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HYPERIDENTITY BASES FOR RECTANGULAR BANDS AND OTHER SEMIGROUP VARIETIES

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Abstract

Hyperidentities were introduced by Taylor, and their properties have been studied by Taylor, Bergman, Penner, Graczynska, Schweigert, Wismath, and others. In particular, Penner has produced bases for the hyperidentities of various types satisfied by the variety of semilattices.

In this paper, we look at bases for the hyperidentities satisfied by some other varieties of semigroups. We first investigate solid and hyper-associative varieties of semigroups, and use a result of Graczynska's to form a basis for the hyperidentities of type $\langle 2 \rangle$ for varieties with these properties. We then produce bases for the hyperidentities satisfied by the variety RB of rectangular bands, and prove that the collection of all hyperidentities satisfied by RB is not finitely based. Finally, we use the RB results to give hyperidentity bases for some varieties of nilpotent semigroups.

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1. Introduction

Let $\mathscr{T}$ be a type of algebras. An identity $P = Q$ of type $\mathscr{T}$ is said to be hypersatisfied by a variety V, not necessarily of the same type, if whenever the operation symbols of P and Q are replaced by terms of V of the appropriate arity, the identity which results holds in V. (Here we use "term" to mean any polynomial expression formed using the operations of V and the variables from P and Q respectively.) In this case, we refer to $P = Q$ as a hyperidentity

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satisfied by V. For example, the type $\langle 2\rangle$ identity $F(x,x) = x$ is satisfied by any variety all of whose binary terms are idempotent.

Hyperidentities have been studied by many authors, including Bergman, Penner, Graczynska, Schweigert, Taylor, and Wismath. In particular, in [6] Penner has given bases for the hyperidentities of each type satisfied by the variety SL of semilattices. In this paper we produce bases for the hyperidentities of some other varieties of semigroups, focusing especially on the variety RB of rectangular bands.

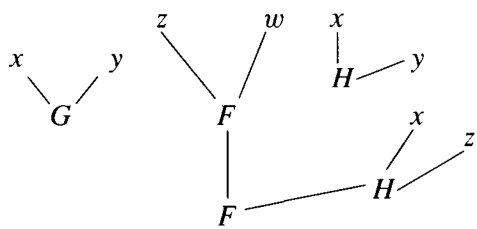
Section 2 considers some special properties of varieties: solidity, which was defined by Graczynska and Schweigert in [4], and the new properties of associativity and basis-solidity. We use a theorem of Graczynska relating solidity and basis-solidity to produce some new examples of solid semigroup varieties.

In Section 3 we focus on some varieties of bands. Using results from [8] we first reduce to looking at the countably infinite chain of closed (self-dual, non-semilattice) varieties of bands. The first non-trivial variety in this chain is the variety RB of rectangular bands. It is particularly important in the study of hyperidentities, since any hyperidentity satisfied by any non-trivial variety must also be satisfied by RB; equivalently, RB is contained in the closure of any non-trivial variety V. The next variety in the chain is NB, the variety of normal bands, which is the closure of the variety SL. It is an easy consequence of the definition of closure that the bases obtained by Penner in [6] for hyperidentities in SL are in fact also bases for the hyperidentities of NB. Penner also notes that a basis for the collection of all hyperidentities satisfied by RB may be formed by adding one hyperidentity to the union of the n-ary bases for $H(SL)$, $n \geq 1$. In Section 3 we produce directly simpler bases for the hyperidentities of various types for RB, and prove that $H(RB)$ is not finitely based.

Section 4 deals with hyperidentity bases for the varieties of k-nilpotent semigroups, for $k \geq 2$. The case $k = 2$ of zero semigroups is dealt with by straightforward modifications to results in Section 3; similar techniques can then be used for $k > 3$.

We conclude this section with some notation and terminology, and some preliminary results to be used in the rest of the paper. We use $H(V)$ for the collection of all hyperidentities (of any type) satisfied by a variety V. A hyperidentity containing k operation symbols each of arity n is said to have type $\underbrace{\langle n, n, \ldots, n\rangle}_{k \text{ factors}}$ and we use $H(V)\langle n\rangle$ and $H(V)\langle n\rangle$ for the hyperidentities of $H(V)$ of type $\langle n\rangle$ and $\underline{n} = \langle n, n, n, \ldots\rangle$ respectively. $H^m(V)$ and $H_n(V)$ will denote the sets of hyperidentities from $H(V)$ with operation symbols of arity at most m, and with at most n variables respectively.

We will call an expression used on one side of a hyperidentity a hyperterm. It will sometimes be useful to consider hyperterms in terms of their formation trees. In particular, if x is a variable of a hyperterm P, we record the sequence of operation symbols and turnings in the path in the tree from the root to the variable x, in the string for x. For example, in the hyperterm shown below the string for the second occurrence of x is recorded as $(F2, F3, H1, x)$. The height of a variable x is the number of operation symbols (not necessarily distinct) listed in its string. The



height of a hyperterm P is the maximum of the heights of the variables in P. For instance, in the previous example, the second occurrence of x has height 3, as does the hyperterm itself.

