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CONTINUOUS PROGRAMMING CONTAINING ARBITRARY NORMS

S. CHANDRA, B. D. CRAVEN and I. HUSAIN

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Abstract

Optimality conditions and duality results are obtained for a class of cone constrained continuous programming problems having terms with arbitrary norms in the objective and constraint functions. The proofs are based on a Fritz John theorem for constrained optimization in abstract spaces. Duality results for a fractional analogue of such continuous programming problems are indicated and a nondifferentiable mathematical programming duality result, not explicitly reported in the literature, is deduced as a special case.

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1. Introduction

A detailed study of the duality aspects of a class of constrained variational problems has been presented by Mond and Hanson [22]. Recently a number of duality theorems for different forms of continuous programming or control problems have appeared in the literature, notably by Rockafellar [28, 29, 30], Abrham and Buie [1, 2, 3], Reiland [25], Reiland and Hanson [26] and other references cited in these papers.

The present authors in [6] studied duality aspects of a nondifferentiable analogue of the problem treated in [22], the nondifferentiability entering due to the square root of a quadratic form appearing in the integrand of the objective functional. In this paper, optimality conditions and duality results for a more general class of continuous programming problems are given. These continuous programming problems are cone constrained and contain terms with arbitrary norms of linear functions in the objective and constraint functions. The dual problem considered is a modified Wolfe dual rather than a Lagrangian or conjugate convex dual considered by some other authors.

Here the problem is expressed very directly as a mathematical programming problem in function spaces and then a Fritz John theorem [15] for constrained optimization is applied. This approach readily leads to duality and converse duality. In this context it is remarked that optimal control results of Clarke [8, 9] could be used, instead of the Fritz John theorem of [15], leading to similar formulas under somewhat different hypothesis. However, nontrivial questions of representing subdifferentials arise with either approach. Also the present approach could allow some weakening of the convexing hypothesis along the lines of Weir, Hanson and Mond [33] and Bector, Chandra and Husain [5] but this is, however, not pursued here.

The results of this paper also give duals to the (static) mathematical programming problems of Fletcher and Watson [18] (and some of its variants), which have not been reported in the literature explicitly. As a special case of this, we get the duality results of Mond [2], Mond and Schechter [23] and Watson [32].

Finally, a continuous fractional programming problem containing arbitrary norms is also considered and duality results are presented. These duality results generalize some results of Abrham and Buie [2] for the differentiable case and give a dynamic analogue of certain nondifferentiable fractional programming problems considered by Mond [20]. They also give duality results to the fractional analogues of problems considered by Watson [32], Fletcher and Watson [18], which have not been studied explicitly in the literature.

2. Notations and statement of the problems

Let $I = [a, b]$ be a real interval; let $f\colon I \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ and $g\colon I \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^m$ be continuously differentiable functions. In order to consider $f(t, x(t), \dot{x}(t))$, where $x\colon I \to \mathbf{R}^n$ is differentiable with derivative $\dot{x}$, denote the partial derivatives of f by

$$(1) \qquad f_t, f_1 = \left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right], \quad f_2 = \left[\frac{\partial f}{\partial \dot{x}_1}, \dots, \frac{\partial f}{\partial \dot{x}_n}\right].$$

The partial derivatives of g are similarly written, using matrices with m rows instead of one. Let $\mathbf{R}_+ = [0, \infty)$; let $\mathbf{R}^n_+$ be the nonnegative orthant of $\mathbf{R}^n$. Let $C(I, \mathbf{R}^m)$ denote the space of continuous functions $\phi\colon I \to \mathbf{R}^n$, with the uniform norm; let $C_+(I, \mathbf{R}^m)$ denote the cone of non-negative functions in $C(I, \mathbf{R}^m)$. Let X

denote the space of piecewise smooth functions x: $I \to \mathbf{R}^n$, with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operation D is given by

$$(2) \qquad y = Dx \Leftrightarrow x(t) = \alpha + \int_a^t y(s)\, ds,$$

and $x(a) = \alpha$, $x(b) = \beta$ are given boundary values. (Thus $D = d/dt$ except at discontinuities.) Denote by V the space of piecewise smooth functions λ: $I \to \mathbf{R}^m$ (written as column vectors). Superscript T denotes matrix transpose.

