J. Austral. Math. Soc. Ser. B 28 (1986), 246–259

THE LONG JUMP RECORD REVISITED

NEVILLE DE MESTRE[1]

(Received 27 September 1985; revised 20 December 1985)

Abstract

A mathematical model is presented in which the long jump is treated as the motion of a projectile under gravity with slight drag. The first two terms of a perturbation solution are obtained and are shown to be more accurate than earlier approximate analytical solutions. Results from the perturbation analysis are just as accurate as results from various numerical schemes, and require far less computer time.

The model is modified to include the observation that a long-jumper's centre of mass is forward of his feet at take-off and behind his feet on landing.

The modified model is used to determine the take-off angle for the current world long jump record, resulting in several interesting observations for athletic coaches.

1. Introduction

Every person interested in athletics in general, and field events in particular, knows that the long jump record has stood since 1968. In the Mexico City Olympics of that year Bob Beamon of the U.S.A. raised the record from 8.35 metres to 8.90 metres at his one and only attempt. This remarkable achievement has been the subject of a number of mathematical analyses. Brearley [1] and Frohlich [3] used a series of approximations to solve the governing equations. Burghes, Huntley and McDonald [2] solved the original equations numerically and also produced a slightly different approximate theoretical analaysis which was later repeated by Ward-Smith [7]. They conclude that the difference between the numerical and theoretical solutions cannot be detected when the corresponding trajectories are drawn.

However when approximations are made to approximate equations there are invariably some contributions from the omitted parts of the original equations that should also have been included. This is the case in all the theoretical

[1] Mathematics Department, University College, Australian Defence Force Academy

https://doi.org/10.1017/S0334270000005348 Published online by Cambridge University Press

considerations mentioned above. It is my aim to produce a theoretical solution which is correct to a specified order, and compare the values that can be computed from it with those from the previous analyses.

2. Long jump model

The long jump athlete can be regarded as a point projectile travelling under the influence of constant gravity and variable air drag. His initial velocity is assumed to be known since it can usually be reasonably deduced from known characteristics of his run-up and take-off. Therefore the essential equations governing the athlete's jump are

$$m\ddot{\bar{\mathbf{r}}} = m\mathbf{g} - \frac{1}{2}\rho A\bar{v}^2 C_D \hat{\mathbf{v}} \tag{1}$$

where m is his mass, $\bar{\mathbf{r}}$ is his position vector at any time $\bar{t}$ after take-off, $\mathbf{g}$ is the acceleration due to gravity, ρ is the density of the air at the athletics field, A is the cross-sectional area of the athlete in a plane normal to his velocity, C_D is the drag coefficient, $\bar{\mathbf{v}}(=\dot{\bar{\mathbf{r}}})$ is his velocity, $\hat{\mathbf{v}}$ denotes the unit vector in the direction of $\bar{\mathbf{v}}$ and a dot denotes differentiation with respect to $\bar{t}$.

The take-off conditions are $\bar{\mathbf{v}} = (V\cos\alpha, V\sin\alpha)$ and $\bar{\mathbf{r}} = \mathbf{0}$ when $\bar{t} = 0$.

For an athlete the density and the combination $\frac{1}{2}AC_D$ can be taken as constants during the long jump.

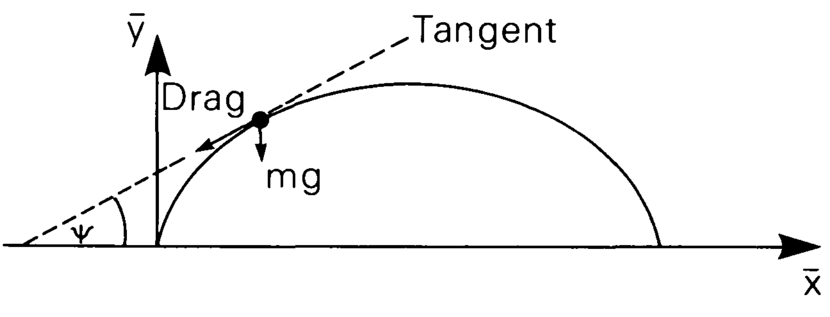

Figure 1. The forces on a long jumper.