If V is a variety and Σ a set of hyperidentities, we use $V \models \Sigma$ to mean that V satisfies all the hyperidentities in Σ. For Σ and Γ both sets of hyperidentities, we use $\Sigma \vdash \gamma$ to mean that each hyperidentity in Γ is a consequence of those in Σ.

Most of the varieties considered here will be varieties of semigroups. If α is a finite set of semigroup identities, we use $V(\alpha)$ for the variety of semigroups defined by α. In particular, we will refer throughout to the following varieties:

$$B_{1,1} = V(x(yz) = (xy)z,\ x^2 = x), \qquad \text{the variety of bands.}$$

$$B_{n,m} = V(x(yz) = (xy)z,\ x^n = x^{n+m}), \qquad n, m \geq 1.$$

$$A = V(x(yz) = (xy)z,\ xy = yx), \qquad \text{the variety of abelian semigroups.}$$

$$A_{n,m} = V(x(yz) = (xy)z,\ xy = yx,\ x^n = x^{n+m}), \qquad n, m \geq 1.$$

$$Tr = V(x = y), \qquad \text{the variety of trivial semigroups.}$$

$$RB = V(x(yz) = (xy)z,\ xyz = xz), \qquad \text{the variety of rectangular bands.}$$

$$SL = A_{1,1} = V(x(yz) = (xy)z,\ xy = yx,\ x^2 = x), \qquad \text{the variety of semilattices.}$$

$$NB = V(x(yz) = (xy)z,\ x^2 = x,\ xyzw = xzyw),$$

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the variety of normal bands.

$\mathrm{Reg}B = V(x(yz) = (xy)z,\ x^2 = x,\ xyzx = xyxzx)$,

the variety of regular bands.

$Z = V(xy = zw)$, the variety of zero semigroups.

$N_k = V(x(yz) = (xy)z, x_1 \cdots x_k = y_1 \cdots y_k)$,

the variety of k -nilpotent semigroups, $k \geq 2$.

For any variety V of semigroups, there is a largest variety $\overline{V}$ of semigroups which satisfies exactly the same hyperidentities as V does. We will call V closed if $V = \overline{V}$. This defines a closure opeator on varieties of semigroups; the reader is referred to [8] and [9] for more information about this operator, and for proofs of the following results.

THEOREM 1.1. *For any non-trivial variety V of semigroups, $RB \vee V \subseteq \overline{V}$.*

THEOREM 1.2. *The closed varieties of bands are precisely the self-dual varieties except SL, and $\overline{SL} = NB$. Thus the closed varieties of bands form a countably infinite chain with supremum $B_{1,1}$:*

$$Tr \subsetneqq RB \subsetneqq NB = \overline{SL} \subsetneqq \mathrm{Reg}B \subsetneqq \cdots \subseteq B_{1,1}.$$

THEOREM 1.3. *For each $k \geq 2$, $\overline{N_k} = N_k \vee RB$; for $k \geq 3$, $N_k \vee RB = V((xy)z = x(yz), x_1 \cdots x_k = x_1 y_2 \cdots y_{k-1} x_k)$.*

The following observation is an immediate consequence of the definition of closure:

LEMMA 1.4. *For any non-trivial variety V, any set of hyperidentities which forms a basis for $H(V)$ (of whatever type) is a basis for $H(\overline{V})$, and vice versa.*

2. Solid and hyper-associative varieties

Given any semigroup identity, there is an obvious way to "translate" it into a hyperidentity of type $\langle 2 \rangle$. For example, the identity $x(xy) = (xy)y$ gives the binary hyperidentity $F(x, F(x, y)) = F(F(x, y), y)$. Graczynska and Schweigert in [4] have called a variety V solid if it satisfies the hyperidentity translation of each of its identities. They give one example of a solid variety,

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the variety NB. In this section we define the related concepts of associativity and basis-solidity, and show how a basis for $H(V)\langle 2\rangle$ may be constructed for such semigroup varieties V.

In looking for examples of solid semigroup varieties, we are led naturally to the following definitions.

DEFINITION 2.1. A variety V of semigroups is called (1) *hyper-associative* if V satisfies the associative hyperidentity $F(F(x, y), z) = F(x, F(y, z))$, and (2) *basis-solid* if there is a basis α of semigroup identities for V, including the (usual) associative identity, such that V satisfies all the hyperidentity translations of α.

EXAMPLES AND OBSERVATIONS 2.2. 1. It is easily verified that any variety V which is hyper-associative must be a subvariety of $B_{2,2}$ since it must satisfy $x^2 = x^4$. However $B_{2,2}$ itself is not hyper-associative.

Note also that no abelian variety can be solid, since the hyperidentity translation of $xy = yx$ is not satisfied by any non-trivial variety.

2. The variety $A_{2,2}$ and its subvarieties are hyper-associative; for $n > 2$ and $m > 2$, $A_{n,m}$ is not hyper-associative.

3. The variety $Z = N_2$ of zero semigroups is hyper-associative, but for $k \geq 3$, N_k is not. However, not even Z is solid, since it does not satisfy the hyperidentity translation of $xy = zw$ for instance. Thus hyper-associativity does not imply solidity.

