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THE TURNPIKE RESULT FOR APPROXIMATE SOLUTIONS OF NONAUTONOMOUS VARIATIONAL PROBLEMS

ALEXANDER J. ZASLAVSKI

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Abstract

In this work we study the structure of approximate solutions of variational problems with continuous integrands $f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1$ which belong to a complete metric space of functions. The main result in this paper deals with the turnpike property of variational problems. To have this property means that the approximate solutions of the problems are determined mainly by the integrand, and are essentially independent of the choice of interval and endpoint conditions, except in regions close to the endpoints.

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1. Introduction and main results

In this paper we analyse the structure of solutions of the variational problems

$$\text{(P)} \quad \int_{T_1}^{T_2} f(t, z(t), z'(t))\, dt \to \min, \quad \begin{cases} z(T_1) = x,\ z(T_2) = y,\ z : [T_1, T_2] \to \mathbb{R}^n \text{ is} \\ \text{an absolutely continuous (a.c.) function,} \end{cases}$$

where $T_1 \geq 0$, $T_2 > T_1$, $x, y \in \mathbb{R}^n$ and $f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1$ belongs to a space of integrands described below.

Let $T_1 \geq 0$, $T_2 > T_1$, $x, y \in \mathbb{R}^n$, $f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1$ be an integrand and let δ be a positive number. We say that an absolutely continuous (a.c.) function $u : [T_1, T_2] \to \mathbb{R}^n$ satisfying $u(T_1) = x$, $u(T_2) = y$ is a δ-approximate solution of the problem (P) if

$$\int_{T_1}^{T_2} f(t, u(t), u'(t))dt \leq \int_{T_1}^{T_2} f(t, z(t), z'(t))dt + \delta$$

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for each a.c. function $z : [T_1, T_2] \to \mathbb{R}^n$ satisfying $z(T_1) = x$, $z(T_2) = y$.

The main results in this paper deal with the so-called turnpike property of the variational problems (P). To have this property means, roughly speaking, that the approximate solutions of the problems (P) are determined mainly by the integrand (cost function), and are essentially independent of the choice of interval and endpoint conditions, except in regions close to the endpoints.

Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [12]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). This property was further investigated for optimal trajectories of models of economic dynamics (see, for example, [2, 3, 5–11] and the references mentioned there). In control theory turnpike properties were studied in [18, 19] for linear control systems with convex integrands.

Denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$. Let $a > 0$ be a positive constant and let $\psi : [0, \infty) \to [0, \infty)$ be an increasing function such that $\psi(t) \to +\infty$ as $t \to \infty$. Denote by $\mathcal{M}$ the set of all continuous functions $f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1$ which satisfy the following assumptions:

(A.i) The function f is bounded on $[0, \infty) \times E$ for any bounded set $E \subset \mathbb{R}^n \times \mathbb{R}^n$.

(A.ii) $f(t, x, u) \geq \max\{\psi(|x|), \psi(|u|)|u|\} - a$ for each $(t, x, u) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$.

(A.iii) For each $M, \epsilon > 0$, there exist $\Gamma, \delta > 0$ such that

$$|f(t, x_1, u) - f(t, x_2, u)| \leq \epsilon \max\{f(t, x_1, u), f(t, x_2, u)\}$$

for each $t \in [0, \infty)$ and each $u, x_1, x_2 \in \mathbb{R}^n$ which satisfy

$$|x_i| \leq M, \ i = 1, 2, \ |u| \geq \Gamma, \quad |x_1 - x_2| \leq \delta.$$

(A.iv) For each $M, \epsilon > 0$, there exists $\delta > 0$ such that

$$|f(t, x_1, u_1) - f(t, x_2, u_2)| \leq \epsilon$$

for each $t \in [0, \infty)$ and each $u_1, u_2, x_1, x_2 \in \mathbb{R}^n$ which satisfy

$$|x_i|, |u_i| \leq M, \ i = 1, 2, \quad \max\{|x_1 - x_2|, |u_1 - u_2|\} \leq \delta.$$

In [16, 17] we studied the subset of the set $\mathcal{M}$ which consists of all $f \in \mathcal{M}$ satisfying the following assumptions:

- for each $(t, x) \in [0, \infty) \times \mathbb{R}^n$ the function $f(t, x, \cdot) : \mathbb{R}^n \to \mathbb{R}^1$ is convex;
- for each $M, \epsilon > 0$ there exist $\Gamma, \delta > 0$ such that

$$|f(t, x_1, u_1) - f(t, x_2, u_2)| \leq \epsilon \max\{f(t, x_1, u_1), f(t, x_2, u_2)\}$$

for each $t \in [0, \infty)$ and each $u_1, u_2, x_1, x_2 \in \mathbb{R}^n$ which satisfy

$$|x_i| \le M, \ |u_i| \ge \Gamma, \ i = 1, 2, \qquad \max\{|x_1 - x_2|, |u_1 - u_2|\} \le \delta$$

(see (A.iii)).