When equation (1) is resolved into its tangential and normal components, the governing equations for the athlete's motion become

$$m\dot{\bar{v}} = -mg\sin\psi - \frac{1}{2}\rho A C_D \bar{v}^2 \tag{2}$$

and

$$m\bar{v}\dot{\psi} = -mg\cos\psi \tag{3}$$

https://doi.org/10.1017/S0334270000005348 Published online by Cambridge University Press

with $\bar{v} = V$, $\psi = \alpha$, $\bar{x} = 0$, $\bar{y} = 0$ when $\bar{t} = 0$. Here ψ is the angle made by the tangent with the positive direction of the $\bar{x}$-axis. The $\bar{x}$-axis is horizontal and the $\bar{y}$-axis is vertical as shown in Figure 1. The position of the athlete at any time $\bar{t}$ is $(\bar{x}, \bar{y})$ and can be obtained from the solution of equations (2) and (3) using $\dot{\bar{x}} = \bar{v} \cos\psi$ and $\dot{\bar{y}} = \bar{v}\sin\psi$.

Now equation (3) can be multiplied by $\cos\psi$ and rewritten as

$$m\bar{v}\, d(\sin\psi)/d\bar{t} = mg(\sin^2\psi - 1). \tag{4}$$

The equations (2) and (4) and the initial conditions are nondimensionalised by writing

$$\bar{v} = Vv,\ \bar{t} = Vt/g,\ \bar{x} = V^2x/g,\ \bar{y} = V^2y/g.$$

With $\Psi = \sin\psi$, the non-dimensional equations for an athlete's long jump become

$$v' = -\Psi - \varepsilon v^2 \tag{5}$$

$$v\Psi' = \Psi^2 - 1 \tag{6}$$

where $\varepsilon = C_D\rho A V^2/(2mg)$, a dash denotes differentiation with respect to t, and

$$v = 1,\ \Psi = \sin\alpha,\ x = 0,\ y = 0 \text{ when } t = 0. \tag{7}$$

Here ε measures the importance of the drag force contribution in comparison with the gravity effect. For an athlete performing the long jump the value of ε is small and certainly much less than unity.

3. Solution of the long jump equations

Although the initial-value problem defined by equations (5)–(7) is well-posed and has a unique solution, no technique is known which produces this solution exactly. However a number of techniques can be employed to produce various approximations to the exact solution.

Method 1: Since the system is non-linear the most direct method is to use a fourth-order Runge-Kutta standard library package and solve this initial-value problem numerically for v and Ψ. Then x and y can be determined from the numerical solution of the non-dimensional equations

$$x' = v(1 - \Psi^2)^{1/2},$$
$$y' = v\Psi,$$

with $x = y = 0$ when $t = 0$.

It is just as easy numerically to solve equation (1) in its Cartesian components

$$\left.\begin{aligned}\ddot{\bar{x}} &= -K\dot{\bar{x}}(\dot{\bar{x}}^2 + \dot{\bar{y}}^2)^{1/2}\\ \ddot{\bar{y}} &= -g - K\dot{\bar{y}}(\dot{\bar{x}}^2 + \dot{\bar{y}}^2)^{1/2}\end{aligned}\right\} \tag{8}$$

with $\bar{x} = 0$, $\bar{y} = 0$, $\dot{\bar{x}} = V\cos\alpha$, $\dot{\bar{y}} = V\sin\alpha$ when $\bar{t} = 0$, and where $K = C_D A\rho/(2m)$. With $\xi = \dot{\bar{x}}$ and $\eta = \dot{\bar{y}}$ the IMSL package DVERK applied to the two sets of ordinary differential equations for (ξ, η) and $(\bar{x}, \bar{y})$ will produce values for these variables. When the $\bar{y}$-value reaches zero again, the corresponding $\bar{x}$-value will give the range on the horizontal plane through the projection point.

Method 2: Although the equations (5) and (6) are non-linear, partial progress can be made with an analytical solution for any values of the parameter ε. Equation (5) can be divided by equation (6) and the resulting Bernoulli-type differential equation solved to yield

$$v(\Psi) = \Bigg[(1-\Psi^2)\sec^2\alpha + \varepsilon\left\{\left(\sin\alpha\sec^2\alpha + \frac{1}{2}\ln\left|\frac{1+\sin\alpha}{1-\sin\alpha}\right| - \frac{1}{2}\ln\left|\frac{1+\Psi}{1-\Psi}\right|\right)(1-\Psi^2) - \Psi\right\}\Bigg]^{-1/2}. \tag{9}$$

Although x, y and t can be written in closed form as integrals containing v and Ψ, they can only be evaluated numerically because of the complicated form of equation (9). Also x or y cannot be calculated directly for a given t, as Ψ has to be determined first of all by direct numerical integration.