4. Within the lattice of varieties of bands, the only hyper-associative varieties are the 13 subvarieties of the variety Reg$B$ of regular bands. Since there are only 6 possible binary terms to check, it is easy to verify that Reg$B$, and hence its subvarieties, does satisfy the associative hyperidentity. To see that no other variety of bands satisfies this hyperidentity, it suffices to show that the variety

$$V = V(x(yz) = (xy)z,\ x^2 = x,\ xyzwzyx = xyzxzwzxzyx)$$

does not. This is the variety labelled as $\overline{G}_4G_4 = \overline{H}_4H_4$ by Gerhard and Petrich in [2], and using the solution they have given there to the word problem for this variety, we can verify that substitution of the term xyx for F in the associative hyperidentity leads to an identity not satisfied by this variety. (For more detail on the structure of the lattice of varieties of bands, the reader is referred to [2].)

5. Any solid variety must be closed, but the converse is not true, since, for example, B_{11} is closed (see [8]), but by Example 4 above is not even

hyperassociative. The join of any hyper-associative (solid) varieties is hyper-associative (solid). Thus for example, $Z \vee W$ is hyper-associative, for W any of the varieties of bands from Example 4.

It is clear that solidity implies basis-solidity, which in turn implies hyper-associativity. The importance of basis-solidity lies in the following lemma, due to Graczynska, which shows that in fact basis-solidity is equivalent to solidity.

LEMMA 2.3. (Graczynska [3]) *Let V be a variety of type $\mathcal{T}$, and let α be a basis for the identities of V. Let Σ be the set of hyperidentity translation of α. If V satisfies Σ, then V is a solid variety, and moreover Σ forms a basis for the hyperidentities of type $\mathcal{T}$ satisfied by V.*

This lemma gives a method for showing a variety to be solid (other than testing all identity translations), and allows us to produce the first new examples of solid semigroup varieties.

EXAMPLES 2.4. 1. It was shown in [9] that $\overline{A_{2,2}} = A_{2,2} \vee RB$ has as a basis the set α consisting of $(xy)z = x(yz)$, $xyzw = xzyw$ and $x^2 = x^4$, and that $\overline{A_{2,2}}$ satisfies the hyperidentities formed by translating these 3 identities. Thus $\overline{A_{2,2}}$ is solid, and we have a basis of size 3 for $H(\overline{A_{2,2}})\langle 2\rangle$. Similar results from [9] lead to a basis of size 3 for $H(\overline{A_{2,1}})\langle 2\rangle$.

2. Among the 13 hyper-associative varieties of bands, we need only consider the 3 non-trivial closed varieties, RB, $NB = \overline{SL}$, and Reg$B$. In each case, the variety has a basis consisting of $(xy)z = x(yz)$, $x^2 = x$, and one further identity $u = v$. From Example 2.2.4 and [8], we know that the variety satisfies the corresponding hyperidentities in each case, so is solid, with a basis of size 3 for the hyperidentities of type $\langle 2\rangle$. We note that for $NB = \overline{SL}$, this gives the basis obtained by Penner in [6] for $H(SL)\langle 2\rangle$.

3. We have seen in Example 2.2.3 that of the nilpotent varieties N_k, $k \geq 2$, only $N_2 = Z$ is hyper-associative. By Lemma 1.4, we may consider instead hyperidentities for $\overline{Z} = Z \vee RB$, which is also hyper-associative. From [9], $Z \vee RB$ is defined by the 4 identities

$$(xy)z = x(yz), \qquad xyx = x^2, \qquad x^2 = xy, \qquad xy^2 = xy,$$

and satisfies their hyperidentity translations, which hence form a basis for $H(\overline{Z})\langle 2\rangle$.

In Section 4 we will give another basis, of size 2, for $H(\overline{Z})\langle 2\rangle$, along with bases for $H(\overline{Z})\langle \underline{n}\rangle$ and $H(\overline{Z})$. Section 4 also examines $H(\overline{N_k})$ for $k \geq 3$.

4. The varieties $Z \vee NB$ and $Z \vee \text{Reg}B$ are also closed and hyper-associative, and we may use Lemma 2.3 to give bases for their hyperidentities of type $\langle 2 \rangle$. These consist of translations of

$$(xy)z = (xy)z, \qquad x^2 = x^4, \qquad x^2 y = (xy)^2, \qquad xy = (xy)^2$$

for both, plus

$$F(F(x, y), F(z, w)) = F(F(x, z), F(y, w))$$

for $Z \vee NB$, and

$$F(F(x, y), F(z, x)) = F(F(x, F(y, x)), F(z, x))$$

for $Z \vee \text{Reg}B$. (See [8] for details.)

Bases for $H(Z \vee NB)$ for other types are discussed in Section 4.

