. *Let Δ be a pure simplicial complex of dimension d and $\sigma_1, \sigma_2, \cdots, \sigma_n$ faces of Δ satisfying $\sigma_i \cup \sigma_j \notin \Delta$ for all $i \neq i$. Also, let $\Delta' = \Delta - \{\tau \in \Delta;\ \tau \supset \sigma_i$ for some $i\}$.*

a) *If Δ is Cohen-Macaulay and* $\dim \Delta' < d$, *then* $\dim \Delta' = d - 1$ *and* Δ' *is also Cohen-Macaulay.*

b) *If* $\text{star}_\Delta(\sigma_i)$ *are Cohen-Macaulay for all i and* Δ' *is Cohen-Macaulay of dimension d, then* Δ *is also Cohen-Macaulay.*

Next, we will state two basic results on Buchsbaum rings and modules from Stückrad-Vogel [31] and Schenzel [25].

LEMMA A (Stückrad-Vogel). *Let* $A = \bigoplus_{n \geq 0} A_n$ *be a noetherian graded ring defined over a field* $k = A_0$, $m = \bigoplus_{n > 0} A_n$ *and* $M = \bigoplus_{n \in Z} M_n$ *a finitely generated graded A-module of dimension d. Then M is a Buchsbaum A-module if the natural map*

$$\underline{\text{Ext}}^i_A(A/m, M) \longrightarrow \underline{H}^i_m(M)$$

is surjective for all $i \neq d$. *Moreover, if A is a polynomial ring over k then the converse is also true.*

The above lemma is called the *surjectivity criterion* for Buchsbaum modules.

A simplicial complex Δ is called *Buchsbaum* over k if $k[\Delta]$ is a Buchsbaum ring, which is equivalent to the fact that $k[\Delta]$ is a Buchsbaum A-module, where $A = k[v; v \in V]$ as usual.

LEMMA B (Schenzel). *A simplicial complex* Δ *is Buchsbaum if and only if, for every* $\sigma \in \Delta$, *except* $\sigma = \emptyset$, $\text{link}_\Delta(\sigma)$ *is Cohen-Macaulay.*

A Buchsbaum complex Δ is also called *almost Cohen-Macaulay* in Baclawski [2].

(1.4) Thanks to the exact sequence (*) of (1.2) and Corollary (1.3), we can give quite simple ring-theoretical proofs to important combinatorial results, which we shall achieve in Section 2. Here we treat only two extreme cases.

EXAMPLE A. Let Δ be a pure simplicial complex of dimension d, $\sigma_1, \sigma_2, \cdots, \sigma_n$ the set of all facets (maximal faces) of Δ and

$$\Delta_{d-1} = \{\tau \in \Delta; \sharp(\tau) \leq d - 1\}.$$

In this case, the exact sequence (*) is nothing but

$$0 \longrightarrow \bigoplus_{i=1}^{n} k[v; v \in \sigma_i] \longrightarrow k[\Delta] \longrightarrow k[\Delta_{d-1}] \longrightarrow 0$$

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which is considered in Baclawski-Garsia [4, P. 178]. Since $\dim k[\Delta_{d-1}] = d-1$, by Corollary (1.3) a), $k[\Delta_{d-1}]$ is Cohen-Macaulay if so is Δ.

On the other hand, suppose that Δ is Buchsbaum. We have the following commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{Ext}}^i_A(A/m, k[\Delta]) & \longrightarrow & \underline{\mathrm{Ext}}^i_A(A/m, k[\Delta_{d-1}]) \\ \phi\downarrow & & \psi\downarrow \\ \underline{H}^i_m(k[\Delta]) & \xrightarrow{\sim} & \underline{H}^i_m(k[\Delta_{d-1}]) \end{array}$$

with natural maps if $i < d-1$, where m is the irrelevant ideal of the polynomial ring A over k whose indeterminates are the vertices of Δ. Since ϕ is surjective by Lemma A of (1.3), ψ must be surjective, hence $k[\Delta_{d-1}]$ is also Buchsbaum again by Lemma A of (1.3).

Repeating the above procedure, we see that if Δ is Cohen-Macaulay (resp. Buchsbaum) of dimension d, then

$$\Delta_i = \{\tau \in \Delta;\ \sharp(\tau) \leq i\}$$

is also Cohen-Macaulay (resp. Buchsbaum) for all $i < d$, see Baclawski-Garsia [4, Theorem 5.5].

Example B. Let Δ be a pure simplicial complex on the vertex set V and $v \in V$. Then, by (*) we have

$$0 \longrightarrow k[\mathrm{star}_\Delta(\{v\})] \longrightarrow k[\Delta] \longrightarrow k[\Delta]/(v) \longrightarrow 0\,.$$

Hence if both the localization $k[\Delta]_v$ $(= k[\mathrm{link}_\Delta(\{v\})][v, v^{-1}])$ and the quotient $k[\Delta]/(v)$ are Cohen-Macaulay, then $k[\Delta]$ is also Cohen-Macaulay by Corollary (1.3) b). Note that if $\dim (k[\Delta]/(v)) < \dim k[\Delta]$ then $\Delta = \mathrm{star}_\Delta(\{v\})$.

This result is false in general. For example, let $A = k[x, y]/(xy, y^2)$, $v = x$ (resp. $A = k[x, y, z]/(xz, yz)$, $v = x + y$), then A is not Cohen-Macaulay, however, both $A_v = k[x, x^{-1}]$ (resp. $k[x, y, 1/(x+y)]$) and $A/(v) = k[y]/(y^2)$ (resp. $k[x, z]/(xz)$) are Cohen-Macaulay.

(1.5) Let V be a finite set which is a union (not necessarily a disjoint union) $\bigcup_{i=1}^n V_i$ of non-empty subsets V_i of V and Δ_i simplicial complexes on the vertex sets V_i $(1 \leq i \leq n)$. Define $\Delta = \bigcup_{i=1}^n \Delta_i$, which is a simplicial complex on the vertex set V. Note that $\dim \Delta = \max\{\dim \Delta_i;\ 1 \leqq i \leqq n\}$ and that Δ is pure if each Δ_i is pure with $\dim \Delta_i = \dim \Delta$.

When $\Delta = \bigcup_{i=1}^n \Delta_i$ turns out to be Cohen-Macaulay? For simplicity, we treat the case of $n = 2$.

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LEMMA. *Let $n = 2$. We have the natural exact sequence*

$$(**) \qquad 0 \longrightarrow k[\Delta_1 \cup \Delta_2] \longrightarrow k[\Delta_1] \oplus k[\Delta_2] \longrightarrow k[\Delta_1 \cap \Delta_2] \longrightarrow 0$$

of graded modules over $A = k[v; v \in V]$.

The above lemma was first considered in Hochster [19].

Again, combining the above exact sequence (**) with the local cohomology theory of (1.3), we immediately have the following

COROLLARY. *Let Δ_1, Δ_2 be Cohen-Macaulay complexes of dimension d. Then $\Delta_1 \cup \Delta_2$ is Cohen-Macaulay if and only if* $\operatorname{depth} k[\Delta_1 \cap \Delta_2] \geq d - 1$.

EXAMPLE A. Let Δ be a simplicial complex on the vertex set V and $\sharp(V) \geq 2$. Then $\operatorname{depth} k[\Delta] \geq 2$ if and only if the *geometric realization* $|\Delta|$ is connected.

To see why this is true, first assume that $|\Delta|$ is not connected. Then, there exist non-empty subsets V_1 and V_2 of V and simplicial complexes Δ_i on V_i $(i = 1, 2)$ such that $V = V_1 \cup V_2$, $V_1 \cap V_2 = \varnothing$ and $\Delta = \Delta_1 \cup \Delta_2$. Hence by (**) we have the exact sequence

$$0 \longrightarrow k[\Delta] \longrightarrow k[\Delta_1] \oplus k[\Delta_2] \longrightarrow k \longrightarrow 0\,,$$

so $\underline{H}^1_m(k[\Delta]) \neq 0$.

Conversely, suppose that $|\Delta|$ is connected. We shall prove $\operatorname{depth} k[\Delta] \geq 2$ by induction on $\sharp(V)$. We can choose $v \in V$ such that the geometric realization of the subcomplex $\Delta' = \{\tau \in \Delta;\, v \notin \tau\}$ on the vertex set $V - \{v\}$ is connected. By assumption of induction $\operatorname{depth}(k[\operatorname{star}_\Delta(\{v\})]) \geq 2$ and $\operatorname{depth} k[\Delta'] \geq 2$, hence $\underline{H}^i_m(k[\Delta]) = 0$ if $i < 2$, see the exact sequence contained in Example B of (1.4).