The integrals for x, y and t are

$$t = \int_{\Psi}^{\sin\alpha} \frac{v(\phi)}{1-\phi^2}\, d\phi, \tag{10}$$

$$x = \int_{\Psi}^{\sin\alpha} \frac{v^2(\phi)}{(1-\phi^2)^{1/2}}\, d\phi, \tag{11}$$

$$y = \int_{\Psi}^{\sin\alpha} \frac{\phi v^2(\phi)}{1-\phi^2}\, d\phi, \tag{12}$$

where $v(\phi)$ is given by equation (9). This method has not been previously applied to the long jump although it is commonly used for mortar projectiles. The step size for accurate numerical integration is not as fine as that required for the Runge-Kutta approach in Method 1, and therefore there is a considerable saving in computing time by using Method 2.

Method 3: This approximate method was applied to long jumpers by Burghes et. al. [2] and Ward-Smith [7], but was originally developed by Lamb [6]. The Cartesian form given by equations (8) is approximated by

$$\ddot{\bar{x}} = -K\dot{\bar{x}}^2 \tag{13}$$

$$\ddot{\bar{y}} = -g - K\dot{\bar{x}}\dot{\bar{y}} \tag{14}$$

where it has been assumed that $\dot{\bar{x}}^2$ is much greater than $\dot{\bar{y}}^2$.

The solution of equation (13) is

$$\bar{x} = \frac{1}{K}\ln(1 + KV\bar{t}\cos\alpha).$$

When this is substituted into equation (14) the resulting differential equation can be solved to yield

$$\bar{y} = \frac{g\sec^2\alpha}{V^2K^2}\left[\left(\frac{1}{2} + \frac{KV^2\sin 2\alpha}{2g}\right)\ln(1 + KV\bar{t}\cos\alpha) - \frac{KV\bar{t}\cos\alpha}{4}(2 + KV\bar{t}\cos\alpha)\right],$$

and hence

$$\bar{y} = \frac{g\sec^2\alpha}{K^2V^2}\left[\left(\frac{1}{2} + \frac{KV^2\sin 2\alpha}{2g}\right)K\bar{x} - \frac{1}{4}(e^{2K\bar{x}} - 1)\right]. \tag{15}$$

The range on the horizontal plane is given by the solution of equation (15) with $\bar{y} = 0$. Although this is a transcendental equation it can be solved to any order of accuracy by the Newton-Raphson algorithm. However for $KV^2 \ll g$ the result is

$$\bar{x} \approx (V^2\sin 2\alpha)/g - (2KV^4\sin^2 2\alpha)/3g^2.$$

Method 4: Brearley [1] uses a similar approach but makes a further approximation by replacing equation (14) by

$$\ddot{\bar{y}} = -g. \tag{16}$$

He therefore obtains the time of flight as $2V\sin\alpha/g$. He states that numerical checks shows that there is very little difference in using equation (14) or equation (16) for this problem.

The range on the horizontal plane is therefore given by

$$\bar{x} = \frac{1}{K}\ln\left(1 + \frac{KV^2\sin 2\alpha}{g}\right).$$

He makes a further approximation by expanding the logarithmic function in terms of its Maclaurin Series, since $KV^2 \ll g$, and considers only the first two terms. His final approximate expression for the range is

$$\bar{x} \approx (V^2\sin 2\alpha)/g - (KV^4\sin^2 2\alpha)/2g^2,$$

which differs from the result in Method 3 only in the numerical coefficient of the second term.

Method 5: Since $\varepsilon \ll 1$ for the long jump trajectory a perturbation procedure can be used. With

$$\left.\begin{aligned} v(t;\,\varepsilon) &= v_0(t) + \varepsilon v_1(t) + 0(\varepsilon^2) \\ \text{and}\qquad \Psi(t;\,\varepsilon) &= \Psi_0(t) + \varepsilon\Psi_1(t) + 0(\varepsilon^2) \end{aligned}\right\} \tag{17}$$

https://doi.org/10.1017/S0334270000005348 Published online by Cambridge University Press

substituted into equations (5), (6) and (7), the equating of terms of like powers of ε yields

$$v_0' = -\Psi_0, \tag{18}$$

$$v_0\Psi_0' = \Psi_0^2 - 1, \tag{19}$$

$$v_1' = -\Psi_1 - v_0^2, \tag{20}$$

$$v_0\Psi_1' + v_1\Psi_0' = 2\Psi_0\Psi_1, \tag{21}$$

with initial conditions

$$v_0 = 1,\ \Psi_0 = \sin\alpha,\ v_1 = 0,\ \Psi_1 = 0 \text{ when } t = 0. \tag{22}$$