This subset will be denoted by $\mathcal{M}_{co}$.

It is easy to show that an integrand $f = f(t, x, u) \in C^1([0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n)$ belongs to $\mathcal{M}$ if f satisfies assumption (A.ii), and if $\sup\{|f(t, 0, 0)| : t \in [0, \infty)\} < \infty$ and also there exists an increasing function $\psi_0 : [0, \infty) \to [0, \infty)$ such that

$$\sup\left\{\left|\frac{\partial f(t, x, u)}{\partial x}\right|, \left|\frac{\partial f(t, x, u)}{\partial u}\right|\right\} \le \psi_0(|x|)(1 + \psi(|u|)|u|)$$

for each $t \in [0, \infty)$ and each $x, u \in \mathbb{R}^n$.

For the set $\mathcal{M}$, we consider the uniformity which is determined by the following base:

(1.1) $E(N, \epsilon, \lambda) = \{(f, g) \in \mathcal{M} \times \mathcal{M} : |f(t, x, u) - g(t, x, u)| \le \epsilon$ for each $t \in [0, \infty)$ and each $x, u \in \mathbb{R}^n$ satisfying $|x|, |u| \le N$ and $(|f(t, x, u)| + 1)(|g(t, x, u)| + 1)^{-1} \in [\lambda^{-1}, \lambda]$ for each $t \in [0, \infty)$ and each $x, u \in \mathbb{R}^n$ satisfying $|x| \le N\}$,

where $N > 0, \epsilon > 0, \lambda > 1$.

Clearly, the space $\mathcal{M}$ with this uniformity is metrizable (by a metric ρ_w). It was established in [13, Proposition 2.2] that the metric space $(\mathcal{M}, \rho_w)$ is complete. Note that this uniformity was introduced in [16] for the subset $\mathcal{M}_{co}$ of $\mathcal{M}$. The metric ρ_w induces in $\mathcal{M}$ a topology.

We consider functionals of the form

$$I^f(T_1, T_2, x) = \int_{T_1}^{T_2} f(t, x(t), x'(t))\, dt \tag{1.2}$$

where $f \in \mathcal{M}, 0 \le T_1 < T_2 < +\infty$ and $x : [T_1, T_2] \to \mathbb{R}^n$ is an a.c. function.

For $f \in \mathcal{M}$, $y, z \in \mathbb{R}^n$ and numbers T_1, T_2 satisfying $0 \le T_1 < T_2$ we set

$$U^f(T_1, T_2, y, z) = \inf\{I^f(T_1, T_2, x) \mid x : [T_1, T_2] \to \mathbb{R}^n \text{ is an a.c. function satisfying } x(T_1) = y, x(T_2) = z\}. \tag{1.3}$$

It is easy to see that $-\infty < U^f(T_1, T_2, y, z) < +\infty$ for each $f \in \mathfrak{M}$, each $y, z \in \mathbb{R}^n$ and all numbers T_1, T_2 satisfying $0 \le T_1 < T_2$.

Let $f \in \mathcal{M}$. A locally absolutely continuous (a.c.) function $x : [0, \infty) \to \mathbb{R}^n$ is called an (f)-*good function* if for any a.c. function $y : [0, \infty) \to \mathbb{R}^n$ there is a number M_y such that

$$I^f(0, T, y) \ge M_y + I^f(0, T, x) \quad \text{for each } T \in (0, \infty). \tag{1.4}$$

In [14, Proposition 1.1] we proved the following result.

PROPOSITION 1.1. *Let $f \in \mathcal{M}$ and let $x : [0, \infty) \to \mathbb{R}^n$ be a bounded a.c. function. Then the function x is (f)-good if and only if there is $M > 0$ such that*

$$I^f(0, T, x) \le U^f(0, T, x(0), x(T)) + M \quad \text{for any } T > 0.$$

The next result will be proved in Section 3.