3. Hyperidentity bases for RB

In [6] and [5], Penner has given bases for $H(SL)\langle n \rangle$, $H(SL)\langle \underline{n} \rangle$ and $H(SL)$, and shown that $H(SL)$ is not finitely based. He also notes that by taking the union for $n \geq 1$ of the bases for $H(SL)\langle \underline{n} \rangle$, plus one additional hyperidentity in $H(RB)$ but not in $H(SL)$, one obtains a basis for $H(RB)$. In this section we produce directly bases for $H(RB)\langle n \rangle$ and $H(RB)\langle \underline{n} \rangle$, and from them a basis for $H(RB)$ which has a simpler form. These results will be used later to find bases for other varieties as well. We also prove that $H(RB)$ is not finitely based.

The key to dealing with hyperidentities for RB is whether or not variables may be reached using only projection terms. We will say that a variable x is *accessible by projection* (abbreviated abp) in a hyperterm P if there is a choice of projection terms $x_i (i \geq 1)$ for the operation symbols in P, such that evaluating P with that choice of terms results in x. Otherwise, x is said to be *not accessible by projection*, or nabp. Note that x is nabp precisely if in the string for x, there is at least one operation symbol F repeated twice with different indices. Penner has said that a hyperidentity $P = Q$ models projections if any choice of projection terms leads to a trivial identity $x_i = x_i$, for some variable x_i; that is, if whenever a variable x is abp in P, the same choice of projection terms leads to the same variable x in Q, and vice versa.

LEMMA 3.1. (Penner [6]) *The variety RB satisfies a hyperidentity $P = Q$ iff $P = Q$ models projections.*

DEFINITION 3.2. For $n \geq 1$, let $\Gamma\langle n\rangle$ consist of the two n-ary hyperidentities

$$F(x, \ldots, x) = x$$

and

$$F(F(x_{11}, \ldots, X_{1n}), \ldots, F(x_{n1}, \ldots, x_{nn})) = F(x_{11}, \ldots, x_{nn}).$$

LEMMA 3.3. $RB \models \Gamma\langle n\rangle$.

PROOF. It is clear that both hyperidentities model projections.

The first of these hyperidentities is called the (n-ary) idempotent hyperidentity. The second essentially says that all variables in a type $\langle n\rangle$ hyperidentity for RB which are nabp may be eliminated. This is perhaps more clearly seen in the equivalent set of $n + 1$ hyperidentities

$$F(x, \ldots, x) = x$$
$$F(f(x_{11}, \ldots, x_{1n}), x_2, \ldots, x_n) = F(x_{11}, x_2, \ldots, x_n)$$
$$\vdots$$
$$F(x_1, \ldots, x_{n-1}, F(x_{n1}, \ldots, x_{nn})) = F(x_1, \ldots, x_{n-1}, x_{nn}).$$

which would also give a basis for $H(RB)\langle n\rangle$. This longer formulation will actually be useful in Section 4 when we look at $H(Z)\langle n\rangle$.

THEOREM 3.4. *For $n \geq 1$, $\Gamma\langle n\rangle$ is a basis of size 2 for $H(RB)\langle n\rangle$.*

PROOF. Let $P = Q$ be any hyperidentity satisfied by RB. If P and Q both consist only of a single variable x_i, then $P = Q$ must be trivial. Thus we will assume that at least one of P or Q involves at least one occurrence of the operation symbol F.

Now there is a unique hyperterm $P^*(Q^*)$ of height 1, such that $\Gamma\langle n\rangle \vdash P = P^*(Q = Q^*)$. For if P involves no occurrences of F, use the idempotent hyperidentity from $\Gamma\langle n\rangle$ to introduce one occurrence of F; if P involves more than one occurrence of F, use the two hyperidentities in $\Gamma\langle n\rangle$ to eliminate all but one occurrence of F.

Since $RB \models P = Q$ and $RB \models \Gamma\langle n\rangle$, we have $RB \models P^* = Q^*$, and $P^* = Q^*$ must model projections. But by definition all variables in $P^* = Q^*$ are abp, so $P^* = Q^*$ must be trivial. Therefore $\Gamma\langle n\rangle \vdash P = Q$, as required.

DEFINITION 3.5. 1. The hyperidentity

$$F(G(x_{11},\ldots,x_{1m}),\ldots,G(x_{n1},\ldots,x_{nm})) \\ = G(F(x_{11},\ldots,x_{n1}),\ldots,F(x_{1m},\ldots,x_{nm}))$$

will be referred to as $M(n,m)$, the medial hyperidentity of type $\langle n,m\rangle$. It is clear that RB satisfies $M(n,m)$ for any $m,n \geq 1$.

2. Let $\Gamma\langle \underline{n}\rangle = \Gamma\langle n\rangle \cup \{M(n,n)\}$.

We will prove that $\Gamma\langle \underline{n}\rangle$ is a basis for $H(RB)\langle \underline{n}\rangle$ by means of two lemmas. The first one deals with the special case of hyperidentities for RB in which all variables are abp, and the second one shows how any hyperidentity may be reduced to one of this special kind using $\Gamma\langle \underline{n}\rangle$.