When Ψ_0 is eliminated from equations (18) and (19) the differential equation for v_0 becomes

$$v_0v_0'' = 1 - (v_0')^2.$$

This has a solution

$$v_0 = \left\{(t - \sin\alpha)^2 + \cos^2\alpha\right\}^{1/2} \tag{23}$$

since $v_0 > 0$ and $v_0(0) = 1$. Therefore from equation (18)

$$\Psi_0 = \frac{-(t - \sin\alpha)}{\left\{(t - \sin\alpha)^2 + \cos^2\alpha\right\}^{1/2}}. \tag{24}$$

When Ψ_1 is eliminated from equations (20) and (21) the differential equation is

$$v_0v_1'' + 2v_0'v_1' + v_1v_0'' = -4v_0^2v_0',$$

which has the solution

$$\begin{aligned} v_1 = {} & \frac{\frac{1}{3}(t - \sin\alpha) - \frac{1}{2}\cos^2\alpha\sin\alpha}{\left\{(t - \sin\alpha)^2 + \cos^2\alpha\right\}^{1/2}} - \tfrac{1}{2}\cos^2\alpha(t - \sin\alpha) \\ & - \frac{\frac{1}{2}\cos^4\alpha}{\left\{(t - \sin\alpha)^2 + \cos^2\alpha\right\}^{1/2}}\ln\left|\frac{(t - \sin\alpha) + \left\{(t - \sin\alpha)^2 + \cos^2\alpha\right\}^{1/2}}{(1 - \sin\alpha)}\right| \\ & - \frac{1}{3}(t - \sin\alpha)\left\{(t - \sin\alpha)^2 + \cos^2\alpha\right\}. \end{aligned} \tag{25}$$

https://doi.org/10.1017/S0334270000005348 Published online by Cambridge University Press

Therefore equation (20) yields

$$\Psi_1 = \frac{-\frac{1}{3}\cos^2\alpha - \frac{1}{2}\cos^2\alpha\sin\alpha(t-\sin\alpha)}{\left\{(t-\sin\alpha)^2+\cos^2\alpha\right\}^{3/2}} + \frac{\frac{1}{2}\cos^4\alpha}{\left\{(t-\sin\alpha)^2+\cos^2\alpha\right\}} - \frac{1}{6}\cos^2\alpha$$
$$- \frac{\frac{1}{2}\cos^4\alpha(t-\sin\alpha)}{\left\{(t-\sin\alpha)^2+\cos^2\alpha\right\}^{3/2}}\ln\left|\frac{(t-\sin\alpha)+\left\{(t-\sin\alpha)^2+\cos^2\alpha\right\}^{1/2}}{(1-\sin\alpha)}\right|. \tag{26}$$

Now in terms of non-dimensional variables

$$\begin{aligned} y' &= v\Psi \\ &= v_0\Psi_0 + \varepsilon(v_0\Psi_1 + v_1\Psi_0) + O(\varepsilon^2) \\ &= -(t-\sin\alpha) + \varepsilon\left[-\frac{1}{3} + \frac{1}{3}\left\{(t-\sin\alpha)^2+\cos^2\alpha\right\}^{3/2}\right] + O(\varepsilon^2), \end{aligned}$$

and so

$$\begin{aligned} y = &-\frac{1}{2}t^2 + t\sin\alpha \\ &+\varepsilon\Bigg[\frac{1}{12}(t-\sin\alpha)\left\{(t-\sin\alpha)^2+\cos^2\alpha\right\}^{3/2} - \frac{1}{3}t + \frac{1}{12}\sin\alpha + \frac{1}{8}\cos^2\alpha\sin\alpha \\ &+\frac{1}{8}\cos^2\alpha(t-\sin\alpha)\left\{(t-\sin\alpha)^2+\cos^2\alpha\right\}^{1/2} \\ &+\frac{1}{8}\cos^4\alpha\ln\left|\frac{(t-\sin\alpha)+\left\{(t-\sin\alpha)^2+\cos^2\alpha\right\}^{1/2}}{(1-\sin\alpha)}\right|\Bigg] + 0(\varepsilon^2). \end{aligned} \tag{27}$$