For each $t \in I$, let $P(t)$ and $Q_j(t)$ $(j = 1, 2, \ldots, m)$ be $p \times m$ and $q \times n$ matrices respectively, with $P(\cdot)$ and $Q_j(\cdot)$ continuous on I. The norms $\|\cdot\|_A$ and $\|\cdot\|_{B(j)}$ are arbitrary norms in their appropriate spaces; $\|\cdot\|_A^*$ denotes the dual norm to $\|\cdot\|_A$. The generalized Cauchy inequality (Fletcher and Watson [18]) states that $x^T u \leqslant \|x\|\,\|u\|^*$. Following the notation in [14], $\|Q_.(t)x(t)\|_{B.}$ shall denote the vector whose jth component is $\|Q_j(t)x(t)\|_{B(j)}$. By convention, let $w_.(t)^T q_.(t)$ denote the matrix whose jth row is $w_j(t)^T Q_j(t)$, for $j = 1, 2, \ldots, m$.

Let $S \subset \mathbf{R}^m$ be a convex cone, with nonempty interior. For the definitions of S-convex functions and S^*-nearly convex functions, we shall refer to [13, 15]. If K is a convex cone, its dual cone K^* is the set of continuous linear functionals mapping K into $\mathbf{R}_+$.

The primal and dual problems discussed are the following:

$$(3) \qquad (\mathrm{P}): \quad \text{Minimize} \quad \Phi(x) = \int_I \left[f(t, x(t), \dot{x}(t)) + \|P(t)x(t)\|_A \right] dt$$

$$(4) \qquad \text{subject to} \quad x(a) = \alpha, \quad x(b) = \beta,$$

$$(5) \qquad g(t, x(t), \dot{x}(t)) + \|Q_.(t)x(t)\|_{B.} \in -S,$$

$$(\mathrm{D}): \quad \text{Maximize} \quad \Psi(u, z, \lambda, w)$$

$$(6) \qquad = \int_I \Big[f(t, u(t), \dot{u}(t)) + u(t)^T P(t) z(t) + \lambda(t)^T \big[g(t, u(t), \dot{u}(t)) + w_.(t)^T Q_.(t) u(t) \big] \Big] dt$$

$$(7) \qquad \text{subject to} \quad u(a) = 0, \quad u(b) = 0,$$

$$f_1(t, u(t), \dot{u}(t)) + z(t)^T P(t) + \lambda(t)^T \big[g_1(t, u(t), \dot{u}(t)) + w_.(t)^T Q_.(t) \big]$$
$$(8) \qquad - D\big[f_2(t, u(t), \dot{u}(t)) + \lambda(t)^T g_2(t, u(t), \dot{u}(t)) \big] = 0,$$

$$(9) \qquad \|z(t)\|_A^* \leqslant 1, \quad \|w_j(t)\|_{B(j)}^* \leqslant 1 \quad (j = 1, 2, \ldots, m), \lambda(t) \in S^* \; (t \in I).$$

In the following theorems, it will further be assumed that $G(x)(t) := g(t, x(t), \dot{x}(t)) + \|Q_.(t)x(t)\|_{B.}$ is an S^*-nearly convex function of $x \in X$, as defined in [15]. If, in particular, $S = \mathbf{R}_+^m$, then G is the sum of a (Fréchet) differentiable function of x and an S-convex function of x, hence G is S^*-nearly convex as required.

If $P(\cdot) = 0$ and each $Q_j(\cdot) = 0$, then (P) and (D) reduce to the pair of variational problems considered by Mond and Hanson [22]. If $Q_j(\cdot) = 0$ and $\|\cdot\|_A$ is the L^2-norm, the results of [6] are obtained with $B(t) = P(t)^T P(t)$. However, the present method of proving converse duality differs from that of [6]. The present method assumes a solvability hypothesis, rather than using the Fritz John conditions, and thus does not require second order derivatives of f and g, as [6] required.

It is convenient, as in [13, 6], to shift the origin in X to make the boundary conditions $x(a) = 0 = x(b)$; this is assumed in the proof; the original problem is recovered by a shift of origin.