LEMMA 3.6. *If $P = Q$ is a hyperidentity of type $\langle \underline{n}\rangle$ which models projections and has all variables abp, then*

$$\{F(x,\ldots,x) = x,\ M(n,n)\} \vdash P = Q$$

PROOF. If $P = Q$ has the form $F(-) = x$, the condition that all variables are abp ensures that all variables in $F(-)$ are x's, so that $P = Q$ is a consequence of idempotence. Hence we may now assume that $P = Q$ has the form $F() = G()$. Again by the abp condition, after the first occurrence of F at the root of P, there can be no further F's in P. Moreover, if $F \neq G$, we may assume that every branch in P contains exactly one occurrence of G, since more than one G would lead to variables nabp, while if a branch has no G's we can always inflate the final variable on the branch, x say, into $G(x,\ldots,x)$ using the idempotent hyperidentity.

Using these observations, we proceed by induction on the height of $P = Q$. Any hyperidentity of height 1 meeting these conditions must be trivial, so we begin with height 2. Then $P = Q$ would look like

$$F(H_1(\overline{x}_1),\ldots,H_n(\overline{x}_n)) = G(K_1(\overline{y}_1),\ldots,K_n(\overline{y}_n)),$$

where $\overline{x}_i$ and $\overline{y}_i$, $1 \leq i \leq n$, represent n-tuples of variables. If F and G are the same operation symbol, we substitute for F the n-ary projection terms, to obtain n new hyperidentities $H_i(\overline{x}_i) = K_i(\overline{x}_i)$ of height 1, which must then be trivial. Thus $P = Q$ is trivial in this case. If $F \neq G$, the observations above show that we must have $H_i = G$ and $K_i = F$, for all $1 \leq i \leq n$; that is, $P = Q$ is actually $M(n,n)$.

Now consider $P = Q$ of height $K > 2$. If Q also has the form $F(-)$, we use the n projection terms to reduce to n hyperidentities of height $K - 1$ with the same properties. Then, $P = Q$ is a consequence of these, so by induction, $M(n, n)$ and $F(x, \dots, x) = x$ yield $P = Q$. So we now suppose Q has the form $G(-)$, where $G \neq F$. We give a procedure for forming a new hyperterm P^* from P. As above, every branch of P must contain exactly one occurrence of G. For each such branch, count the number of operation symbols other than G occurring on the path from F to G. Choose any such G where this number is maximal, say p. Now go back along the branch of this G to the previous operation symbol, say H. Each branch coming out of H must contain an occurrence of G, and by maximality of p these occurrences must also be at height p. So this part of P looks like $H(G(-), \dots, G(-))$, and we can use the medial identity to change it to $G(H(-), \dots, H(-))$. In this new identity, we repeat this process, first with any remaining G's at height p, then with G's at lower height. Eventually we reach a new hyperterm P^* of the form $G(-)$, such that $M(n, n) \vdash P = P^*$. Now the hyperidentity $P^* = Q$ still models projections and has all variables abp, and it has the form $G(\ ) = G(\ )$, so by the earlier case it is a consequence of $M(n, n)$ and idempotence. Thus $M(n, n)$ and idempotence yield $P = Q$, as required.

LEMMA 3.7. *For any hyperterm P, there is a hyperterm P^* with no variables nabp, such that $\Gamma\langle \underline{n} \rangle \vdash P = P^*$.*

PROOF. Obviously if P has no variables nabp, we may take P^* to be P. We show how the hyperidentities in $\Gamma\langle \underline{n} \rangle$ may be used to eliminate any variable x nabp in P. For any such variable x, there is an operation symbol F and indices $i \neq j$ $(j \leq n)$ such that the path from the root of P to x involves first F_i, then F_j.

If the 2 occurrence of F are adjacent, then part of P looks like

$$-F(-, F(-, R, -), -)-$$

where the second F occurs in the i-th place of the first F, R is a hyperterm involving x which occurs in the j-th place of the second F, and $-$ indicates other hyperterms in P. We use the idempotent hyperidentity to inflate so that all n entries in the first F have the form $F(-)$, then use the other hyperidentity in $\Gamma\langle n \rangle$ to reduce to

$$-F(-, \dots, -, -, \dots -),$$

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thereby eliminating the nabp variable x.

If the two occurrences of F are separated by one or more other operation symbols, say $G_1, \ldots, G_k$, $k \geq 1$, then P has the form

$$-F(-, \ldots -, G_1(\ldots G_k(-, \ldots F(-, R-)-) \ldots) \ldots) - .$$

Here we again use idempotence to inflate so that the last operation symbol before the second F has all entries of the form $F(-)$; then use $M(n, n)$ to replace the part $G_k(F(-), \ldots, F(-R-), \ldots F(-))$ by $F(G(-), \ldots, G(\,))$. This moves the second occurrence of F one step closer to the first. By repeating this process we eventually reach a stage where the two occurrences of F are adjacent, when the method of the previous paragraph may be used to eliminate x.

In this way all nabp variables in P may be eliminated, giving us P^* as required.