For impact on the horizontal plane through the projection point the time of flight correct to $O(\varepsilon)$ is obtained by writing

$$t = t_0 + \varepsilon t_1 + O(\varepsilon^2)$$

and substituting into equation (27) with $y = 0$. One solution is $t_0 = t_1 = 0$ as expected, while the other is

$$\left.\begin{aligned} t_0 &= 2\sin\alpha, \\ t_1 &= -\frac{1}{2} + \frac{1}{4}\cos^2\alpha + \frac{1}{8}\frac{\cos^4\alpha}{\sin\alpha}\ln\left|\frac{1+\sin\alpha}{1-\sin\alpha}\right|. \end{aligned}\right\} \tag{28}$$

https://doi.org/10.1017/S0334270000005348 Published online by Cambridge University Press

To find the range the time of flight is substituted into the appropriate expression for x. Now

$$\begin{aligned} x' &= v(1-\Psi^2)^{1/2} \\ &= v_0\left(1-\Psi_0^2\right)^{1/2} + \varepsilon\left[v_1\left(1-\Psi_0^2\right)^{1/2} - \frac{v_0\Psi_0\Psi_1}{\left(1-\Psi_0^2\right)^{1/2}}\right] + O(\varepsilon^2) \\ &= \cos\alpha + \varepsilon\left[-\frac{1}{2}\cos\alpha\sin\alpha - \frac{1}{2}\cos\alpha(t-\sin\alpha)\left\{(t-\sin\alpha)^2+\cos^2\alpha\right\}^{1/2}\right. \\ &\quad \left. -\frac{1}{2}\cos^3\alpha\ln\left|\frac{(t-\sin\alpha)+\left\{(t-\sin\alpha)^2+\cos^2\alpha\right\}^{1/2}}{(1-\sin\alpha)}\right|\right] + O(\varepsilon^2). \end{aligned}$$

Integration yields

$$\begin{aligned} x = t\cos\alpha + \varepsilon&\left[-\frac{t}{2}\cos\alpha\sin\alpha - \frac{1}{6}\cos\alpha\left\{(t-\sin\alpha)^2+\cos^2\alpha\right\}^{3/2}\right. \\ &+\frac{1}{2}\cos^3\alpha\left\{(t-\sin\alpha)^2+\cos^2\alpha\right\}^{1/2} + \frac{1}{6}\cos\alpha - \frac{1}{2}\cos^3\alpha \\ &\left.-\frac{1}{2}\cos^3\alpha(t-\sin\alpha)\ln\left|\frac{(t-\sin\alpha)+\left\{(t-\sin\alpha)^2+\cos^2\alpha\right\}^{1/2}}{(1-\sin\alpha)}\right|\right] + O(\varepsilon^2). \end{aligned} \tag{29}$$

Substitution of equation (28) into equation (29) produces finally

$$\begin{aligned} x = \sin 2\alpha + \varepsilon&\left[-\frac{1}{4}\cos\alpha - \frac{5}{4}\cos\alpha\sin^2\alpha + \frac{\cos^5\alpha}{8\sin\alpha}\ln\left|\frac{1+\sin\alpha}{1-\sin\alpha}\right|\right. \\ &\left.-\frac{1}{2}\cos^3\alpha\sin\alpha\ln\left|\frac{1+\sin\alpha}{1-\sin\alpha}\right|\right] + O(\varepsilon^2). \end{aligned} \tag{30}$$

When the ranges predicted by Methods 3 and 4 are rewritten in non-dimensional variables the expressions are respectively

$$x \approx \sin 2\alpha + \varepsilon\left[-\frac{2}{3}\sin^2 2\alpha\right]$$

and

$$x \approx \sin 2\alpha + \varepsilon\left[-\frac{1}{2}\sin^2 2\alpha\right].$$

https://doi.org/10.1017/S0334270000005348 Published online by Cambridge University Press

Table 1 shows the values of the coefficients of ε for different values of α using the x-components given by Methods 3, 4 and 5. The value $\alpha = 45^\circ$ is included because it gives the maximum range when $\varepsilon = 0$. This value of α also gives the maximum range for the analyses of Methods 3 and 4 for any small value of ε. This points to an inherent weakness in Methods 3 and 4 since the addition of drag should upset the symmetry of the trajectory, not preserve it. On the other hand the rigorous perturbation analysis of Method 5 leads to a maximum range occurring at values of α which are just less than 45°.

TABLE 1. Comparison of $O(\varepsilon)$ coefficients from Methods 3, 4 and 5.