EXAMPLE B. Let Δ be a simplicial complex on the vertex set V. The *extension* of Δ by an element $v \in V$, denoted by $\Delta \propto v$, is a simplicial complex on $V \cup \{v'\}$ $(v' \notin V)$ such that (i) $\sigma' = \{v', v_1, \cdots, v_i\} \in \Delta \propto v$ if and only if $\sigma = \{v, v_1, \cdots, v_i\} \in \Delta$, where $v_1, \cdots, v_i \in V - \{v\}$, (ii) for $\sigma \subset V - \{v\}$, $\sigma \in \Delta \propto v$ if and only if $\sigma \in \Delta$, and (iii) $\{v, v'\} \notin \Delta \propto v$. If Δ is Cohen-Macaulay, then so is $\Delta \propto v$.

To see this, let $\Delta' = \{\tau \in \Delta \propto v;\, v \notin \tau\}$. Then $\Delta \propto v = \Delta \cup \Delta'$. Hence, thanks to the above corollary, we have only to check that $\operatorname{depth} k[\Delta \cap \Delta'] \geq \dim k[\Delta] - 1$. This inequality follows by the exact sequence contained in Example B of (1.4), since $k[\Delta \cap \Delta'] = k[\Delta]/(v)$.

§ 2. Rank selection theorem, shellable complexes and G-complexes

Based on the exact sequences (*) and (**) of Section 1, in this section, we shall give quite elementary proofs to some combinatorial results on Cohen-Macaulay posets and complexes.

(2.1) We begin with some definitions and terminology on partially ordered sets.

All posets to be considered are finite. The *length* of a chain (totally ordered set) C is the cardinality $\sharp(C)$ as a set. The *rank* of a poset Q, denoted by $\mathrm{rank}(Q)$, is the supremum of lengths of chains contained in Q. The *rank* of an element α in a poset is the supremum of lengths of chains descending from α, and written by $r(\alpha)$. If two elements α and β in a poset are incomparable, then we will write $\alpha \not\sim \beta$. A *clutter* is a poset in which no two elements are comparable. A *poset ideal* in a poset Q is a subset I such that $\alpha \in I$, $\beta \in Q$ and $\beta \leq \alpha$ together imply $\beta \in I$. A poset P is called a *subposet* of Q if $P \subset Q$ and, for $\alpha, \beta \in P$, $\alpha < \beta$ in P if and only if $\alpha < \beta$ in Q.

When we regard a poset Q only as a finite set V forgetting the partial order, we call V the *underlying set* of Q and Q is called a poset on V.

Let Q be a poset on V and define $\Delta(Q)$, called the *order complex* of Q, to be the simplicial complex on V whose faces are the chains of Q. We will use such terminology as "Q is pure" or "Q is Cohen-Macaulay" to mean the corresponding statement for $\Delta(Q)$. Thus Q is pure if and only if all maximal chains of Q have the same length. Note that $\mathrm{rank}(Q) = \dim \Delta(Q)$. Moreover, a Cohen-Macaulay (resp. Buchsbaum) poset Q is one for which $\Delta(Q)$ is a Cohen-Macaulay (resp. Buchsbaum) complex. The Stanley-Reisner ring of a poset Q, denoted by $k[Q]$, is the Stanley-Reisner ring $k[\Delta(Q)]$ of $\Delta(Q)$, thus

$$k[Q] = k[\alpha;\, \alpha \in Q]/(\alpha\beta;\, \alpha \not\sim \beta)\,.$$

Also, as an analogue of simplicial complexes, if C is a chain of a poset Q then we define

$$\begin{aligned} \mathrm{link}_Q(C) &= \{\alpha \in Q;\, \alpha \notin C \text{ and } C \cup \{\alpha\} \in \Delta(Q)\} \\ \mathrm{star}_Q(C) &= \{\alpha \in Q;\, C \cup \{\alpha\} \in \Delta(Q)\}\,, \end{aligned}$$

which should be considered as subposets of Q. Note that $\mathrm{link}_{\Delta(Q)}(C) = \Delta(\mathrm{link}_Q(C))$ and $\mathrm{star}_{\Delta(Q)}(C) = \Delta(\mathrm{star}_Q(C))$.

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(2.2) *Rank selection theorem.* Probably one of the most important results in a theory of Cohen-Macaulay posets is the "rank selection theorem", which first appeared in Baclawski [2, Theorem 5.4] and Munkres [23, Theorem 6.4] with purely topological proofs. On the other hand, starting from the definition [28, P. 63] of Cohen-Macaulay rings by the Hilbert functions, Stanley [29, Theorem 4.3] also gave an elementary ring-theoretical proof to this theorem.

Let Q be a pure poset of rank d. Suppose that S is any subset of $[d] := \{1, 2, \cdots, d\}$. Define the rank-selected subposet Q_S with respect to S to be

$$Q_S = \{\alpha \in Q;\ r(\alpha) \in S\}.$$

Note that $\operatorname{rank}(Q_S) = \sharp(S)$.

The rank selection theorem states that if Q is Cohen-Macaulay then so is Q_S for any $S \subset [d]$.

We shall prove this theorem by using (*) of (1.2). We may assume $S = [d] - \{i\}$, $i \in [d]$. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be the set of all rank i elements of Q. Then the exact sequence (*) asserts

$$0 \longrightarrow \bigoplus_{i=1}^{n} k[\operatorname{star}_Q(\{\alpha_i\})] \longrightarrow k[Q] \longrightarrow k[Q_S] \longrightarrow 0.$$

Hence Q_S is Cohen-Macaulay by Corollary (1.3) a).

Moreover, with the same argument as in Example A of (1.4), we see that if Q is Buchsbaum then so is Q_S for all $S \subset [d]$, since $k[\operatorname{star}_Q(\{\alpha_i\})]$ is Cohen-Macaulay by Lemma B of (1.3). This Buchsbaum analogue of the rank selection theorem was proved, in topological sence, in Baclawski [2, Theorem 6.5].

(2.3) *Shellable complexes.* A pure complex Δ of dimension d is called *shellable* if its all facets (maximal faces) can be listed $F_1, F_2, \cdots, F_s$ in such a way that

$$\left(\bigcup_{j=1}^{i-1} \bar{F}_j\right) \cap \bar{F}_i$$

is pure of dimension $d-1$ for all $i > 1$, where $\bar{F}_i = \{\sigma \in \Delta;\ \sigma \subset F_i\}$.

If we define

$$\sigma = \left\{v \in F_s;\ F_s - \{v\} \in \left(\bigcup_{j=1}^{s-1} \bar{F}_j\right) \cap \bar{F}_s\right\},$$

then

$$\operatorname{star}_{\Delta}(\sigma) = \bar{F}_s\,,$$
$$\Delta - \{\tau \in \Delta;\, \tau \supset \sigma\} = \bigcup_{j=1}^{s-1} \bar{F}_j\,.$$

In fact, if $\tau \in \operatorname{star}_{\Delta}(\sigma)$ and $\tau \notin \bar{F}_s$ then $\sigma \cup \tau \in \bar{F}_j$ for some $j < s$, hence $\sigma \in \bar{F}_j$, thus $\sigma \in (\bigcup_{j=1}^{s-1} \bar{F}_j) \cap \bar{F}_s$, in other words, $\sigma \subset F_s - \{v\}$ for some $v \in \sigma$, a contradiction. On the other hand, $\{\tau \in \Delta;\, \tau \supset \sigma\}$ coincides with $\bar{F}_s - (\bigcup_{j=1}^{s-1} \bar{F}_j) \cap \bar{F}_s$, thus $\Delta - \{\tau \in \Delta;\, \tau \supset \sigma\} = \bigcup_{j=1}^{s-1} \bar{F}_j$.

Hence, thanks to (*) of (1.2), we have

$$0 \longrightarrow k[\bar{F}_s] \longrightarrow k[\Delta] \longrightarrow k\left[\bigcup_{j=1}^{s-1} \bar{F}_j\right] \longrightarrow 0\,,$$

so, by induction on s, we see that a shellable complex is Cohen-Macaulay by Corollary (1.3) b).

The notion of shellability originated in the study of polytopes. A fundamental result of Bruggesser-Main [9] concerning the shellability of the boundary complexes of simplicial polytopes is essential in McMullen [22] and Hochster [19]. The Cohen-Macaulayness of shellable complexes is essentially due to Folkman [11] in topological sense and Hochster [19] in ring-theoretical sense. See also, Björner [6], [7], Garsia [12] and Kind-Kleinschmidt [21].

(2.4) *G-complexes.* Let Δ be a simplicial complex on the vertex set V. For $W \subset V$, define

$$\Delta_W = \{\sigma \in \Delta;\, \sigma \subset W\}\,,$$

which is a simplicial complex on W.

We call Δ a *G-complex* (or "matroid") if Δ_W is pure for all $W \subset V$. Refer to Stanley [27, § 7] for further information.