3. Conditions necessary or sufficient for optimality

THEOREM 1 (F. John conditions). *If* (P) *attains a local minimum at* $x = x_0 \in X$, *then there exist Lagrange multipliers* $\tau \in \mathbf{R}_+$ *and piecewise smooth* $\lambda\colon I \to S^*$, $z\colon I \to \mathbf{R}^p$, $w_j\colon I \to \mathbf{R}^{q_j}$ $(j = 1, 2, \ldots, m)$, *with* τ *and* λ *not both zero, such that*

$$\begin{aligned} &\tau\left[f_1(t, x_0(t), \dot{x}_0(t)) + z(t)^T P(t)\right] \\ &\qquad + \lambda(t)^T\left[g_1(t, x_0(t), \dot{x}_0(t)) + w.(t)^T Q.(t)\right]; \end{aligned}$$

$$- D\left[\tau f_2(t, x_0(t), \dot{x}_0(t)) + \lambda(t)^T g_2(t, x_0(t), \dot{x}_0(t))\right] = 0; \tag{10}$$

$$\lambda(t)^T\left[g(t, x_0(t), \dot{x}_0(t)) + w.(t)^T Q.(t) x_0(t)\right] = 0; \tag{11}$$

$$\|z(t)\|_A^* \leqslant 1; \quad \|w_j(t)\|_{B(j)}^* \leqslant 1 \qquad (j = 1, 2, \ldots, m); \tag{12}$$

$$x_0(t)^T P(t) z(t) = \|P(t) x_0(t)\|_A; \tag{13}$$

$$x_0(t)^T Q_j(t) w_j(t) = \|Q_j(t) x_0(t)\|_{B(j)} \qquad (j = 1, 2, \ldots, m); \tag{14}$$

holds for all $t \in I$, *where* $w(t) \equiv \{w_j(t)\colon j = 1, 2, \ldots, m\}$.

Conversely, if (10) *to* (14) *hold with* $\tau = 1$ *and* $x_0(\cdot)$ *feasible for* (P), *and if* $f(t, \cdot, \cdot)$ *is convex and* $g(t, \cdot, \cdot) + \|Q.(\cdot)x(\cdot)\|_{B.}$ *is S-convex for each* $t \in I$, *then* x_0 *minimizes* (P).

PROOF. The problem (P) may be expressed as (EP):

$$\text{Minimize} \quad \Phi(x) = F(x) + J(x) \quad \text{subject to } G(x) \in -K, \tag{15}$$

in which

$$F(x) = \int_I f(t, x(t), \dot{x}(t))\, dt, \quad J(x) = \int_I \|P(t) x(t)\|_A\, dt, \tag{16}$$

(17)
$$G: X \to C(I, \mathbf{R}^m) \text{ is given by } (\forall t \in I, x \in X)$$
$$G(x)(t) = g(t, x(t), \dot{x}(t)) + \|Q_{\cdot}(t)x(t)\|_{B_{\cdot}},$$

and the convex cone $K = \{y \in C(I, \mathbf{R}^m)\colon (\forall t \in I)\, y(t) \in S\}$. From [15, Theorem 3], the Fritz John necessary conditions for (EP) to attain a local minimum at $x = x_0$ are the existence of Lagrange multipliers $\tau \in \mathbf{R}_+$ and $\rho \in K^*$, not both zero, satisfying

(18)
$$0 \in \tau \partial \Phi(x_0) + \partial(\rho G)(x_0); \quad 0 = \rho G(x_0).$$

The cited theorem requires certain convex sets to be weak $*$ compact; this holds for (P). Here $\partial(\rho G)(x_0)$ denotes a subgradient for nearly convex functions (see [15]).