THEOREM 3.8. *$\Gamma\langle \underline{n} \rangle$ forms a basis of size 3 for $H(RB)\langle \underline{n} \rangle$.*

PROOF. Let $P = Q$ be any hyperidentity from $H(RB)\langle \underline{n} \rangle$. By Lemma 3.7, there are hyperterms P^* and Q^*, with no variables nabp, such that $\Gamma\langle \underline{n} \rangle \vdash P = P^*$, $Q = Q^*$. Since $RB \models P = Q$, and $RB \models \Gamma\langle \underline{n} \rangle$, we have $RB \models P^* = Q^*$. By Lemma 3.6, this gives $\Gamma\langle \underline{n} \rangle \vdash P^* = Q^*$. Therefore, $\Gamma\langle \underline{n} \rangle \vdash P = Q$, as required.

The results of Theorem 3.8 may now be extended to deal with hyperidentities for RB of arbitrary type. Merely by adding to our basis all the medial hyperidentities $M(n, m)$, we may carry out the proofs of Lemmas 3.6 and 3.7 in this more general setting. Thus we obtain the following corollary.

COROLLARY 3.9. *Let $\Gamma = \bigcup_{n \geq 1} \Gamma\langle n \rangle \cup \{M(n, m)\ n, m \geq 1\}$. Then Γ is a countably infinite basis for $H(RB)$.*

THEOREM 3.10. *$H(RB)$ is not finitely based.*

PROOF. We will prove that for any 2 positive integers m and n, there is a hyperidentity H such that RB satisfies H, but H is not a consequence of $H^m(RB) \cup H_n(RB)$.

Take $k = \max\{m, n\} + 1$. Define H to be the following hyperidentity, with one k-ary operation symbol F:

(H) $F(F(x_1, x_2, \ldots, x_k), x_1, \ldots, x_1) = F(F(x_1, x_1, x_3, \ldots, x_k), x_1, \ldots, x_1)$

Since H models projections, it is clear that $RB \models H$.

Now define an algebra $\underline{A} = (A; f)$ as follows. Take

$$A = \{a_1, \ldots, a_k, a_1a_2, \ldots, a_ka_{k-1}\},$$

the free rectangular band on the k generators $a_1, \ldots, a_k$. f is k-ary, given by

$$f(x_1, \ldots, x_k) = \begin{cases} x_1x_2, & \text{if } \{x_1, \ldots x_k\} = \{a_1, \ldots, a_k\} \\ x_1, & \text{otherwise.} \end{cases}$$

Using f for the operation symbol in H leads to an identity which does not hold in $\underline{A}$, since the valuation $x_i = a_i$, $1 \leq i \leq k$, in the identity leads to

$$a_1a_2 = a_1.$$

Therefore $\underline{A}$ does not satisfy H. However, we claim that $\underline{A}$ does satisfy all the hyperidentities in $H^m(RB) \cup H_n(RB)$.

For if a hyperidentity involves at most n variables, or operation symbols all of arity $\leq m$, since $k > m, n$ it follows that the only $\underline{A}$-terms used in the hyperidentity amount to projections. So in this case $\underline{A}$ satisfies the hyperidentity iff RB does.

Thus we see that H is a hyperidentity satisfied by RB, which is not a consequence of $H^m(RB) \cup H_n(RB)$. Therefore $H(RB)$ is not finitely based.

As noted in Examples 2.4.2, there are only three closed and hyper-associa tive varieties of bands: RB, NB, and Reg$B$. Our results here, and Penner's results for SL (which extend to $\overline{SL} = NB$ by Lemma 1.4) give a complete picture of the hyperidentities for the first two of these three. For Reg$B$, however, we have only the basis for $H(\text{Reg}B)\langle 2\rangle$ obtained in Example 2.4.2.

OPEN PROBLEM 3.11. *Find bases for* $H(\text{Reg}B)\langle n\rangle$, $H(\text{Reg}B)\langle\underline{n}\rangle$, *and* $H(\text{Reg } B)$.

4. Hyperidentity bases for nilpotent varieties

In this section we extend the results of the previous section to deal with hyperidentities for the nilpotent varieties N_k, $k \geq 2$. We saw in Example 2.4.3 a basis of size four for $H(\overline{N}_2)\langle 2\rangle$; a slight modification of Theorems 3.4 and 3.8

now gives a basis of size n for $H(\overline{N}_2)\langle n\rangle$, and one of size $n+2$ for $H(\overline{N}_2)\langle \underline{n}\rangle$. Similar results may be obtained for $H(N_2 \vee NB)$ by modifying proofs from [6]. We also show that $H(N_2 \vee RB)$ is not finitely based. Finally, we examine hyperidentities for the varieties $\overline{N}_k = N_k \vee RB$, for $k \geq 3$.