Note that if Δ is a G-complex and $\sigma \in \Delta$ then $\operatorname{link}_{\Delta}(\sigma)$ is, hence $\operatorname{star}_{\Delta}(\sigma)$ is also, a G-complex. To see why $\operatorname{link}_{\Delta}(\sigma)$ is a G-complex, let W be a subset of the vertex set of $\operatorname{link}_{\Delta}(\sigma)$, and τ_1, τ_2 facets of $(\operatorname{link}_{\Delta}(\sigma))_W$. Then $\tau_1 \cup \sigma$, $\tau_2 \cup \sigma$ are facets of $\Delta_{W \cup \sigma}$, hence $\sharp(\tau_1 \cup \sigma) = \sharp(\tau_2 \cup \sigma)$, thus $\sharp(\tau_1) = \sharp(\tau_2)$.

Now, using (*) and (**) of Section 1, we shall give a direct proof, without the requirement of some combinatorial properties of G-complexes, to the Cohen-Macaulayness of G-complexes. Let Δ be a G-complex of dimension d on the vertex set V. We shall prove Δ is Cohen-Macaulay by induction on $\sharp(\Delta)$. Let $v_1, v_2 \in V$ $(v_1 \neq v_2)$ and $V_i = V - \{v_i\}$, $\Delta_i = \Delta_{V_i}$ $(i = 1, 2)$. We may assume $\dim \Delta_i = d$, otherwise $\Delta = \operatorname{star}_{\Delta}(\{v_i\})$ $(i = 1, 2)$.

Since $\Delta_1 \cap \Delta_2 = \Delta_{V-\{v_1,v_2\}}$, $\Delta_1 \cap \Delta_2$ is a G-complex and $\dim(\Delta_1 \cap \Delta_2) \geq d-1$. By assumption of induction, $\Delta_1 \cap \Delta_2$ is Cohen-Macaulay, hence, thanks to Corollary (1.5), $\Delta_1 \cup \Delta_2$ is Cohen-Macaulay. If $\sigma = \{v_1, v_2\} \notin \Delta$, then $\Delta = \Delta_1 \cup \Delta_2$ and we have done. On the other hand, if $\sigma \in \Delta$ then

$$\Delta - \{\tau \in \Delta;\ \tau \supset \sigma\} = \Delta_1 \cup \Delta_2\,,$$

hence we have the exact sequence

$$0 \longrightarrow k[\mathrm{star}_\Delta(\sigma)] \longrightarrow k[\Delta] \longrightarrow k[\Delta_1 \cup \Delta_2] \longrightarrow 0$$

by (*) of (1.2). Since $\mathrm{star}_\Delta(\sigma)$ is a G-complex, $\mathrm{star}_\Delta(\sigma)$ is Cohen-Macaulay by assumption of induction. Hence Δ is also Cohen-Macaulay by Corollary (1.3) b).

If Δ is a G-complex, then the Stanley-Reisner ring $k[\Delta]$ is a "level ring" defined in Stanley [27, § 3]. The concept of level rings intermediates between Cohen-Macaulay and Gorenstein which will be of interest from viewpoints of both commutative algebra and combinatorics. See [3], [16] and [35] for some results on level rings.

§ 3. Comparability graphs and order complexes of partially ordered sets

In order to apply Corollary (1.3) and Corollary (1.5) concerning with the Cohen-Macaulayness of simplicial complexes to finite partially ordered sets, in this section, we shall define explicitly an "intersection", a "union" and the "glueing" of a family of finite partially ordered sets.

(3.1) First, recall some definitions and terminology from graph theory.

A *graph* G is a finite set $V \neq \emptyset$ together with a (possibly empty) set E of two-element subsets of distinct elements of V. Each element of V is referred as a *vertex* and V itself as the *vertex set* of G and the members of the *edge set* E are called *edges*. In general, we represent the vertex set and edge set of a graph G by $V(G)$ and $E(G)$, respectively. Note that, by our definition, all graphs considered in this section have no loops and multiple edges.

The edge $e = \{u, v\}$ is said to be *join* the vertices u and v. If $e = \{u, v\}$ is an edge of a graph G, then u and v are *adjacent vertices*, while u and e are *incident*, as are v and e. Furthermore, if e_1 and e_2 are distinct edges of G incident with a common vertex, then e_1 and e_2 are *adjacent edges*.

A graph H is called a *subgraph* of a graph G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. If H is a subgraph of G, then we write $H \subset G$. Also, a graph G is called *complete* if $\{u, v\} \in E(G)$ for all $u, v \in V(G)$, $u \neq v$,

Let V be a finite set which is a union (not necessarily a disjoint union) $\bigcup_{i=1}^{n} V_i$ of non-empty subsets V_i of V and G_i graphs on the vertex sets V_i $(1 \leq i \leq n)$. Then we define the *union* $\bigcup_{i=1}^{n} G_i$ (resp. the *intersection* $\bigcap_{i=1}^{n} G_i$) to be a graph on the vertex set V (resp. $\bigcap_{i=1}^{n} V_i$) whose edge set is $\bigcup_{i=1}^{n} E(G_i)$ (resp. $\bigcap_{i=1}^{n} E(G_i)$).

(3.2) A graph G is said to be a *comparability graph* if there exists a partial order on $V(G)$ such that, for $u, v \in V(G)$, $u \neq v$, $\{u, v\} \in E(G)$ if and only if u and v are comparable with respect to the partial order on $V(G)$.

By a *cycle* of a graph G is meant here any finite sequence of vertices $v_1, v_2, \cdots, v_n$ of G such that $\{v_i, v_{i+1}\}$, $1 \leq i < n$, and $\{v_n, v_1\}$ are in $E(G)$, and for no vertices α and β and integer $i, j < n$, $i \neq j$, $\alpha = v_i = v_j$, $\beta = v_{i+1} = v_{j+1}$ or $\alpha = v_i = v_n$, $\beta = v_{i+1} = v_1$. A cycle is odd or even depending on whether n is odd or even. By a *triangular chord* of a cycle $v_1, v_2, \cdots, v_n$ of G is meant any one of the edges of the form $\{v_i, v_{i+2}\}$, $1 \leq i \leq n-2$, or $\{v_{n-1}, v_1\}$ or $\{v_n, v_2\}$.

LEMMA (Gilmore-Hoffmann [13]). *A graph G is a comparability graph if and only if each odd cycle has at least one triangular chord.*

EXAMPLE. The graph G of Figure 2 is not a comparability graph since an odd cycle p, z, b, c, y has no triangular chord.

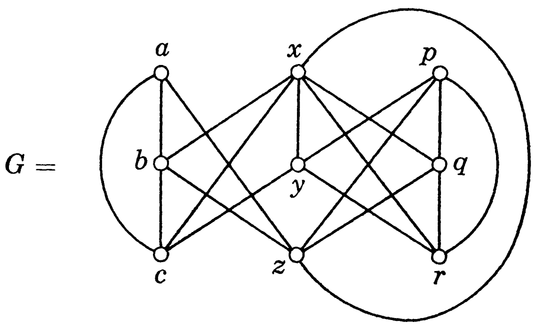

Figure 2

(3.3) Let Δ be a simplicial complex on the vertex set V. We will associate Δ with a graph $G(\Delta)$, called a *skelton*, whose vertex set is V and edge set is the set of all faces of Δ with $\sharp(\sigma) = 2$.

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For which simplicial complexes Δ, does there exist a poset Q such that $\Delta = \Delta(Q)$? The answer was obtained in Stanley [29], namely,

LEMMA (Stanley). *A simplicial complex Δ is of the form $\Delta = \Delta(Q)$ for some poset Q if and only if Δ satisfies the following*:

(i) *the ideal I_Δ is generated by quadratic monomials, and*

(ii) *the skelton $G(\Delta)$ is a comparability graph.*

In fact, the "only if" part is obvious. To see the "if" part, let the skelton $G(\Delta)$ be a comparability graph, say $G(\Delta) = G(\Delta(Q))$ for some poset Q. Then it is easy to see that $\Delta \subset \Delta(Q)$. To show $\Delta \supset \Delta(Q)$, let $\sigma \in \Delta(Q)$, namely σ is a chain of Q. Then every two-element subset $\{x, y\}$ of σ is an edge of $G(\Delta(Q)) = G(\Delta)$, hence $\{x, y\} \in \Delta$. This $\sigma \in \Delta$ since I_Δ is generated by quadratic monomials.

(3.4) Let V be a finite set which is a union $\bigcup_{i=1}^n V_i$ of non-empty subsets V_i of V and Q_i posets whose underlying sets are V_i $(1 \le i \le n)$. Then a poset Q whose underlying set is $\bigcap_{i=1}^n V_i$ is called an *intersection* of a family $\{Q_i\}_{1 \le i \le n}$, if $\Delta(Q) = \bigcap_{i=1}^n \Delta(Q_i)$. Of course, an intersection is not necessarily unique even though it exists.