PROPOSITION 1.2. *Let $f \in \mathcal{M}$ and let for each $(t, x) \in [0, \infty) \times \mathbb{R}^n$ the function $f(t, x, \cdot) : \mathbb{R}^n \to \mathbb{R}^1$ be convex. Then for each $z \in \mathbb{R}^n$ there is a bounded (f)-good function $Z : [0, \infty) \to \mathbb{R}^n$ such that $Z(0) = z$ and that for each $T > 0$,*

$$I^f(0, T, Z) = U^f(0, T, Z(0), Z(T)).$$

Let $f \in \mathcal{M}$. We say that f has the *strong turnpike property*, or briefly (STP), if there exists a bounded a.c. function $X_f : [0, \infty) \to \mathbb{R}^n$ which satisfies the following condition: For each $K, \epsilon > 0$ there exist constants $\delta, L > 0$ such that for each $T_1 \ge 0$, $T_2 \ge T_1 + 2L$ and each a.c. function $v : [T_1, T_2] \to \mathbb{R}^n$ which satisfies $|v(T_1)|, |v(T_2)| \le K$ and $I^f(T_1, T_2, v) \le U^f(T_1, T_2, v(T_1), v(T_2)) + \delta$

(i) there are $\tau_1 \in [T_1, T_1 + L]$ and $\tau_2 \in [T_2 - L, T_2]$ for which

$$|v(t) - X_f(t)| \le \epsilon, \quad t \in [\tau_1, \tau_2];$$

(ii) if $|v(T_1) - X_f(T_1)| \le \delta$, then $\tau_1 = T_1$ and if $|v(T_2) - X_f(T_2)| \le \delta$, then $\tau_2 = T_2$.

The function X_f is called *the turnpike of f*.

If the integrand f has the strong turnpike property, then the solutions of variational problems with f are essentially independent of the choice of time interval and values at the endpoints except in regions close to the endpoints of the time interval. If a point t does not belong to these regions, then the value of a solution at t is closed to a trajectory ('turnpike') which is defined on the infinite time interval and depends only on f. This phenomenon has the following interpretation. If one wish to reach a point A from a point B by a car in an optimal way, then one should turn to a turnpike, spend most of time on it and then leave the turnpike to reach the required point.

If in the definition above condition (ii) is not assumed, then we say that the integrand f has the turnpike property [14, 15, 17].

In the sequel we use the following definition [4].

Let $f \in \mathcal{M}$. We say that an a.c. function $x : [0, \infty) \to \mathbb{R}^n$ is (f)-overtaking optimal if for each a.c. function $y : [0, \infty) \to \mathbb{R}^n$ satisfying $y(0) = x(0)$,

$$\limsup_{T \to \infty} [I^f(0, T, x) - I^f(0, T, y)] \le 0.$$

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Assume that $f \in \mathcal{M}$ and $X : [0, \infty) \to \mathbb{R}^n$ is a bounded a.c. function. How to verify if the integrand f has (STP) and X is its turnpike? In this paper we introduce three properties (P1), (P2) and (P3) and show that f has (STP) if and only if f possesses properties (P1), (P2) and (P3). Property (P1) means that all (f)-good functions have the same asymptotic behavior while property (P2) means that X is a unique (f)-overtaking optimal function whose value at zero is $X(0)$. Property (P3) means that if an a.c. function $v : [0, T] \to \mathbb{R}^n$ is an approximate solution and T is large enough, then there is $\tau \in [0, T]$ such that $v(\tau)$ is close to $X(\tau)$. In [14] we establish that f has the turnpike property if and only if f possesses properties (P1) and (P3).

The next theorem is the main result of the paper.

THEOREM 1.3. *Let $f \in \mathcal{M}$, for each $(t, x) \in [0, \infty) \times \mathbb{R}^n$ the function $f(t, x, \cdot) : \mathbb{R}^n \to \mathbb{R}^1$ be convex and let $X_f : [0, \infty) \to \mathbb{R}^n$ be a bounded a.c. function. Then f has the strong turnpike property with X_f being the turnpike if and only if the following three properties hold*:

(P1) *For each pair of (f)-good functions $v_1, v_2 : [0, \infty) \to \mathbb{R}^n$,*

$$|v_1(t) - v_2(t)| \to 0 \quad \text{as } t \to \infty.$$

(P2) *X_f is an (f)-overtaking optimal function and if an (f)-overtaking optimal function $v : [0, \infty) \to \mathbb{R}^n$ satisfies $v(0) = X_f(0)$, then $v = X_f$.*