We begin by looking at hyperidentities satisfied by the variety $N_2 = Z$ of zero semigroups, or equivalently by its closure $\overline{Z} = Z \vee RB$. Z has only $n+1$ terms of arity n, for $n \geq 1$: $x_1, \ldots, x_n$, and x_1^2; and Z satisfies any identity $u = v$ where u and v both have length greater than or equal to 2. From this it is clear that Z satisfies all the hyperidentities in $H(RB)$ except those of the form which equate a single variable to a hyperterm involving at least one operation symbol. In particular, Z satisfies the elimination hyperidentity in the basis $\Gamma\langle n\rangle$, but not the idempotent one, $F(x, \ldots, x) = x$. Examination of the proof of Theorem 3.4 (that $\Gamma\langle n\rangle$ is a basis for $H(RB)\langle n\rangle$) reveals that the idempotent hyperidentity is used there in two ways. The first is to deal with hyperidentities $P = Q$ of the form $F(-) = x$, which we need no longer be concerned with for Z. The second use of idempotence is to inflate hyperterms such as

$$F(F(x_1, \ldots, x_n), y_2, \ldots, y_n)$$

to

$$F(F(x_1, \ldots, x_n), F(y_2, \ldots, y_2), \ldots F(y_n, \ldots, y_n))$$

in order to use the other hyperidentity to eliminate the variables nabp. However, we may do away with this need for idempotence for Z by replacing the one elimination hyperidentity by the following:

$$\begin{aligned} F(F(x_1, \ldots, x_n), y_2, \ldots, y_n) &= F(x_1, y_2, \ldots, y_n) \\ F(y_1, F(x_1, \ldots, x_n), y_3, \ldots, y_n) &= F(y_1, x_2, y_3, \ldots, y_n) \\ F(y_1, \ldots, y_{n-1}, F(x_1, \ldots, x_n)) &= F(y_1, \ldots, y_{n-1}, x_n). \end{aligned}$$

Let us call the set containing these n hyperidentities $\Sigma\langle n\rangle$. By repeating the proof of Theorem 3.4 for Z instead of RB with modifications just described, we prove the following:

THEOREM 4.1. *For $n \geq 1$, $\Sigma\langle n\rangle$ is a basis of size n for $H(Z)\langle n\rangle$.*

Similarly we may modify the hyperidentities and proof given for $H(RB)\langle \underline{n}\rangle$. We still need the medial hyperidentity $M(n, n)$; we must delete the idempotent hyperidentity; and we must modify the one elimination hyperidentity so that we can still eliminate all nabp variables without having to invoke idempotency. Define $\Sigma\langle \underline{n}\rangle$ to be the set

$$\{M(n,n)\} \cup \Sigma\langle n\rangle \bigcup \{F(x_1,\ldots,x_n) \\ = F(G_1(x_1,\ldots,x_1),\ldots,G_n(x_n,\ldots,x_n))\}.$$

Note that the last new hyperidentity used here essentially allows the use of idempotency, as long as the context of its use guarantees words of length ≥ 2. This is sufficient to allow us to carry over the proofs of Lemmas 3.6 and 3.7, and Theorem 3.8, with appropriate changes, since any use we need to make in them of idempotence for the case of Z is within this special context. This gives

THEOREM 4.2. *For $n \geq 1$, $\Sigma\langle \underline{n}\rangle$ is a basis of size $n+2$ for $H(Z)\langle \underline{n}\rangle$.*

Finally, the proof of Theorem 3.9 also carries over to deal with Z. The hyperidentity (H) given in that proof is satisfied by Z but not by the algebra $\underline{A}$ given there. We saw that for any $n, m < k$, that algebra $\underline{A}$ satisfied any hyperidentity in $H^m(RB) \cup H_n(RB)$, which includes $H^m(Z) \cup H_n(Z)$. Thus we have also proved

THEOREM 4.3. *$H(Z)$ is not finitely based.*

We note that similar modifications may be made to Penner's proofs in [6], where a basis is given for $H(Sl)\langle n\rangle$ and $H(SL)\langle \underline{n}\rangle$, to obtain bases for $H(NB \vee Z)\langle n\rangle$ and $H(NB \vee Z)\langle \underline{n}\rangle$.

We next consider hyperidentities for the nilpotent varieties N_k, and their closures $N_k \vee RB$, for $k \geq 3$. We will show how to produce a basis for $H(\overline{N}_k)\langle \underline{n}\rangle$, for $k \geq 3$ and $n \geq 2$. Since the hyperidentities in the arbitrary case are rather cumbersome, we will illustrate with the situation for $H(\overline{N}_3)\langle 2\rangle$, then discuss the generalization.

DEFINITION 4.4. let $\Delta_3\langle 2\rangle$ be the set of hyperidentities:

$$\begin{aligned}
F(F(x,y),z &= F(F(x,w),z) && (\Delta_1)\\
F(x,F(y,z)) &= F(x,F(w,z)) && (\Delta_2)\\
F(F(F(x,y),z),w) &= F(F(x,z),w) && (\Delta_3)\\
F(x,F(y,F(z,w))) &= F(x,F(y,w)) && (\Delta_4)\\
F(F(x,F(y,z),w &= F(F(x,y),w) && (\Delta_5)\\
\text{and} \quad f(x,F(F(y,z),w)) &= F(x,F(y,w)). && (\Delta_6)
\end{aligned}$$

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Before proving that $\Delta_3\langle 2\rangle$ is a basis for $H(\overline{N}_3)\langle 2\rangle$, we discuss what these hyperidentities "mean". Recall that the hyperidentities used in the basis for $H(RB)\langle n\rangle$ essentially encoded the fact that variables nabp could always be eliminated, since they were arbitrary. Here, however, because N_3 imposes a length restriction on the hyperidentities, variables nabp cannot always be eliminated, although they are always still arbitrary. Thus hyperidentities Δ_1 and Δ_2 above encode the fact that in "short" hyperidentities (height 2), variables nabp are arbitrary, while $\Delta_3 - \Delta_6$ show that any variable which has height 3 and is nabp can be eliminated, decreasing the height of its hyperterm by 1. (We note that there are actually $6 = 2^3 - 2$ ways for a variable to be nabp and at height 3. If we use R and L to describe right left turnings in the hypertree, these 6 ways can be described as RRL, LLR, RLR, LRL, RLL, and LRR. However, 2 of the corresponding hyperidentities can be easily deduced from the others, so only 4 such hyperidentities are included in $\Delta_3\langle 2\rangle$.)