Since the ideal $I_{\bigcap_{i=1}^n \Delta(Q_i)}$ of $A = k[v; v \in V]$ is generated by quadratic monomials, there exists an intersection of $\{Q_i\}_{1 \le i \le n}$ if and only if

$$G\left(\bigcap_{i=1}^n \Delta(Q_i)\right) = \bigcap_{i=1}^n G(\Delta(Q_i))$$

is a comparability graph.

Note that the intersection of comparability graphs is not necessarily a comparability graph. For example,

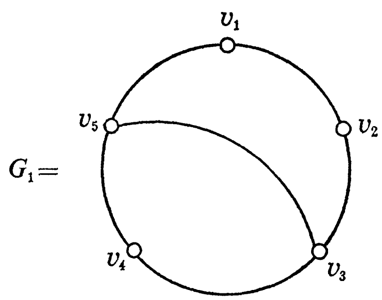

Figure 3

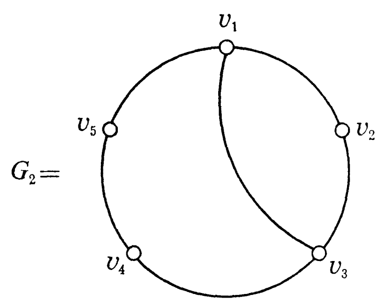

Figure 4

are comparability graphs, however, $G_1 \cap G_2$ is not a comparability graph.

So, it is natural to find a sufficient condition for the existence of an intersection of $\{Q_i\}_{1\le i\le n}$.

LEMMA. *Suppose that, for* $\alpha, \beta \in \bigcap_{i=1}^n V_i$, *there exist no* i *and* j, $i \neq j$, *such that* $\alpha < \beta$ *in* Q_i *while* $\alpha > \beta$ *in* Q_j. *Then there exists an intersection of* $\{Q_i\}_{1\le i\le n}$.

In fact, for $\alpha, \beta \in \bigcap_{i=1}^n V_i$, define $\alpha \le \beta$ if $\alpha \le \beta$ in all Q_i, and we obtain a poset Q whose underlying set is $\bigcap_{i=1}^n V_i$ satisfying $\Delta(Q) = \bigcap_{i=1}^n \Delta(Q_i)$.

Of course, the condition of the above lemma is not a necessary condition.

(3.5) Now, as an analogue of a union of simplicial complexes, we will define a "union" and "glueing" of a family of partially ordered sets.

DEFINITION. Let V be a finite set which is a union $\bigcup_{i=1}^n V_i$ of non-empty subsets V_i of V. Also, let Q_i be posets whose underlying sets are V_i $(1 \le i \le n)$.

a) A poset Q whose underlying set is V is called a *union* of a family $\{Q_i\}_{1\le i\le n}$ if $\Delta(Q) = \bigcup_{i=1}^n \Delta(Q_i)$.

b) A poset Q whose underlying set is V is called a *glueing* of a family $\{Q_i\}_{1\le i\le n}$ if each Q_i is a subposet of Q and $\Delta(Q) = \bigcup_{i=1}^n \Delta(Q_i)$.

When does there exist a union or a glueing of a family of partially ordered sets?

PROPOSITION. *Work in the same notation as in the above definition.*

a) *There exists a union of* $\{Q_i\}_{1\le i\le n}$ *if and only if* (i) *the graph* $G(\bigcup_{i=1}^n \Delta(Q_i)) = \bigcup_{i=1}^n G(\Delta(Q_i))$ *is a comparability graph and* (ii) *every complete subgraph of* $\bigcup_{i=1}^n G(\Delta(Q_i))$ *is a subgraph of* $G(\Delta(Q_i))$ *for some* i.

b) *There exists a glueing of* $\{Q_i\}_{1\le i\le n}$ *if and only if* (i) *for all* $i \neq j$ *and* $\alpha, \beta \in V_i \cap V_j$, $\alpha \le \beta$ *in* Q_i *if and only if* $\alpha \le \beta$ *in* Q_j *and* (ii) *if* $\alpha_1, \alpha_2 \in V_{i_1}$, $\alpha_2, \alpha_3 \in V_{i_2}, \cdots, \alpha_s, \alpha_{s+1} \in V_{i_s}$ *and* $\alpha_1 \le \alpha_2$ *in* Q_{i_1}, $\alpha_2 \le \alpha_3$ *in* $Q_{i_2}, \cdots, \alpha_s \le \alpha_{s+1}$ *in* Q_{i_s}, *then* $\{\alpha_1, \alpha_2, \cdots, \alpha_{s+1}\} \subset V_t$ *for some* t.

Proof. a) Let $I = I_{\bigcup_{i=1}^n \Delta(Q_i)}$. The problem is whether I is generated by quadratic monomials or not.

First, assume that I is not generated by quadratic monomials. Let $\sigma = \{v_1, v_2, \cdots, v_p\}$ $(p \ge 3)$ be a subset of V such that $\sigma \notin \bigcup_{i=1}^n \Delta(Q_i)$ and $\sigma - \{v_j\} \in \bigcup_{i=1}^n \Delta(Q_i)$ for all j. Then, there exists a complete subgraph G' of $\bigcup_{i=1}^n G(\Delta(Q_i))$ whose vertex set is σ. If G' is contained in $G(\Delta(Q_i))$ for some i, then σ is a chain of Q_i, hence $\sigma \in \Delta(Q_i) \subset \bigcup_{i=1}^n \Delta(Q_i)$, a contradiction.

Conversely, assume that I is generated by quadratic monomials and that G' is a complete subgraph contained in $\bigcup_{i=1}^{n} G(\Delta(Q_i))$ whose vertex set is $\sigma = \{v_1, v_2, \cdots, v_p\}$ $(p \geq 3)$. Then, $v_i v_j \notin I$ for all i, j, hence $v_1 v_2 \cdots v_p \notin I$. So, $\sigma \subset \Delta(Q_i)$ for some i. Thus G' is a subgraph of $G(\Delta(Q_i))$.

b) First, we shall prove "if" part. Suppose that the condition (i) and (ii) are satisfied. Then, we can define a partial order $\leq_V$ on V as follows. Let $v_1, v_2 \in V$. Define $v_1 \leq_V v_2$ if and only if $v_1, v_2 \in V_i$ for some i and $v_1 \leq v_2$ in Q_i. By (i), this definition does not depend on the choice of V_i containing v_1 and v_2. Also, let $v_1, v_2, v_3 \in V$ and $v_1 \leq_V v_2$, $v_2 \leq_V v_3$. By the definition of $\leq_V$ we have $v_1, v_2 \in V_i$, $v_2, v_3 \in V_j$ for some i, j and $v_1 \leq v_2$ in Q_i, $v_2 \leq v_3$ in Q_j. Then, $v_1, v_2, v_3 \in V_t$ for some t by (ii), and $v_1 \leq v_2$, $v_2 \leq v_3$ in Q_t by (i). Hence $v_1 \leq v_3$ in Q_t, this $v_1 \leq_V v_3$. So, $\leq_V$ is a partial order on V. Let Q be a poset with this partial order $\leq_V$ whose underlying set is V. Then we see immediately that $\Delta(Q) = \bigcup_{i=1}^{n} \Delta(Q_i)$ by (ii).

Secondly, we shall prove "only if" part. Suppose that a family $\{Q_i\}_{1 \leq i \leq n}$ has a glueing Q. Since each Q_i is a subposet of Q, for $\alpha, \beta \in V_i$, $\alpha \leq \beta$ in Q_i if and only if $\alpha \leq \beta$ in Q, hence (i) holds. Also, $\alpha_1, \alpha_2 \in V_{i_1}$, $\alpha_2, \alpha_3 \in V_{i_2}, \cdots, \alpha_s, \alpha_{s+1} \in V_{i_s}$ and $\alpha_1 \leq \alpha_2$ in Q_{i_1}, $\alpha_2 \leq \alpha_3$ in $Q_{i_2}, \cdots, \alpha_s \leq \alpha_{s+1}$ in Q_{i_s}, then $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{s+1}$ in Q, hence $\{\alpha_1, \alpha_2, \cdots, \alpha_{s+1}\} \in \Delta(Q) = \bigcup_{i=1}^{n} \Delta(Q_i)$. So, $\{\alpha_1, \alpha_2, \cdots, \alpha_{s+1}\} \in \Delta(Q_i)$ for some i. Q.E.D.