(P3) *For each $K, \epsilon > 0$ there exist $\gamma, l > 0$ such that for each $T \geq 0$ and each a.c. function $w : [T, T + l] \to \mathbb{R}^n$ which satisfies $|w(T)|, |w(T + l)| \leq K$ and $I^f(T, T + l, w) \leq U^f(T, T + l, w(T), w(T + l)) + \gamma$ there is $\tau \in [T, T + l]$ for which $|X_f(\tau) - v(\tau)| \leq \epsilon$.*

2. Auxiliary results

We have the following result (see Berkovitz [1]).

PROPOSITION 2.1. *Assume that $f \in \mathcal{M}$ and $f(t, x, \cdot) : \mathbb{R}^n \to \mathbb{R}^1$ is a convex function for each $(t, x) \in \mathbb{R}^n \times [0, \infty)$. Then for each pair of numbers T_1, T_2 satisfying $0 \leq T_1 < T_2$ and each $z_1, z_2 \in \mathbb{R}^n$ there exists an a.c. function $x : [T_1, T_2] \to \mathbb{R}^n$ such that*

$$x(T_i) = z_i, \quad i = 1, 2, \quad I^f(T_1, T_2, x) = U^f(T_1, T_2, z_1, z_2).$$

In [13] we analyzed the properties of (f)-good functions and established the following results.

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PROPOSITION 2.2 ([13, Theorem 1.1]). *For each $h \in \mathcal{M}$, each $\delta \in (0, 1)$ and each $z \in \mathbb{R}^n$, there exists an (h)-good function $Z_\delta^h : [0, \infty) \to \mathbb{R}^n$ satisfying $Z_\delta^h(0) = z$ such that the following assertions hold*:

(1) *Let $f \in \mathcal{M}$, $\epsilon \in (0, 1)$, $z \in \mathbb{R}^n$ and let $y : [0, \infty) \to \mathbb{R}^n$ be an a.c. function. Then one of the following properties holds*:

(i) $I^f(0, T, y) - I^f(0, T, Z_\epsilon^f) \to \infty$ *as* $T \to \infty$;

(ii) $\sup\{|I^f(0, T, y) - I^f(0, T, Z_\epsilon^f)| : T \in (0, \infty)\} < \infty$ *and*

$$\sup\{|y(t)| : t \in [0, \infty)\} < \infty.$$

(2) *For each $f \in \mathcal{M}$ and each positive number M, there exist a neighborhood U of f in $\mathcal{M}$ and a number $Q > 0$ such that* $\sup\{|Z_\epsilon^g(t)| : t \in [0, \infty)\} \le Q$ *for each $g \in U$, each $\epsilon \in (0, 1)$ and each $z \in \mathbb{R}^n$ satisfying $|z| \le M$.*

(3) *For each $f \in \mathcal{M}$ and each positive number M, there exist a neighborhood U of f in $\mathcal{M}$ and a number $Q > 0$ such that for each $g \in U$, each $z \in \mathbb{R}^n$ satisfying $|z| \le M$, each $\epsilon \in (0, 1)$, each $T_1 \ge 0$, $T_2 > T_1$ and each a.c. function $y : [T_1, T_2] \to \mathbb{R}^n$ satisfying $|y(T_1)| \le M$ the following relation holds*:

$$I^g(T_1, T_2, Z_\epsilon^g) \le I^g(T_1, T_2, y) + Q.$$

(4) *For each $f \in \mathcal{M}$, $\epsilon > 0$, $z \in \mathbb{R}^n$, $T_1 \ge 0$ and $T_2 > T_1$,*

$$I^f(T_1, T_2, Z_\epsilon^f) \le U^f(T_1, T_2, Z_\epsilon^f(T_1), Z_\epsilon^f(T_2)) + \epsilon.$$

(5) *For each $f \in \mathcal{M}$, $z \in \mathbb{R}^n$ and an integer $i \ge 0$,*

$$Z_{\epsilon_1}^f(i) = Z_{\epsilon_2}^f(i) \quad \textit{for each } \epsilon_1, \epsilon_2 \in (0, 1).$$

Proposition 2.2 is an extension of [16, Theorem 1.1] which was established for the space $\mathcal{M}_{co}$. In [16] we have shown that for each $f \in \mathcal{M}_{co}$ and $z \in \mathbb{R}^n$,

$$Z_{\epsilon_1}^f = Z_{\epsilon_2}^f \quad \text{for each } \epsilon_1, \epsilon_2 \in (0, 1)$$

and

$$U^f(T_1, T_2, Z_\epsilon^f(T_1), Z_\epsilon^f(T_2)) = I^f(T_1, T_2, Z_\epsilon^f)$$

for each $T_1 \ge 0$, $T_2 > T_1$ and each $\epsilon \in (0, 1)$.

