Notes on polynomials in schoolwork

By G. Lawson.

1. Teachers tell me that in the final proof of the Remainder Theorem, where lurks the possible fallacy of dividing by $x - h$ which cannot be zero, and then putting x equal to h, they adopt Hyslop's (Math. Notes 25, 1930) redefinition. I here suggest an approach and treatment which I do not find in the textbooks.

Example:—To evaluate $4x^3 + 7x^2 - 3x - 2$ when $x = 5$, multiply 4 by 5 and add 7, 27; multiply 27 by 5 and subtract 3, 132; multiply 132 by 5 and subtract 2, 658, the answer.

Explanation. Pupils may recognise the resemblance to the familiar business of reducing pounds, shillings, etc., to farthings; and this suggests an explanation as follows:—

Just as 1 pound = 20 shillings
1 shilling = 12 pence
1 penny = 4 farthings

so also, when $x = 5$,

$$1x^3 = 5x^2$$
$$1x^2 = 5x$$
$$1x = 5 \text{ units}$$

and evaluation of the polynomial becomes the familiar thing, reduction.

Another explanation, once the laws of algebra are known:—

Multiply a by x and add b, building $ax + b$,
multiply $ax + b$ by x and add c, building $ax^2 + bx + c$,
multiply $ax^2 + bx + c$ by x and add d, building $ax^3 + bx^2 + cx + d$
and so the value of the polynomial may be built up.

2. In this upbuilding of the polynomial we may find the remainder theorem. A preliminary drill is advisable:—

$$mx - 7 = m(x - h) + mh - 7$$
$$(l^2 - mn + 3)x + 5 = (l^2 - mn + 3)(x - h) + l^2h - mnh + 3h + 5.$$

Hence

Degree 1 .... $ax + b = a(x - h) + ah + b$ .... with remainder $ah + b$.

Degree 2 .... multiply both sides of the identity by x and add c

$$ax^2 + bx + c = ax(x - h) + (ah + b)x + c$$
$$= ax(x - h) + (ah + b)(x - h) + ah^2 + bh + c \ldots.$$

with remainder $ah^2 + bh + c$.

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Degree 3. . . .multiply both sides of the identity by x and add d

$$ax^3 + bx^2 + cx + d = ax^2(x-h) + (ah+b)x(x-h) + (ah^2 + bh + c)x + d$$
$$= ax^2(x-h) + (ah+b)x(x-h) + (ah^2+bh+c)(x-h)$$
$$+ ah^3 + bh^2 + ch + d$$

and so on to higher degrees. Observe that at each degree this process shows quotient as well as remainder.

3. One of the simplest ways to the remainder theorem is:—

$$x^4 = (x^4 - x^3h) + (x^3h - x^2h^2) + (x^2h^2 - xh^3) + (xh^3 - h^4) + h^4$$
$$= x^3(x-h) + x^2h(x-h) + xh^2(x-h) + h^3(x-h) + h^4$$
$$ax^4 = ax^3(x-h) + \text{etc.} \qquad\qquad + ah^4$$

therefore ax^4 leaves the remainder ah^4, similarly bx^3 leaves bh^3 and so the general theorem may be proved.

4. Horner's synthetic division of a polynomial is not usually found in schoolbooks. Is the transition from ordinary to synthetic division too troublesome? An old method which might now be christened the quantum-method I have found useful.

Say I have a fund of 379 pounds which I must distribute to as many persons as possible, each to receive exactly 8 pounds. But for some reason or another I can only hand out the money in quanta of 10 pounds. Then each recipient of a quantum must give back 2 pounds to the fund. It is division, not by 8, but by $10 - 2$.

Of the 379 pounds I give 300 in quanta to 30 people, who return 2 pounds each, *i.e.* 60 pounds, to the fund, making it $79 + 60 = 139$ pounds. I now give out 130 pounds in quanta to 13 people, who return 2 each, or a total of 26 to the fund, making it $9 + 26 = 35$ pounds. Of these I give out 30 in quanta to 3 people, who return 6, making the fund $5 + 6$, or 11 pounds. Of these 11 I give out a ten-pounds quantum to 1 person who returns 2 to the fund, making the fund 3 pounds. No quantum remains, no further division is possible. The quotient is $30 + 13 + 3 + 1 = 47$ persons; the remainder 3 pounds.

More generally, suppose the fund in a fund-box is

		$6x^4 +$	$11x^3 +$	$7x^2 +$	$2x +$	4
Returned by each	$+ 2x$		$4x^3$	C		
recipient	$+ 5$			$10x^2$	D	
Quantum is $3x^2$		A	B			
Persons		$2x^2 +$	$5x$	etc.		

Each share is $3x^2 - 2x - 5$, the quantum being $3x^2$; each recipient of a quantum must return $+ 2x + 5$ to the fund-box.

Out of the fund-box move $6x^4$ pounds to A, and there distribute in $3x^2$-quanta to $2x^2$ persons, who each return $2x + 5$ and therefore a total of $4x^3 + 10x^2$ to the box as shown. Next move $15x^2$ out of the box to position B and there give out in $3x^2$ quanta to $5x$ persons who each return $2x + 5$, that is, a total of $10x^2$ at C and $25x$ at D. Next move out $27x^2$ etc. until the fund is reduced to $45x + 49$ and further quantum distribution is impossible. The quotient is $2x^2 + 5x + 9$ people and the remainder $45x + 49$ pounds.

I found, long ago, a pupil calculating without division the remainder of a polynomial divided by $x^2 - 2x + 3$; he was substituting $2x - 3$ for x^2 wherever x^2 occurred in the dividend. The boy's argument was that he was applying the remainder theorem. He was right, but I might have seen that the boy was applying Horner's method. For obviously quantum-division consists in pushing every x^2-quantum out of the fund-box and receiving, for every quantum pushed out, an exchange of $2x - 3$. And have we not here a new way to the remainder theorem? Teach quantum-division, say in two steps as outlined above, emphasising perhaps the exchange for every quantum distributed till the degradation of the fund leaves a remainder with quantum output impossible. Then come to the remainder theorem; the remainder after division by $x - h$ will be the degraded fund after every x-quantum in it has been exchanged for h.

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Elementary methods in the theory of numbers

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Introduction.

§ 1. The importance of proving inequalities of an essentially algebraic nature by "elementary" methods has been emphasised by Hardy

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