EXAMPLE A. Let $V = \{v_i\}_{1 \leq i \leq 8}$ and $V_1 = \{v_i\}_{1 \leq i \leq 6}$, $V_2 = \{v_i\}_{4 \leq i \leq 8}$. Also, let

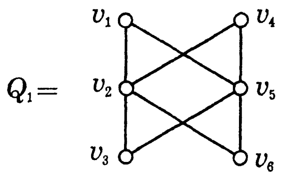

Figure 5

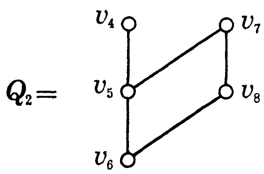

Figure 6

Then $G = G(\Delta(Q_1)) \cup G(\Delta(Q_2))$ looks like the graph of Figure 7, which is a comparability graph. In fact, the partial order on V induced by the following poset Q of Figure 8 produces the above graph G.

So, thanks to the above proposition, we see that there exists a union of Q_1 and Q_2, however, there exists no glueing of Q_1 and Q_2.

Note that a union of a family of posets is not necessarily unique

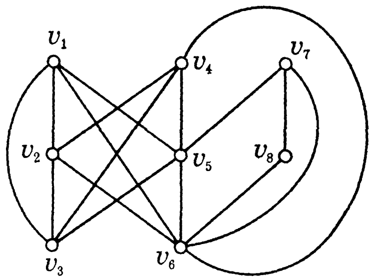

Figure 7

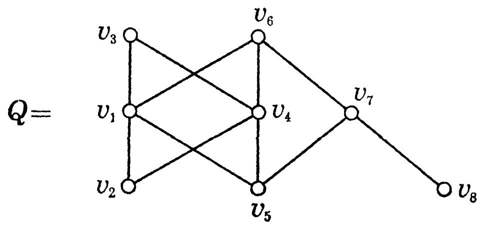

Figure 8

even though it exists, while a glueing of a family of posets is unique if it exists. Hence, from now on, we call *the* glueing of a family of posets.

COROLLARY. *Let V be a finite set which is a union $V_1 \cup V_2$ of non-empty subsets V_i of V and Q_i posets whose underlying sets are V_i $(i = 1, 2)$. Suppose that* (i) *for $\alpha, \beta \in V_1 \cap V_2$, $\alpha \le \beta$ in Q_1 if and only if $\alpha \le \beta$ in Q_2 and* (ii) *$I_i := V_i - (V_1 \cap V_2)$ are poset ideals of Q_i $(i = 1, 2)$. Then, there exists the glueing of Q_1 and Q_2.*

This corollary is treated in [32, § 4]. So, our concept of the glueing of a family of posets is a generalization of that in [32].

EXAMPLE B. Let Q be a poset and I, J poset ideals of Q. Then, the subposet $Q - (I \cap J)$ of Q may be considered the glueing of $Q - I$ and $Q - J$. In particular, if $I \cap J = \emptyset$ then Q itself is the glueing of $Q - I$ and $Q - J$.

§ 4. Criteria of Cohen-Macaulay partially ordered sets

The purpose of this final section is to state criteria of Cohen-Macaulay posets and to show some examples.

(3.1) To begin with, we consider

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EXAMPLE. Let Q be a pure poset of rank d, $i \in [d]$, and $\{\alpha_1, \alpha_2, \cdots, \alpha_n, \beta_1, \beta_2, \cdots, \beta_m\}$ the set of rank i elements of Q $(n, m \geq 1)$.

Thanks to Proposition (3.5) b), two families $\{\mathrm{star}_Q(\{\alpha_i\})\}_{1 \leq i \leq n}$ and $\{\mathrm{star}_Q(\{\beta_j\})\}_{1 \leq j \leq m}$ have the glueings $Q_{[\alpha]}$ and $Q_{[\beta]}$, respectively. Also, Q is the glueing of the family $\{\mathrm{star}_Q(\{\alpha_i\}), \mathrm{star}_Q(\{\beta_j\})\}_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$.

However, $Q_{[\alpha]}$ and $Q_{[\beta]}$ are not necessarily subposets of Q, hence Q is not necessarily the glueing of $Q_{[\alpha]}$ and $Q_{[\beta]}$. For example, if

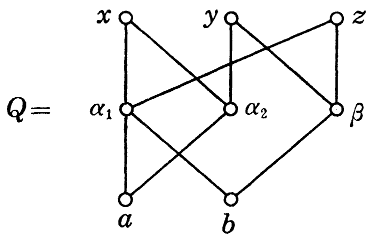


Figure 9

then

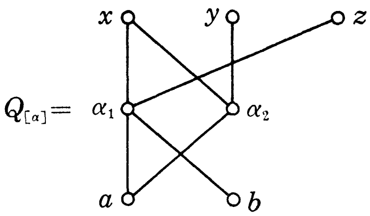


Figure 10

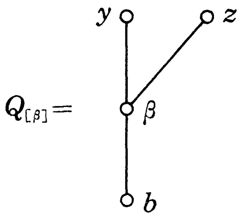


Figure 11

in this case y and b are comparable in $Q_{[\beta]}$ while incomparable in $Q_{[\alpha]}$, hence there exists no glueing of $Q_{[\alpha]}$ and $Q_{[\beta]}$. Of course, in general, Q is a union of $Q_{[\alpha]}$ and $Q_{[\beta]}$.

Nevertheless, if $i = 1$ or d then $Q_{[\alpha]}$ and $Q_{[\beta]}$ are subposets of Q and Q is the glueing of $Q_{[\alpha]}$ and $Q_{[\beta]}$.

Note that, for each $i \in [d]$, there exists an intersection $Q^\wedge$ of $Q_{[\alpha]}$ and $Q_{[\beta]}$ by Lemma (3.4). For example,

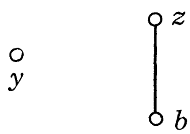


Figure 12

is an intersection of $Q_{[\alpha]}$ of Figure 10 and $Q_{[\beta]}$ of Figure 11. In general, an intersection $Q^\wedge$ of $Q_{[\alpha]}$ and $Q_{[\beta]}$ is not necessarily a subposet of Q, however, we can take $Q^\wedge$ as a subposet of Q if $i = 1$ or d.

Before starting our next work, we had better introduce one notation. Let P and Q be posets whose underlying sets are V_P and V_Q, respectively. We define a new poset $P \oplus Q$, called the *sum*, whose underlying set is the disjoint union of V_P and V_Q and the partial order of $P \oplus Q$ is defined by $x \leq y$ if (i) $x \leq y$ in P, or (ii) $x \leq y$ in Q, or (iii) $x \in P$ and $y \in Q$. For example,

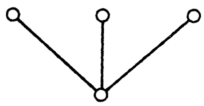


Figure 13

$\oplus$

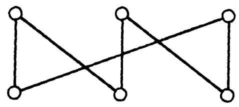
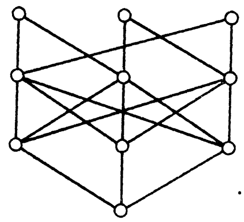


Figure 14

$=$



.

Figure 15

If P and Q are Cohen-Macaulay over a field k, then so is $P \oplus Q$ since $k[P \oplus Q] = k[P] \otimes_k k[Q]$.

(4.2) Now, we will state our main

THEOREM. *Let k be a field and Q a pure poset of rank d.*

a) (*Extension lemma* A) *Suppose that two elements $x, y \in Q$ satisfy $x < y$ and $r(y) - r(x) = 2$, and that Q is Cohen-Macaulay. Then, the new poset Q with $\alpha, \alpha \notin Q$, whose partial order preserves that of Q and, in addition, $x' < \alpha < y'$ for all $x' \leq x$, $y' \geq y$, is also Cohen-Macaulay.*

b) (*Extension lemma* B) *Let x, y be two elements of Q satisfying $x \not< y$ and $r(y) - r(x) = 1$. Suppose that any element $\xi < x$ (resp. $\eta > y$) is comparable with y (resp. x), and that Q is Cohen-Macaulay. Then, the new poset Q with an additional comparable relation $x < y$ is also Cohen-Macaulay.*

c) (*Union criterion*) *Let $i \in [d]$ and $\{\alpha_1, \alpha_2, \cdots, \alpha_n, \beta_1, \beta_2, \cdots, \beta_m\}$ the set of rank i elements of Q ($n, m \geq 1$). Also, let $Q_{[\alpha]}$ (resp. $Q_{[\beta]}$) be the glueing of the family $\{\mathrm{star}_Q(\{\alpha_i\})\}_{1 \leq i \leq n}$ (resp. $\{\mathrm{star}_Q(\{\beta_j\})\}_{1 \leq j \leq m}$), and $Q^\wedge$ an intersection of $Q_{[\alpha]}$ and $Q_{[\beta]}$. Suppose that both $Q_{[\alpha]}$ and $Q_{[\beta]}$ are Cohen-Macaulay over k. Then, Q is Cohen-Macaulay over k if and only if* $\operatorname{depth} k[Q^\wedge] = d - 1$.