PROPOSITION 2.3 ([13, Proposition 2.6]). *Let $f \in \mathcal{M}$, $0 < c_1 < c_2 < \infty$ and let $M, \epsilon > 0$. Then there exists $\delta > 0$ such that for each $T_1, T_2 \ge 0$ satisfying $T_2 - T_1 \in [c_1, c_2]$ and each $y_1, y_2, z_1, z_2 \in \mathbb{R}^n$ satisfying*

$$|y_i|, |z_i| \le M, \quad i = 1, 2, \quad |y_1 - y_2|, |z_1 - z_2| \le \delta$$

the relation $|U^f(T_1, T_2, y_1, z_1) - U^f(T_1, T_2, y_2, z_2)| \le \epsilon$ holds.

PROPOSITION 2.4 ([13, Theorem 1.3]). *Let $f \in \mathcal{M}$ and let M_1, M_2, c be positive numbers. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$ and a number $S > 0$ such that for each $g \in \mathcal{U}$, each $T_1 \in [0, \infty)$ and each $T_2 \in [T_1 + c, \infty)$ the following property holds: For each $x, y \in \mathbb{R}^n$ satisfying $|x|, |y| \le M_1$ and each a.c. function $v : [T_1, T_2] \to \mathbb{R}^n$ satisfying*

$$v(T_1) = x, \quad v(T_2) = y, \quad I^g(T_1, T_2, v) \le U^g(T_1, T_2, x, y) + M_2,$$

inequality $|v(t)| \le S$ is valid for $t \in [T_1, T_2]$.

PROPOSITION 2.5 ([13, Proposition 2.4]). *Let $M_1, \epsilon > 0$, $0 < \tau_0 < \tau_1$. Then there exists $\delta > 0$ such that for each $f \in \mathcal{M}$, each $T_1 \in [0, \infty)$, $T_2 \in [T_1 + \tau_0, T_1 + \tau_1]$, each a.c. function $x : [T_1, T_2] \to \mathbb{R}^n$ satisfying $I^f(T_1, T_2, x) \le M_1$ and each $t_1, t_2 \in [T_1, T_2]$ which satisfy $|t_2 - t_1| \le \delta$, relation $|x(t_1) - x(t_2)| \le \epsilon$ holds.*

PROPOSITION 2.6 ([13, Proposition 2.5]). *Let $f \in \mathcal{M}$, $0 < c_1 < c_2 < \infty$ and $c_3 > 0$. Then there exists a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$ such that the set*

$$\{U^g(T_1, T_2, z_1, z_2) : g \in \mathcal{U},\ T_1 \in [0, \infty),\ T_2 \in [T_1 + c_1, T_1 + c_2],\\ z_1, z_2 \in \mathbb{R}^n,\ |z_i| \le c_3,\ i = 1, 2\}$$

is bounded.

PROPOSITION 2.7. *Let $T_1 \ge 0$, $T_2 > T_1$ and let $v : [T_1, T_2] \to \mathbb{R}^n$ be a continuous function. Assume that for each $\tau_1, \tau_2 \in (T_1, T_2)$ satisfying $\tau_1 < \tau_2$ the restriction of v to $[\tau_1, \tau_2]$ is an a.c. function and*

$$I^f(\tau_1, \tau_2, v) = U^f(\tau_1, \tau_2, v(\tau_1), v(\tau_2)). \tag{2.1}$$

Then the function $v : [T_1, T_2] \to \mathbb{R}^n$ is an a.c. function and

$$I^f(T_1, T_2, v) = U^f(T_1, T_2, v(T_1), v(T_2)). \tag{2.2}$$

PROOF. Choose

$$M_0 > \sup\{|v(t)| : t \in [T_1, T_2]\}. \tag{2.3}$$

By (2.1), (2.3) and Proposition 2.6 the set

$$\{I^f(\tau_1, \tau_2, v) : \tau_1, \tau_2 \in (T_1, T_2),\ \tau_2 - \tau_1 \in (0, (T_2 - T_1)/8)\}$$

is bounded. It follows from this fact, (A.ii) and Fatou's lemma that the integral

$$\int_{T_1}^{T_2} f(t, v(t), v'(t))\, dt$$

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is finite. Then (A.ii) implies that $v' \in L^1([T_1, T_2]; \mathbb{R}^n)$ and $v : [T_1, T_2] \to \mathbb{R}^n$ is an a.c. function.