Since $f(t, \cdot, \cdot)$ is continuously differentiable, $F'(x_0)$ is given ([13], page 16) by

(19)
$$(\forall v \in X)\, F'(x_0)v = \int_I \left[f_1(t, x_0(t), \dot{x}_0(t)) v(t) + f_2(t, x_0(t), \dot{x}_0(t)) \dot{v}(t) \right] dt.$$

Assume now, subject to later validation, that $\rho \in K^*$ can be represented by a measurable function $\lambda\colon I \to S^*$ satisfying

(20)
$$(\forall \zeta \in C(I, \mathbf{R}^m)) \quad \rho\zeta = \int_I \lambda(t)^T \zeta(t)\, dt.$$

Let $\xi(t, x) = \|Q_{\cdot}(t)x(t)\|_{B_{\cdot}}$, for $x \in X$, $t \in I$. Then

(21)
$$\rho\xi(\cdot, x) = \int_I \lambda(t)^T \xi(t, x)\, dt.$$

Denote by ∂_C the Clarke generalized subgradient [7] with respect to x. Then

(22)
$$\begin{aligned} \partial_C\left(\lambda(t)^T \xi(t, x)\right) &\subset \sum_{j=1}^m \partial_C\left(\lambda_j(t)\xi_j(t, x)\right) \quad \text{by [7, Proposition 8]} \\ &= \sum_{j=1}^m |\lambda_j(t)| \partial_C\left(\operatorname{sgn}(\lambda_j(t))\xi_j(t, x)\right) \\ &= \sum_{j=1}^m |\lambda_j(t)| \operatorname{sgn}(\lambda_j(t)) \partial_C\left(\xi_j(t, x)\right) \end{aligned}$$

using the representation [7] of $\partial_C(\cdot)$ as the convex hull of limit points of gradients at smooth points near x; Σ denotes here algebraic sum of sets. From Watson [32], since $\xi_j(t, \cdot)$ is convex,

(23)
$$\partial_C \xi_j(t, x) = \partial \xi_j(t, x) = \Big\{ w_j(t)^T Q_j(t)\colon w_j(t) \in \mathbf{R}, \|w_j(t)\|^*_{B(j)} \leqslant 1, \; \xi_j(t, x) = w_j(t)^T Q_j(t) x \Big\}.$$

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From [10], it follows that $q \in \partial(\rho\xi)(\cdot, x)$ if and only if there exists a measurable function $\eta: I \to \mathbf{R}^{m \times n}$ such that

$$(\forall t \in I)\ \eta(t) \in \partial_C \xi(t, x(t)); \qquad (\forall v \in X)\ qv = \int_I \lambda(t)^T \eta(t) v(t)\, dt.$$

Here, from (22) and (23), $\eta(t) = w_.(t)^T Q_.(t)$. Therefore

(24)

$$\partial(\rho G)(x_0)v := \{\theta v : \theta \in \partial(\rho G)(x_0)\}$$
$$\subset \int_I \lambda(t)^T \Big[g_1(t, x_0(t), \dot{x}_0(t)) v(t) + g_2(t, x_0(t), \dot{x}_0(t)) \dot{v}(t) + \big(w_.(t)^T Q_.(t)\big) v(t) \Big]\, dt.$$

Using (18), (19) and (24), necessary conditions for a minimum of (P) at x_0 are that $\tau \geqslant 0$ and $\lambda(\cdot)$ and $w(\cdot)$ exist, satisfying conditions of (23), and also $z: I \to \mathbf{R}^n$ satisfying the similar conditions $\|z(t)\|_A^* \leqslant 1$, $z(t)^T P(t) x_0(t) = \|P(t)x_0(t)\|_A$ such that, for each $v \in X$,

$$\int_I [M(t)v(t) + N(t)\dot{v}(t)]\, dt = 0;$$

$$\int_I \lambda(t)^T \big[g(t, x_0(t), \dot{x}_0(t)) + \|Q_.(t) x_0(t)\|_{B.} \big]\, dt = 0, \tag{25}$$

where

$$M(t) = \tau f_1(t, x_0(t), \dot{x}_0(t)) + z(t)^T P(t) + \lambda(t)^T \big[g_1(t, x_0(t), \dot{x}_0(t)) + w_.(t)^T Q_.(t) \big];$$

$$N(t) = \tau f_2(t, x_0(t), \dot{x}_0(t)) + \lambda(t)^T g_2(t, x_0(t), \dot{x}_0(t)). \tag{26}$$

Integrating the first part of (25) by parts, and using the boundary condition $v(a) = v(b) = 0$, leads to $\int_I [N(t) - H(t)]\dot{v}(t)\, dt = 0$, where H is an indefinite integral of M. This holds whenever $\dot{v}$ is replaced by a piecewise continuous function σ for which $\int_I \sigma(t)\, dt = 0$. Using [11, page 500, Lemma 2] it follows that $N(\cdot) - H(\cdot)$ is constant almost everywhere. (The cited lemma assumes $N - H$ piecewise continuous, but extends readily to measurable.) Hence, for almost all t, $N(t)$ is differentiable, and $\dot{N} = M$. This proves (10) for almost all t. Also (11) follows from the second part of (18), with (14).