THEOREM 4.5. *$\Delta_3\langle 2\rangle$ forms a basis for $H(\overline{N}_3)\langle 2\rangle$.*

PROOF. Let $P = Q$ be any hyperidentity of type $\langle 2\rangle$ satisfied by $\overline{N}_3 = N_3 \vee RB$. We will assume that $\;= $ length of $P \geq$ length of Q.

If $\;= 1$, it is clear that length of Q must also be 1, and that $P = Q$ must be trivial. If $\;= 2$, P must be one of $F(F(x, y), z)$, $F(y, F(z, x))$, or $F(F(x, y), F(z, w))$. In any case, the variable x (or w) must also appear in Q at height 2, since otherwise using the term x^2 for F would lead to the identity $x^4 = x^2$, which doesn't hold for N_3. This restriction means that P and Q must have the same shape, and then we have $\{\Delta_1, \Delta_2\} \vdash P = Q$.

For any height $\;\geq 3$, P must have variables nabp at height ≥ 3. We then use $\Delta_3 - \Delta_6$ to eliminate all such variables, producing a new hyperterm P^* such that $\Delta_3\langle 2\rangle \vdash P = P^*$, and P^* has height 2. Similarly, reduce Q to Q^*. Now $N_3 \vee RB \models P^* = Q^*$, so by the case $\;= 2$, $\Delta_3\langle 2\rangle \vdash P^* = Q^*$. This gives $\Delta_3\langle 2\rangle \vdash P = Q$, as required.

It is now clear how the technique used here may be extended. For $H(N_3 \vee RB)\langle n\rangle$, $n \geq 3$, we need n hyperidentities saying that variables nabp at height 2 are arbitrary; plus (at most) $n^3 - n$ hyperidentities saying that an nabp variable at height 3 may be eliminated, down to height 2. (As before, some of these hyperidentities will not be needed for the basis.)

For values of $k \geq 4$, the situation is only a little more complicated. For each such N_k, there is a minimum height t at which one may begin eliminating nabp variables. (For $k = 3$ above, for instance, we had $t = 3$.) Hence a basis needs a finite number of hyperidentities describing how variables nabp are arbitrary at heights $< t$, plus a finite number more which describe how to reduce from height t to height $t-1$, by eliminating one nabp variable at height t. This proves the following:

THEOREM 4.6. *There is a finite basis for* $H(\overline{N}_k)\langle 2\rangle$, *for any* $n \geq 2$ *and any* $k \geq 2$.

Unfortunately this approach does not seem to extend to hyperidentities in $H(N_k)\langle \underline{n}\rangle$, with two or more operation symbols. It appears that the situation there is more complicated. For instance, the variable z in the hyperterm P,

$$P = F(G(F(H(x, y), z), w), t)$$

is nabp and at height 4, yet N_3 does not satisfy the elimination hyperidentity

$$P = F(G(H(x, y), w), t).$$

In fact, for any given height k, one can construct a hyperidentity of type $\langle 2, 2, \dots, 2\rangle$ which expresses that a variable nabp at height k can be eliminated, yet which is not satisfied by N_3. Moreover, once $k \geq 5$, N_k does not even satisfy the medial hyperidentity.

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References

[1] G. M. Bergman, 'Hyperidentities of groups and semigroups', *Aequationes Math.* **23** (1981), 50–65.

[2] A. Gerhard and M. Petrich, 'Varieties of bands revisited', *Proc. London Math. Soc.* **58** (1989), 323–350.

[3] E. Graczynska, *On normal and regular identities and hyperidentities* (World Scientific, Singapore, 1989) pp. 107–135.
[4] E. Graczynska and D. Schweigert, 'Hypervarieties of a given type', *Algebra Universalis* **27** (1990), 305–318.
[5] P. Penner, 'Hyperidentities of lattices and semilattices', *Algebra Universalis* **13** (1981), 307–314.
[6] ——, 'Hyperidentities of semilattices', *Houston J. Math.* **10** (1984), 81–108.
[7] W. Taylor, 'Hyperidentities and hypervarieties', *Aequationes Math.* **23** (1981), 30–49.
[8] S. Wismath, 'Hyperidentities for some varieties of semigroups', *Algebra Universalis* **27** (1990), 111–127.
[9] ——, 'Hyperidentities for some varieties of commutative semigroups', *Algebra Universalis* **28** (1991), 245–273.

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