d) (*Glueing criterion*) *Let* $\{\alpha_1, \alpha_2, \cdots, \alpha_n, \beta_1, \beta_2, \cdots, \beta_m\}$ *be the set of minimal* (*resp. maximal*) *elements of* Q $(n, m \geq 1)$, *and*

$$Q_{[\alpha]} = \{\xi \in Q;\ \xi \geq \alpha_i \ \textit{for some}\ i\}$$
$$Q_{[\beta]} = \{\eta \in Q;\ \eta \geq \beta_j \ \textit{for some}\ j\}$$
$$\left(\textit{resp.}\ \begin{array}{l} Q_{[\alpha]} = \{\xi \in Q;\ \xi \leq \alpha_i \ \textit{for some}\ i\} \\ Q_{[\beta]} = \{\eta \in Q;\ \eta \leq \beta_j \ \textit{for some}\ j\} \end{array}\right),$$

which will be naturally considered as subposets of Q. *On the other hand, let* $Q^\wedge$ *be the subposet of* Q *consisting of all elements* x *of* Q *satisfying* $x \geq \alpha_i$ *and* $x \geq \beta_j$ (*resp.* $x \leq \alpha_i$ *and* $x \leq \beta_j$) *for some* i *and* j. *Suppose that* $Q_{[\alpha]}$ *and* $Q_{[\beta]}$ *are Cohen-Macaulay over* k. *Then,* Q *is Cohen-Macaulay over* k *if and only if* depth $k[Q^\wedge] = d - 1$.

Proof. In case a) (resp. b)) we denote the new poset by Q'. Since $x < y$ (resp. $\xi < y$ for all $\xi < x$ and $\eta > x$ for all $\eta > y$) in Q, we have

$$\Delta(Q') - \{\tau \in \Delta(Q');\ \tau \supset \{\alpha\}\} = \Delta(Q)$$
$$(\text{resp. } \Delta(Q') - \{\tau \in \Delta(Q');\ \tau \supset \{x, y\}\} = \Delta(Q)),$$

which is Cohen-Macaulay. Also,

$$\text{star}_{Q'}(\{\alpha\}) = \{\xi \in Q;\ \xi \leq x\} \oplus \{\alpha\} \oplus \{\eta \in Q;\ \eta \geq y\}$$
$$(\text{resp. star}_{Q'}(\{x < y\}) = \{\xi \in Q;\ \xi \leq x\} \oplus \{\eta \in Q;\ \eta \geq y\})$$

is Cohen-Macaulay of rank d. Hence, thanks to Corollary (1.3) b), Q' is Cohen-Macaulay.

On the other hand, in case c) (resp. d)), by Example (4.1), the poset Q is a union (resp. the glueing) of $Q_{[\alpha]}$ and $Q_{[\beta]}$, hence the result c) (resp. d)) is an immediate consequence of Corollary (1.5). Q.E.D.

We have some remarks about the above results. First in case a) (resp. b)) the new poset Q' is the glueing (resp. a union) of $\text{star}_{Q'}(\{\alpha\})$ (resp. $\text{star}_{Q'}(\{x < y\})$) and Q, secondly the results a) and d) are special cases of c), and thirdly in case a) if $r(y) = 2$ (resp. $r(x) = d - 1$) then we may consider $Q \cup \{-\infty\}$, $x = -\infty$ (resp. $Q \cup \{\infty\}$, $y = \infty$), where $-\infty$ (resp. ∞) is a least (resp. greatest) element of $Q \cup \{-\infty\}$ (resp. $Q \cup \{\infty\}$) which is not contained in Q.

EXAMPLE A. In cases a) and b) the conditions concerning with x and y are indispensable. In fact,

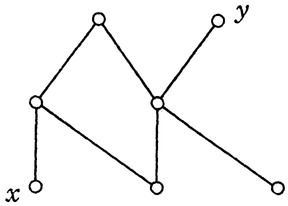


Figure 16

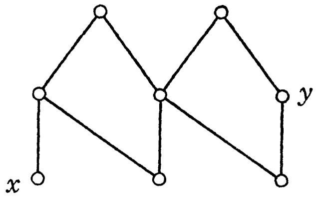


Figure 17

are Cohen-Macaulay, however,

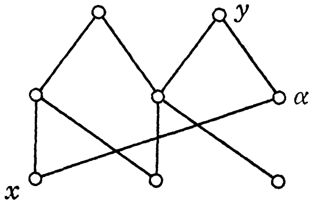


Figure 18

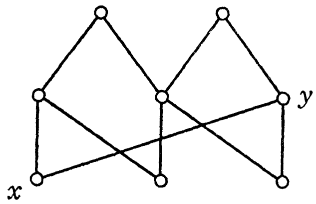


Figure 19

are not Cohen-Macaulay.

EXAMPLE B. By using the union criterion c) and the glueing criterion d), we see that the following posets are not Cohen-Macaulay.

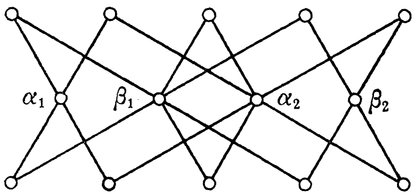


Figure 20

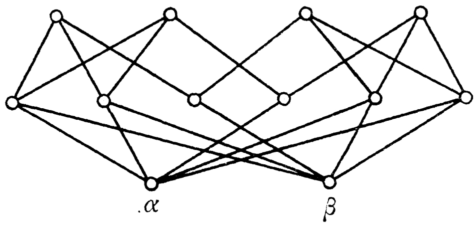


Figure 21

(4.3) Since our results of (4.2) are most fundamental in this paper, we had better consider an elementary combinatorial proof without using (*) or (**) of Section 1.

In [12], Garsia obtained a purely combinatorial and linear algebraic characterization of Cohen-Macaulay posets. Recall some basic results from [12].

Let k be a field, Q a pure poset of rank d and

$$\theta_i = \sum_{r(x)=i} x$$

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the rank i form in the Stanley-Reisner ring $k[Q]$ $(1 \leq i \leq d)$.

If $\mathfrak{c} = \{q_1, q_2, \cdots, q_n\} \subset Q$ then we denote by $\delta(\mathfrak{c})$ the square-free monomial $q_1 q_2 \cdots q_n$ of $k[Q]$. Note that $\delta(\mathfrak{c}) = 0$ unless $\mathfrak{c}$ is a chain.

Let $\mathscr{C}(Q)$ be the set of all chains of Q and $\mathscr{M}(Q)$ the set of all maximal chains of Q. For $\mathfrak{c} \in \mathscr{C}(Q)$, define

$$L(\mathfrak{c}) = \delta(\mathfrak{c}) \prod_{i \in r(\mathfrak{c})} \theta_i ,$$

where $r(\mathfrak{c})$ is the rank set of $\mathfrak{c}$, i.e.,

$$r(\mathfrak{c}) = \{r(x);\ x \in \mathfrak{c}\} .$$

Note that

$$L(\mathfrak{c}) = \sum_{\mathfrak{c} \subset \mathfrak{m} \in \mathscr{M}(Q)} \delta(\mathfrak{m}) ,$$

in particular, $L(\mathfrak{m}) = \delta(\mathfrak{m})$ for $\mathfrak{m} \in \mathscr{M}(Q)$.

Lemma (Garsia). *In the same notation as above, Q is Cohen-Macaulay over k if and only if there exists a collection $\mathscr{B}(Q)$ of chains of Q which satisfies*

(i) $\sharp\{\mathfrak{c} \in \mathscr{C}(Q);\ r(\mathfrak{c}) = S\} = \sharp\{\mathfrak{b} \in \mathscr{B}(Q);\ r(\mathfrak{b}) \subset S\}$ *for all* $S \subset [d]$ *and*

(ii) *in $k[Q]$, $L(\mathfrak{m})$ can be expressed as a linear combination $\sum_{\mathfrak{b} \in \mathscr{B}(Q)} w_{\mathfrak{b}} L(\mathfrak{b})$ $(w_{\mathfrak{b}} \in k)$ for all $\mathfrak{m} \in \mathscr{M}(Q)$.*

Now, based on Garsia's characterization, we shall give an another proof to b) of Theorem (4.2). Presumably, similar proof would be given to c).

We will work in the situation of b) of Theorem (4.2). Let $T = \{r(x), r(y)\}$. Since both

$$P = \{\xi \in Q;\ \xi < x\} \oplus \{\eta \in Q;\ \eta > y\}$$

and Q are Cohen-Macaulay, we can choose $\mathscr{B}_1 = \mathscr{B}(P)$ and $\mathscr{B}_2 = \mathscr{B}(Q)$ which satisfy the conditions (i) and (ii) in the above lemma.