In order to validate the representation of ρ by a function $\lambda(\cdot)$, note that the proof leading to (10) and (11) remains valid, without this assumption, if $\lambda(\cdot)$ is considered as a Schwartz distribution. However, (10) and (11) is a first order linear ordinary differential equation system for $\lambda(\cdot)$, given x_0, z and w, and so possesses a piecewise smooth solution $\lambda(\cdot)$. Then z and w are also piecewise smooth, from (10) and (11). Hence $N - H$ is constant for all t.

The sufficient conditions for a minimum follow immediately [13, page 64], since (P) is then a convex problem.

The local minimum x_0 of (P) may be called *normal* if $\tau = 1$, so that the Fritz John conditions (10) to (14) reduce to Kuhn-Tucker conditions. Sufficient conditions for x_0 to be normal are the Slater conditions, either (i) that (P) is a convex problem and, for some $\xi \in X$, $G(\xi) \in -\operatorname{int} S$, or (ii) that, for some $q \in X$, the directional derivative of G at x_0 in direction q satisfies $G'(x_0; q) \in -\operatorname{int} S$. The proof of (i) is standard; for (ii), if $\tau = 0$ then $0 \neq \rho \in S^*$, hence $\rho G'(x_0; q) < 0$, contradicting $\rho G'(x_0; q) = 0$ from (18).

4. Duality and converse duality

Using (15), problem (P) is equivalently written as the problem (EP):

$$\text{Minimize } \Phi(x) \text{ subject to } G(x) \in -K$$

Since $g(t, x(t)) + \|Q(t)x(t)\|_{B.}$ has been assumed to be an S^*-nearly convex function of $x \in X$, G is K^*-nearly convex. Assume now that G is K-convex; this follows if $S = \mathbf{R}_+^m$ and each component $g_i(t, x(t))$ is a convex function of x. Then, using (18), the problem (D) is equivalently written as the problem (ED):

(27) $\quad$ Maximize $\Phi(u) + \rho G(u)$ subject to $\rho \in K^*, 0 \in \partial(\Phi + \rho G)(u)$.

Here $\Psi(u, z, \lambda, w) = \Phi(u) + \rho G(u) \equiv L(u, \rho)$, and $\partial L(u, \rho)$ means $\partial L(\cdot, \rho)$ at u.

THEOREM 2 (Duality). *Let f be convex, and let G be K-convex. If x is feasible for* (P) *and* (u, z, λ, w) *is feasible for* (D), *then*

$$\Phi(x) \geqslant \Psi(u, z, \lambda, w). \tag{28}$$

If equality holds in (28), *then x optimizes* (P) *and* (u, z, λ, w) *optimizes* (D). *If $\bar{x}$ minimizes* (P) *and* $\tau = 1$, *then there exist* $(\bar{z}, \bar{\lambda}, \bar{w})$ *such that* $(\bar{x}, \bar{z}, \bar{\lambda}, \bar{w})$ *maximizes* (D), *and* $\Phi(\bar{x}) = \Psi(\bar{x}, \bar{z}, \bar{\lambda}, \bar{w})$.

PROOF. The proof of (28) is immediate, since (P) is a convex problem. Theorem 1 gives $(\bar{z}, \bar{\lambda}, \bar{w})$ with the stated properties.

Here, as with a differentiable problem, $\tau = 1$ holds under a suitable constraint qualification, such as Slater's.

For proving converse duality, the following solvability hypothesis will be assumed (with $(\bar{u}, \bar{\rho})$ denoting the optimum for (ED)).

(H): Whenever $\bar{\rho} + d \in K^*$ with $\|d\|$ sufficiently small, there exists a solution $u = \bar{u} + \theta(d)$ to $0 \in \partial(\Phi + (\bar{\rho} + d)G)(u)$ for which $\theta(0) = 0$ and $\theta(\cdot)$ is continuous.