α(degrees)	$(-\frac{1}{2}\sin^2 2\alpha)$	$(-\frac{2}{3}\sin^2 2\alpha)$	(Coefficient of ε from equation (30))
0	0	0	0
10	−0.06	−0.08	−0.08
20	−0.21	−0.28	−0.28
30	−0.37	−0.50	−0.53
40	−0.48	−0.65	−0.73
45	−0.50	−0.67	−0.78
50	−0.48	−0.65	−0.80
60	−0.38	−0.50	−0.72
70	−0.21	−0.28	−0.53
80	−0.06	−0.08	−0.27

4. Applications to Beamon's record jump

The model developed of a long jumper as a point projectile has to be considered in association with the following aspects:

(i) The jumper's centre of mass at the instant of landing is usually about 0.5 to 0.6 metres lower than the centre of mass at take-off.

(ii) The centre of mass at take-off will be forward of the take-off toe.

(iii) The centre of mass at landing will be behind the landing heel.

(iv) The take-off toe will be an unspecified distance behind the zero measuring position at the take-off board.

Not one of these four aspects was recorded for Beamon's world record jump. Nevertheless suitable estimates can be made from available data for other top-class long jumpers.

With regard to aspect (i), values measured at the Australian Institute of Sport Biomechanics Laboratory indicate that for tall long jumpers like Beamon (height 1.90 metres) the difference in height of his centre of mass between the take-off and landing phase should be taken as 0.6 metres.

For the 12 finalists at the 1983 U. S. National Championships in the men's long jump, Hay and Miller [5] report an average value of 0.41 metres for the take-off distance corresponding to aspect (ii).

Landing data seems to have only been measured by Benno M. Nigg on 25 Swiss and West German jumpers. Hay [4] records an average value of 0.53 metres from Nigg's measurements of aspect (iii).

Finally, since Beamon's jump was a record, his take-off toe would have been very close to the front edge of the take-off board. As his jump was not a foul a reasonable estimate of aspect (iv) would be 0.02 metres.

When the estimates from aspects (ii), (iii) and (iv) are combined it appears that the non-projectile part of Beamon's jump contributed at least 0.92 metres to the measured jump-distance. In the first edition of his book, Hay [4] estimated this value at 0.80 metres, but its deletion from later editions suggests that Hay had doubts about its validity. In the absence of a measured value I shall take 0.92 metres as being more realistic for these aspects of Beamon's record jump.

Therefore for his record jump of 8.90 metres the $\bar{x}$-component of Beamon's projectile motion is 7.98 metres while the $\bar{y}$-component is -0.60 metres. Before these can be converted to non-dimensional values an estimate of Beamon's take-off speed is required. The importance of estimating V correctly is emphasised by its role in calculating the downrange distance and its appearance in the parameter ε.

Beamon was a fast sprinter so it is reasonable to assume that his approach speed was probably greater than $10ms^{-1}$ just before take-off. Both Brearley [1] and Burghes et. al. [2] assumed a lesser value of $9.45ms^{-1}$ which was criticised by Ward-Smith [7] who opted for $10.1ms^{-1}$. Moreover the former authors transposed the approach speed to the horizontal component of Beamon's take-off velocity. But evidence by Hay and Miller [5] for world class men and women long-jumpers clearly indicates that in most jumps the take-off speed is slightly less than the approach speed, and therefore the horizontal component of the take-off velocity is significantly less than the approach speed. They give the maximum take-off speed for all the male long-jumpers in their study as $10ms^{-1}$, and this was also used by Frohlich [3]. This indicates that for Beamon's record jump a more realistic estimate of his take-off speed should be $10ms^{-1}$, which I shall use in all further calculations.

At the Mexico City altitude of 2256 metres the values of the other relevant physical properties for Beamon's jump are

$$\rho = 0.984\ \text{kg}\, m^{-3},$$
$$m = 75\text{kg},$$
$$\frac{1}{2} C_D A = 0.18,$$

the last two being given by Ward-Smith [7]. Therefore $\varepsilon = 0.024$ and ε^2 is certainly negligible.