Let

$$\mathscr{B} = \{\mathfrak{b} \cup \{x, y\};\ \mathfrak{b} \in \mathscr{B}_1\} \cup \mathscr{B}_2 .$$

We shall prove that this collection $\mathscr{B}$ of chains of Q' satisfies Garsia's conditions (i) and (ii) in the above lemma. In the following, the rank set of a chain $\mathfrak{c}$ of P is meant the rank set as a chain of Q.

First, if $S \subset [d]$ and $S \not\supset T$ then

$$\begin{aligned}&\#\{\mathfrak{c}\in\mathscr{C}(Q');\ r(\mathfrak{c})=S\}\\ &=\#\{\mathfrak{c}\in\mathscr{C}(Q);\ r(\mathfrak{c})=S\}\\ &=\#\{\mathfrak{b}\in\mathscr{B}_2;\ r(\mathfrak{b})\subset S\}\\ &=\#\{\mathfrak{b}\in\mathscr{B};\ r(\mathfrak{b})\subset S\}\,.\end{aligned}$$

On the other hand, if $S\subset[d]$ and $S\supset T$ then

$$\begin{aligned}&\#\{\mathfrak{c}\in\mathscr{C}(Q');\ r(\mathfrak{c})=S\}\\ &=\#\{\mathfrak{c}\in\mathscr{C}(Q);\ r(\mathfrak{c})=S\}+\#\{\mathfrak{c}\in\mathscr{C}(P);\ r(\mathfrak{c})=S-T\}\\ &=\#\{\mathfrak{b}\in\mathscr{B}_2;\ r(\mathfrak{b})\subset S\}+\#\{\mathfrak{b}\in\mathscr{B}_1;\ r(\mathfrak{b})\subset S-T\}\\ &=\#\{\mathfrak{b}\in\mathscr{B};\ r(\mathfrak{b})\subset S\}\,.\end{aligned}$$

Secondly, let $\mathfrak{m}\in\mathscr{M}(Q')$. If $\mathfrak{m}\supset\{x,y\}$ then

$$\delta(\mathfrak{m}-\{x,y\})\equiv\sum_{\mathfrak{b}\in\mathscr{B}_1}w_{\mathfrak{b}}\delta(\mathfrak{b})\prod_{i\notin r(\mathfrak{b}\cup\{x,y\})}\theta_i$$

in $k[Q]$ modulo the ideal generated by all $z\in Q-P$, hence

$$\delta(\mathfrak{m}-\{x,y\})-\sum_{\mathfrak{b}\in\mathscr{B}_1}w_{\mathfrak{b}}\delta(\mathfrak{b})\prod_{i\notin r(\mathfrak{b}\cup\{x,y\})}\theta_i=\sum_{\substack{\mathfrak{c}\in\mathscr{C}(Q')\\ \mathfrak{c}\not\subset P\\ r(\mathfrak{c})=[d]-T}}w_{\mathfrak{c}}\delta(\mathfrak{c})\,.$$

Thus, multiplying by xy on both sides, we have

$$\delta(\mathfrak{m})=\sum_{\mathfrak{b}\in\mathscr{B}_1}w_{\mathfrak{b}}\delta(\mathfrak{b}\cup\{x,y\})\prod_{i\notin r(\mathfrak{b}\cup\{x,y\})}\theta_i$$

since $\delta(\mathfrak{c})xy=0$ if $\mathfrak{c}\not\subset P$, in other words,

$$L(\mathfrak{m})=\sum_{\mathfrak{b}\in\mathscr{B}-\mathscr{B}_2}w_{\mathfrak{b}}L(\mathfrak{b})$$

as desired. On the other hand, if $\mathfrak{m}\not\supset\{x,y\}$ then

$$L(\mathfrak{m})\equiv\sum_{\mathfrak{b}\in\mathscr{B}_2}w_{\mathfrak{b}}L(\mathfrak{b})$$

in $k[Q]$ modulo the ideal (xy), hence

$$L(\mathfrak{m})-\sum_{\mathfrak{b}\in\mathscr{B}_2}w_{\mathfrak{b}}L(\mathfrak{b})=\sum_{\substack{\mathfrak{m}'\in\mathscr{M}(Q')\\ \mathfrak{m}'\supset\{x,y\}}}w_{\mathfrak{m}'}L(\mathfrak{m}')\,,$$

and the right-hand side is expressed as $\sum_{\mathfrak{b}'\in\mathscr{B}-\mathscr{B}_2}w_{\mathfrak{b}'}L(\mathfrak{b}')$. Q.E.D.

(4.4) Work in the same notation as in Example (4.1).

Let $\mathscr{P}$ be a certain property for pure posets. Suppose that this property $\mathscr{P}$ satisfies the following: (i) if Q is a pure poset with a unique minimal element α and Q has this property $\mathscr{P}$, then the subposet $Q-\{\alpha\}$

($=\mathrm{link}_Q(\{\alpha\})$) also has this property $\mathscr{P}$, (ii) if Q is a pure poset of rank d having this property $\mathscr{P}$ and Q has at least two minimal elements, then for $i=1$ and some $n, m \geq 1$, $Q_{[\alpha]}$, $Q_{[\beta]}$ have $\mathscr{P}$ and $Q^\wedge$ is pure poset of rank $d-1$ having $\mathscr{P}$, and (iii) any chain has this property $\mathscr{P}$.

Then, by an obvious induction and d) of Theorem (4.2), we see that this property $\mathscr{P}$ implies the Cohen-Macaulay property.

Example. An element y of a poset Q is a *cover* of an element $x \in Q$ if $x < y$ and no element of Q is properly between x and y.

A poset Q is called *wonderful* if the following condition holds in the poset $Q \cup \{-\infty, \infty\}$ obtained by adjoining least and greatest elements to Q: If $y_1, y_2 < z$ are covers of an element x, then there exists an element $y \leq z$ which is a cover of both y_1 and y_2.

Now, [10, Lemma 8.2] shows that every wonderful poset is pure and that $\mathscr{P}$ = "wonderful" satisfies the above conditions (i), (ii) and (iii). Hence every wonderful poset is Cohen-Macaulay. Of course, our argument is an axiomatization of the method contained in [10, P. 42].

(4.5) We shall consider some concrete examples.

Example A. The following posets of Figure 22–24 are all Cohen-Macaulay. Indeed, it is a routine work to check that $Q_i - \{x_i\}$ are wonderful. Hence Q_i are Cohen-Macaulay thanks to a) and b) of Theorem (4.2).

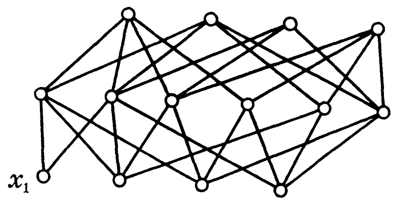
$Q_1 =$ 

Figure 22

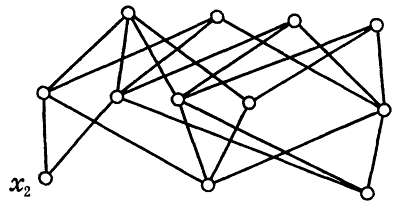
$Q_2 =$ 

Figure 23

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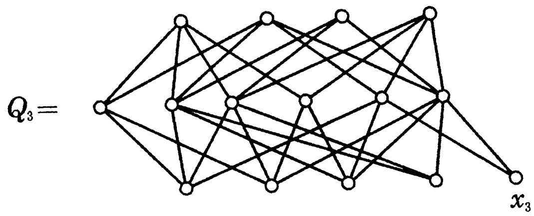


Figure 24

Example B. We shall show that the poset

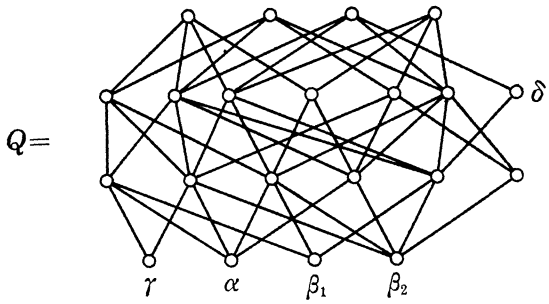


Figure 25

is Cohen-Macaulay. Let $Q' = Q - \{\gamma, \delta\}$. Then, $\mathrm{link}_{Q'}(\{\alpha\})$, $\mathrm{link}_{Q'}(\{\beta_1\})$ and $\mathrm{link}_{Q'}(\{\beta_2\})$ coincide with Q_1, Q_2 and Q_3 in the above Example A, respectively.