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THEOREM 3 (Converse Duality). *Let f be convex, and let G be K-convex. If $(\bar{u}, \bar{\rho})$ maximizes* (ED) *and* (H) *holds then $\bar{u}$ minimizes* (EP), *and $\Phi(u) = L(\bar{u}, \bar{\rho})$.*

PROOF. Since $(\bar{u}, \bar{\rho})$ maximizes (ED), we have

$$[\Phi(\bar{u} + \theta(d)) - (\bar{\rho} + d)G(\bar{u} + \theta(d))] - [\Phi(\bar{u}) + \bar{\rho}G(\bar{u})] \leqslant 0,$$

i.e.,

$$[L(\bar{u} + \theta(d), \bar{\rho}) - L(\bar{u}, \bar{\rho}) + dG(\bar{u} + \theta(d))] \leqslant 0. \tag{29}$$

But $0 \in \partial L(\bar{u}, \bar{\rho})$ is a constraint in (ED) and hence, by definition of subgradient, $L(\bar{u} + \theta(d), \bar{\rho}) - L(\bar{u}, \bar{\rho}) \geqslant 0$. Therefore by (29), $dG(\bar{u} + \theta(d)) \leqslant 0$. This inequality also holds with d replaced by αd for $0 < \alpha \leqslant 1$. Now letting $\alpha \downarrow 0$, we get $\theta(\alpha d) \to 0$ and then using the fact that $0 < \alpha \leqslant 1$, we obtain $dG(\bar{u}) \leqslant 0$. Thus $d \in K^*$ and $G(\bar{u}) \in -K$.

Now setting $d = -\frac{1}{2}\bar{\rho}$, we observe that $\bar{\rho} + \alpha d \in K^*$ for $0 < \alpha \leqslant 1$. Hence $-\frac{1}{2}\bar{\rho}G(\bar{u}) \leqslant 0$, i.e., $\bar{\rho}G(\bar{u}) \geqslant 0$. But $\bar{\rho}G(\bar{u}) \leqslant 0$ as $G(\bar{u}) \in -K$ and $\rho \in K^*$. Therefore $\bar{\rho}G(\bar{u}) = 0$ and consequently $\Phi(\bar{u}) = L(\bar{u}, \bar{\rho})$. The result then follows by Theorem 2.

5. Related problems

As in [22] and [6], these duality results can be extended to the corresponding problem (P1), omitting the boundary conditions $x(a) + \alpha$, $x(b) = \beta$, and (D1) with "natural boundary values". Thus (D1) is the problem (D) together with the additional boundary conditions

$$f_2(t, x(t), \dot{x}(t)) - \lambda(t)^T g_2(t, x(t), \dot{x}(t)) = 0, \tag{30}$$

whenever $t = a$ and $t = b$. The boundary conditions (30) are similar to "natural boundary conditions" in the calculus of variations [12].

6. Certain static cases

If (P1) and (D1) are independent of t, then they reduce to the following nondifferentiable mathematical programming problems:

Primal (P2): Minimize $f(x) + \|Px\|_A$ subject to $g(x) + \|Q.x\|_{B.} \in -S$.

Maximize $f(x) + x^T Pz + \lambda^T\big(g(x) + w_{\cdot}^T Q_{\cdot} x\big)$ subject to

Dual (D2): $\nabla f(x) + z^T P + \lambda^T\left[\nabla g(x) + w_i^T Q_{\cdot}\right] = 0, \quad \|z\|_A^* \leqslant 1, \lambda \in S^*,$

$$\|w_j\|_{B(j)}^* \leqslant 1 \qquad (j = 1, 2, \ldots, m).$$

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Here, it is noted that the primal-dual pair (P2)–(D2) has not been explicitly reported in the literature. In case $Q_j = 0$ for all j and $S = \mathbf{R}^m_+$, (P2) and (D2) reduce to the primal-dual pair studied by Watson [32] which, in turn, includes the problems of Mond [21] and Mond and Schechter [23] as special cases. The pair (P2)–(D2) could also be considered as an extension of Schechter [31] with regard to the converse duality, because in [31], it is given for differentiable constraints only. Also by taking $m = 1$, $S = \mathbf{R}_+$, $g(x) = -\delta$, $(\delta > 0)$ (P2) reduces to the problem of Fletcher and Watson [18], who give optimality conditions only and do not discuss duality.