Consequently Method 5 can be used to produce an accurate estimate of Beamon's angle of take-off (α) when drag has been included. To calculate α it is necessary to slightly extend the results of Method 5. With $y = h$ in equation (27) the time of flight of the projectile is seen to be

$$t_p = \sin\alpha + (\sin^2\alpha - 2h)^{1/2} + \varepsilon\Big[\frac{1}{12}(1-2h)^{3/2} + \frac{1}{8}\cos^2\alpha(1-2h)^{1/2} - \frac{1}{3} + \frac{1}{(\sin^2\alpha - 2h)^{1/2}}\Big\{\frac{1}{8}\cos^4\alpha\ln\Big|\frac{(\sin^2\alpha-2h)^{1/2}+(1-2h)^{1/2}}{1-\sin\alpha}\Big| - \frac{1}{4}\sin\alpha + \frac{1}{8}\cos^2\alpha\sin\alpha\Big\}\Big] + O(\varepsilon^2).$$

Note that when $h = 0$ the results (28) are recovered. With this t_p the dimensionless projectile downrange distance is obtained from equation (29) as

$$x_p = \cos\alpha\{\sin\alpha + (\sin^2\alpha - 2h)^{1/2}\} + \varepsilon\Big[-\frac{1}{12}(1-2h)^{3/2}\cos\alpha + \frac{5}{8}(1-2h)^{1/2}\cos^3\alpha - \frac{2}{3}\cos\alpha - \frac{1}{2}\cos\alpha\sin\alpha(\sin^2\alpha-2h)^{1/2} + \frac{(\cos^3\alpha\sin\alpha - 2\sin\alpha\cos\alpha)}{8(\sin^2\alpha-2h)^{1/2}} + \Big(\frac{\cos^5\alpha}{8(\sin^2\alpha-2h)^{1/2}} - \frac{1}{2}\cos^3\alpha(\sin^2\alpha-2h)^{1/2}\Big)\times\ln\Big|\frac{(\sin^2\alpha-2h)^{1/2}+(1-2h)^{1/2}}{1-\sin\alpha}\Big|\Big] + O(\varepsilon^2). \tag{31}$$

The analogous result corresponding to Brearley's approximation [1] is

$$x_p \approx \cos\alpha\{\sin\alpha + (\sin^2\alpha-2h)^{1/2}\} - \varepsilon\cos^2\alpha\big[\sin\alpha + (\sin^2\alpha-2h)^{1/2}\big]^2/2$$

and to Burghes' approximation [2] is

$$x_p \approx \cos\alpha\{\sin\alpha + (\sin^2\alpha-2h)^{1/2}\} - \frac{\varepsilon\cos^2\alpha\big[\sin\alpha + (\sin^2\alpha-2h)^{1/2}\big]^3}{3(\sin^2\alpha-2h)^{1/2}}.$$

Using $V = 10$ the non-dimensional values for $(x_p, h) = (0.782, -0.059)$ correct to three decimal places. When these are substituted into equation (31) the α-value for Beamon's jump is $20^\circ 39'$.

https://doi.org/10.1017/S0334270000005348 Published online by Cambridge University Press

With this value of α and $V = 10$ the jump distances predicted by the various methods are given below.

Method 1:	(Runge-Kutta numerical, IMSL package DVERK, step-length 0.001 seconds)	8.90m,
Method 2:	(Semi-analytical, Simpson's rule, step-length 0.01 radians)	8.90m,
Method 3:	(Burghes' approximate analysis)	8.90m,
Method 4:	(Brearley's approximate analysis)	8.91m,
Method 5:	(Perturbation analysis)	8.90m,
Method 6:	(Classical no-drag analysis)	8.99m.

For this value of α the jump-distances predicted by Methods 3 and 5 differ only in the third decimal place. However as the angle increases the difference moves to the second decimal place which is a measurable quantity for the long jump.

It would appear that a theoretical analysis which includes drag is valid and comparably accurate for Methods 3, 4 or 5. Method 4 is essentially based on the approximation $\dot{\bar{y}} = 0$, but at the end of Beamon's long jump this value is nearly 5. Method 3 is based on the assumption that $\dot{\bar{x}}^2$ is much greater than $\dot{\bar{y}}^2$, but again at the end of the jump the ratio of these two quantities is less than 4. Both Method 3 and Method 4 are saved—when applied to Beamon's long-jump—by the exceedingly small value of $\varepsilon = 0.024$. For problems where the value of ε is larger (for example in free fall from an aeroplane) or where the trajectory has values of Ψ that are not small the errors in using Methods 3 or 4 become substantial.

For all ε values less than 0.1, the perturbation result (31) is a very accurate and efficient result for calculating long jump characteristics when drag is to be included. The error in using either Method 3 or Method 4 for Beamon's jump is very small, while the error in disregarding air resistance is 1%.