Since an intersection of $\mathrm{star}_{Q'}(\{\beta_1\})$ and $\mathrm{star}_{Q'}(\{\beta_2\})$ looks like

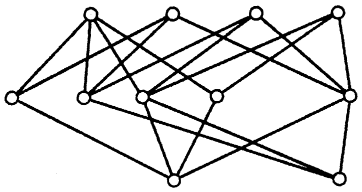
 ,

Figure 26

which is a wonderful poset of rank 3, the glueing $Q'_{[\beta]}$ of $\mathrm{star}_{Q'}(\{\beta_1\})$ and $\mathrm{star}_{Q'}(\{\beta_2\})$ is Cohen-Macaulay. On the other hand, the poset Q_1 of Example A, which is Cohen-Macaulay of rank 3, is an intersection of $Q'_{[\beta]}$ and $\mathrm{star}_{Q'}(\{\alpha\})$, hence the poset Q', which is the glueing of $Q'_{[\beta]}$ and $\mathrm{star}_{Q'}(\{\alpha\})$, is Cohen-Macaulay. So, thanks to a) and b) of Theorem (4.2) again, we see that Q is Cohen-Macaulay.

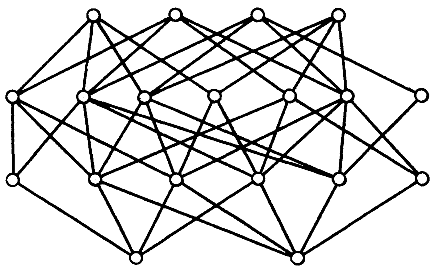
Let Π_n be the partially ordered set of partitions of an integer n ordered by refinement, see Birkhoff [5, I, Example 10, P. 16]. According to Björner [6, Example 6.2], Π_n is wonderful if and only if $n \leq 7$. On the other hand, Π_8 is not wonderful but is shellable, namely, the simplicial complex $\Delta(\Pi_8)$ is shellable. In general, it is proved in [6, Theorem 6.1] that if Q is a wonderful poset then $\Delta(Q)$ is shellable. See also Björner-Wachs [8].

Here, as an application of Theorem (4.2), we shall prove the Cohen-Macaulayness of Π_9 directly.

The poset $Q^{(0)} = \Pi_9 - \{x, y, z\}$, where $x = (\underbrace{1, 1, \cdots, 1}_{9\text{-times}})$, $y = (\underbrace{1, \cdots, 1}_{7\text{-times}}, 2)$, $z = (9)$, is just the poset of Figure 1.

First, we shall prove $Q^{(1)} = Q^{(0)} - \{(1, 1, 1, 1, 1, 1, 3), (1, 1, 1, 1, 1, 2, 2), (1, 1, 1, 1, 1, 4)\}$ is Cohen-Macaulay. Let $\alpha = (1, 1, 1, 1, 2, 3)$ and $\beta = (1, 1, 1, 2, 2, 2)$ of $Q^{(1)}$. Then the poset $\text{link}_{Q^{(1)}}(\{\alpha\})$ is just the poset of Figure 25 of Example B, hence $\text{link}_{Q^{(1)}}(\{\alpha\})$ is Cohen-Macaulay, thus $\text{star}_{Q^{(1)}}(\{\alpha\})$ is Cohen-Macaulay. On the other hand, by the same method as in the case of $\text{star}_{Q^{(1)}}(\{\alpha\})$, we can prove $\text{star}_{Q^{(1)}}(\{\beta\})$ is also Cohen-Macaulay.

Now, an intersection of $\text{star}_{Q^{(1)}}(\{\alpha\})$ and $\text{star}_{Q^{(1)}}(\{\beta\})$ looks like



,

Figure 27

which can be checked to be Cohen-Macaulay by the glueing criterion d) of Theorem (4.2), hence, again thanks to d) of Theorem (4.2), $Q^{(1)}$ is Cohen-Macaulay.

Secondly, apply the extension lemmas a) and b) of Theorem (4.2) to $Q^{(1)}$ and $(1, 1, 1, 1, 1, 4)$, and we see that $Q^{(2)} = Q^{(0)} - \{(1, 1, 1, 1, 1, 1, 3), (1, 1, 1, 1, 1, 2, 2)\}$ is Cohen-Macaulay. Hence $Q^{(3)} = Q^{(0)} - \{(1, 1, 1, 1, 1, 1, 3)\}$ is also Cohen-Macaulay.

Finally, again applying a) and b) of Theorem (4.2) to $Q^{(3)}$ and $(1, 1, 1, 1, 1, 1, 3)$, $Q^{(0)}$ is Cohen-Macaulay, which means Π_9 is Cohen-Macaulay. Q.E.D.

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After this paper was submitted, the author received Björner's letter of May 8, 1986, in which the recent paper Ziegler [33] was referred. According to [33], both Π_9 and Π_{10} are shellable, however, for $n \geq 19$, the posets Π_n are not Cohen-Macaulay.

The author would like to thank Prof. A. Björner for his information about the partition posets Π_n.

(4.6) The final topic of this paper is to consider a question that what conditions of partially ordered sets ensure Buchsbaum posets to be Cohen-Macaulay.

We will propose the following

DEFINITION. A pure poset Q of rank d is called an $L^{(n)}$*-poset*, where n is a positive integer, if for each $i \in [d]$ there exist n elements $\alpha_1^{(i)}, \alpha_2^{(i)}, \cdots, \alpha_n^{(i)}$ of rank i such that (i) $\alpha_1^{(1)}, \alpha_2^{(1)}, \cdots, \alpha_n^{(1)}$ are all the minimal elements of Q and (ii) for each $r \in [d-1]$ and $s > 1$,

$$\left(\bigcup_{j=1}^{s-1} [\alpha_j^{(r)}, \infty)\right) \cap [\alpha_s^{(r)}, \infty) = \bigcup_{j=1}^{s} [\alpha_j^{(r+1)}, \infty),$$

where $[\alpha, \infty) = \{x \in Q; x \geq \alpha\}$ for $\alpha \in Q$.

LEMMA. *Let Q be an $L^{(n)}$-poset. Suppose that the subposets $[\alpha_j^{(1)}, \infty)$ of Q are Cohen-Macaulay for all j, $1 \leq j \leq n$. Then, the subposet $\bigcup_{j=1}^{s} [\alpha_j^{(1)}, \infty)$ of Q is Cohen-Macaulay for every s, $1 \leq s \leq n$. In particular, Q itself is Cohen-Macaulay.*

Proof. We shall prove by induction on d. The case $d = 1$ is trivial. Let $d > 1$. Since $[\alpha_j^{(1)}, \infty)$ is Cohen-Macaulay for every j, $[\alpha_j^{(2)}, \infty)$ is Cohen-Macaulay for every j. Hence, by assumption of induction, the subposets $\bigcup_{j=1}^{s} [\alpha_j^{(2)}, \infty)$ of Q are Cohen-Macaulay for all s. Now, we shall prove each subposet $\bigcup_{j=1}^{s} [\alpha_j^{(1)}, \infty)$ of Q is Cohen-Macaulay by induction on s. Let $s > 1$. Since $\bigcup_{j=1}^{s-1} [\alpha_j^{(1)}, \infty)$ and $[\alpha_s^{(1)}, \infty)$ are Cohen-Macaulay, thanks to the glueing criterion d) of Theorem (4.2), $\bigcup_{j=1}^{s} [\alpha_j^{(1)}, \infty)$ is Cohen-Macaulay, because the subposet

$$\left(\bigcup_{j=1}^{s-1} [\alpha_j^{(1)}, \infty)\right) \cap [\alpha_s^{(1)}, \infty) = \bigcup_{j=1}^{s} [\alpha_j^{(2)}, \infty)$$

of Q, which may be regarded as an intersection of $\bigcup_{j=1}^{s-1} [\alpha_j^{(1)}, \infty)$ and $[\alpha_s^{(1)}, \infty)$, is Cohen-Macaulay of rank $d-1$. Q.E.D.

Now, combining the above lemma with Lemma B of (1.3), we obtain

PROPOSITION. *Let Q be an $L^{(n)}$-poset. Then Q is Cohen-Macaulay if and only if Q is Buchsbaum.*

EXAMPLE. Let Π_n be the partition poset of the integer $n \geq 4$ considered in (4.5). Also, let $\Pi_{(n)} = \Pi_n - \{x_n, y_n, z_n\}$, where

$$x_n = (\underbrace{1, 1, \cdots, 1}_{n\text{-times}}), \quad y = (\underbrace{1, \cdots, 1}_{(n-1)\text{-times}}, 2), \quad z_n = (n).$$

Then, $\Pi_{(n)}$ is an $L^{(2)}$-poset of rank $n-3$. In fact, it is easy to see that

$$\alpha_1^{(i)} = (\underbrace{1, 1, \cdots, 1}_{(n-(i+2))\text{-times}}, i+2)$$
$$\alpha_2^{(i)} = (\underbrace{1, 1, \cdots, 1}_{(n-(i+3))\text{-times}}, 2, i+1),$$

$i \in [n-3]$, satisfy the conditions (i) and (ii) in our definition of $L^{(2)}$-posets.

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