7. A fractional analogue

Taking h: $I \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$, similar to f, and $E(t)$ an $(n \times l)$ matrix with $E(\cdot)$ continuous for each $t \in I$, the following fractional analogue of problem (P) is considered.

Primal (FP):

$$\text{Minimize} \quad \frac{\int_b^a \left[f(t, x(t), \dot{x}(t)) + \|P(t)x(t)\|A_1 \right] dt}{\int_b^a \left[h(t, x(t), \dot{x}(t)) - \|E(t)x(t)\|_{A_2} \right] dt}$$

$$\text{subject to} \quad x(a) = \alpha, \quad x(b) = \beta$$

$$g(t, x(t), \dot{x}(t)) + \|Q.(t)x(t)\|_{B.} \in -S,$$

where over the feasible region of (FP), the denominator of the objective function is strictly positive and the numerator is non-negative. Also $f(t, \cdot, \cdot)$ and $-h(t, \cdot, \cdot)$ are convex functions for each $t \in I$ and $g(t, \cdot, \cdot) + \|Q.(t)(\cdot)\|_{B.}$ is S-convex. Other symbols have the same meanings as in Section 2.

Using an abstract version of Dinkelbach's [17] result, given by Craven and Mond [16], and following techniques similar to Bector, Chandra and Gulati [4] and Jagannathan [19], the following dual problem (FD) is constructed:

Dual (FD):

$$\text{Maximize} \quad \mu$$

$$\text{subject to} \quad u(a) = \alpha, \quad u(b) = \beta,$$

$$\begin{aligned}
&\left[f_1(t, u(t), \dot{u}(t)) + z(t)^T P(t) - \mu h_1(t, u(t), \dot{u}(t)) + \mu \sigma(t)^T E(t) \right] \\
&\quad + \lambda(t)^T \left[g_1(t, u(t), \dot{u}(t)) + w.(t)^T Q.(t) \right] \\
&\quad - D\left[f_2(t, u(t), \dot{u}(t)) - \mu h_2(t, u(t), \dot{u}(t)) + \lambda(t)^T g_2(t, u(t), \dot{u}(t)) \right] = 0,
\end{aligned}$$

$$\begin{aligned}
&\int_a^b \left[f(t, u(t), \dot{u}(t)) - \mu h(t, u(t), \dot{u}(t)) + u(t)^T P(t) z(t) \right. \\
&\quad \left. - \mu u(t)^T E(t) \sigma(t) + \lambda(t)^T \left(g(t, u(t), \dot{u}(t)) + w.(t)^T Q.(t) u(t) \right) \right] dt = 0,
\end{aligned}$$

$$\|z(t)\|_{A_1}^* \leqslant 1, \quad \|\sigma(t)\|_{A_2}^* \leqslant 1,$$
$$\|w_j(t)\|_{B(j)}^* \leqslant 1 \qquad (j = 1, 2, \ldots, m),$$
$$\lambda(t) \in S^* \; (t \in I), \quad \mu \geqslant 0.$$

The duality results between (FP) and (FD) follow essentially on the lines of Theorems 2 and 3 and certain obvious modifications, similar to [4] and [19]. Also for $P(t) = 0$, $Q_j(t) = 0$, $E(t) = 0$ for all $t \in I$, (FP) and (FD) reduce to certain differentiable fractional continuous programs of [2].

If (FP) and (FD) are independent of t, a fractional analogue of (P2) together with its dual will be obtained. Such fractional problems have not been explicitly studied in the fractional programming literature. As a very special case, if $Q_j = 0$ for all j, $\|\cdot\|_{A_1}$ and $\|\cdot\|_{A_2}$ are L^2-norms and $S = \mathbf{R}_+^m$, the fractional problem studied by Mond [20] is derived.

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Mathematics Department
Indian Institute of Technology
Hauz Khas, New Delhi 110016
India

Mathematics Department
University of Melbourne
Parkville, Victoria 3052
Australia

Mathematics Department
Regional Engineering College
Srinagar, Kashmir
India