For a jump at sea-level with the same initial conditions and a change in air density the parameters are

$$\rho = 1.225 \text{ kg} \, m^{-3}, \; \varepsilon = 0.030,$$

and therefore from Method 5 the predicted jump distance is 8.88 metres. Thus the increase in range at Mexico City due solely to the difference in air density is 2*cm*, as discovered by most of the earlier investigators. Frohlich [3] also includes the variation in take-off velocity (2%) due to the differences in air density between Mexico City and sea-level. When this is included through Method 5 the equivalent jump distance at sea-level is now only 8.57 metres.

What may be of interest to long-jumpers is the fact that Method 5 predicts a projection angle of 43°13′ for a maximum range of 11.51 metres when $V = 10$. Therefore although Beamon's record has stood for 17 years, and was a mighty jump, it could easily be eclipsed by a long-jumper with a take-off speed of $10ms^{-1}$ and a take-off angle between 21° and 64°. The strengthening of leg muscles and the development of techniques to achieve such a take-off angle is therefore worthy of further investigation by athletic coaches. Table 2 shows the gains that could have been achieved by Beamon if he could have increased his take-off angle. As the angle gets close to 43°13′ the change in jump length is not as pronounced as for the same increment in angle near 21°.

TABLE 2. Jump length for various take-off angles for Beamon with $V = 10ms^{-1}$.

Take-off angle (α)	Jump length ($\bar{x}$) in metres
20°39′	8.90
25°	9.75
30°	10.55
35°	11.13
40°	11.45
45°	11.49
50°	11.25
55°	10.71
60°	9.91
65°	8.85

This table gives a clue as to why Beamon out-jumped the others by so much. The record of 8.10 metres which he broke could have been achieved exactly by him at Mexico City with a take-off angle of 16°59′. Therefore with an increase of 3°40′ in take-off angle Beamon was able to achieve an immense change in the jump distance.

The analysis in general confirms that the most important characteristics for a successful long jump are still the take-off speed and take-off angle, or equivalently the horizontal and vertical take-off velocity components. Although the current philosophy in modelling long jumps is to disregard drag (Hay [4]), it has been shown that it has a small but significant effect on the jump distance.

It is interesting to seek the take-off speed at an angle near 45° which would still produce a jump of 8.90 metres. This is easily calculated as approximately 8.72 ms^{-1}, and illustrates the importance of maintaining speed as the jumping-pit is approached. Recent studies by Ward-Smith [8] show that during a 100-metre sprint at maximum available power a sprinter does not reach his maximum speed until about 60 metres have been run. Long jumpers of course still need some power in reserve to launch themselves with a reasonable take-off angle. It is this trade-off of launching speed versus launching angle which makes the long jump event fascinating from an optimisation point of view also.

5. Conclusion

When drag is small compared with gravity the perturbation analysis given in Method 5 produces expressions that are more accurate for the downrange than those produced by other approximate analytical approaches (Methods 3 and 4). On the other hand the jump distances predicted by this perturbation analysis are just as accurate as standard numerical techniques (Methods 1 and 2) which are less computer efficient.

The results indicate that it is theoretically possible to greatly increase the long jump world record by developing athletes who can take-off at a larger angle than is currently achieved. The possibility of reducing speed to obtain a larger angle and still increase the jump distance will be the subject of a future report.

References

[1] M. N Brearley, "The long jump at Mexico City", *Function* (Monash University) 3 (1979), 16–19.

[2] D. N. Burghes, I. Huntley and J. McDonald, *Applying mathematics* (Ellis Horwood, Chichester, England, 1982).

[3] C. Frohlich, "Effect of wind and altitude on record performances in foot races, pole vault and long jump", *Amer. J. Phys.* 53 (1985), 726–730.

[4] J. G. Hay, *The biomechanics of sports techniques*, 3rd edition (Prentice-Hall, New Jersey, 1985).

[5] J. G. Hay and J. A. Miller Jr., "Techniques used in the transition from approach to takeoff in the long jump", *Internat. J. Sport Biomech.* 1 (1985), 174–184.

[6] H. Lamb, *Dynamics* (Cambridge University Press, 1923).

[7] A. J. Ward-Smith, "The influence of aerodynamic and biomechanical factors on long jump performance", *J. Biomechanics* 16 (1983), 655–658.

[8] A. J. Ward-Smith, "A mathematical theory of running, based on the first law of thermodynamics, and its application to the performance of world-class athletes", *J. Biomechanics* 18 (1985), 337–349.