We show that (2.2) holds. Assume the contrary. Then there is an a.c. function $u : [T_1, T_2] \to \mathbb{R}^n$ such that

$$(2.4) \qquad u(T_i) = v(T_i), \quad i = 1, 2, \quad I^f(T_1, T_2, v) - I^f(T_1, T_2, u) > 2\Delta$$

with $\Delta > 0$.

It is not difficult to see that there is $\gamma \in (0, (T_2 - T_1)/8)$ such that:

$$(2.5) \qquad \begin{aligned} &|I^f(s_1, s_2, v)| \le \Delta/64 \ \text{ for each } s_1, s_2 \in [T_1, T_1 + \gamma] \text{ satisfying } s_2 > s_1, \\ &|I^f(s_1, s_2, v)| \le \Delta/64 \ \text{ for each } s_1, s_2 \in [T_2 - \gamma, T_2] \text{ satisfying } s_2 > s_1, \end{aligned}$$

$$(2.6) \qquad \begin{aligned} &|I^f(s_1, s_2, u)| \le \Delta/64 \ \text{ for each } s_1, s_2 \in [T_1, T_1 + \gamma] \text{ satisfying } s_2 > s_1, \\ &|I^f(s_1, s_2, u)| \le \Delta/64 \ \text{ for each } s_1, s_2 \in [T_2 - \gamma, T_2] \text{ satisfying } s_2 > s_1. \end{aligned}$$

Choose a number

$$(2.7) \qquad M_1 > \sup\{|v(t)| : t \in [T_1, T_2]\} + \sup\{|u(t)| : t \in [T_1, T_2]\}.$$

By Proposition 2.3 there is $\delta > 0$ such that the following property holds: For each $t_1 \ge 0$, $t_2 \in [t_1 + \gamma/16, t_1 + 16\gamma]$ and each $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$ satisfying

$$(2.8) \qquad |x_i|, |y_i| \le M_1, \quad i = 1, 2, \quad |x_i - y_i| \le \delta, \quad i = 1, 2,$$

the inequality

$$(2.9) \qquad |U^f(t_1, t_2, x_1, x_2) - U^f(t_1, t_2, y_1, y_2)| \le \Delta/64$$

is true.

Choose numbers t_1, t_2 such that

$$(2.10) \qquad t_1 \in (T_1, T_1 + \gamma/4], \quad t_2 \in [T_2 - \gamma/4, T_2],$$

$$(2.11) \qquad \begin{aligned} &|v(T_1) - v(t_1)|, \ |u(T_1) - u(t_1)| \le \delta/4, \\ &|v(T_2) - v(t_2)|, \ |u(T_2) - v(t_2)| \le \delta/4. \end{aligned}$$

Relations (2.11) and (2.4) imply that, for $i = 1, 2$,

$$(2.12) \qquad |v(t_i) - u(t_i)| \le |v(t_i) - v(T_i)| + |v(T_i) - u(T_i)| + |u(T_i) - u(t_i)| \le \delta/2.$$

Consider an a.c. function $\tilde{u} : [t_1, t_2] \to \mathbb{R}^n$ such that

$$\tilde{u}(t) = u(t), \ t \in [T_1 + \gamma, T_2 - \gamma], \quad \tilde{u}(t_i) = v(t_i), \ i = 1, 2,$$

and

$$(2.13)\qquad \begin{aligned} I^f(t_1, T_1+\gamma, \tilde{u}) &\le U^f(t_1, T_1+\gamma, v(t_1), u(T_1+\gamma)) + \Delta/128, \\ I^f(T_2-\gamma, t_2, \tilde{u}) &\le U^f(t_2, T_2-\gamma, u(T_2-\gamma), v(t_2)) + \Delta/128. \end{aligned}$$

It follows from (2.10), the choice of γ (see (2.5), (2.6)) and (2.4) that

$$(2.14)\qquad \begin{aligned} I^f(t_1, t_2, v) - I^f(t_1, t_2, u) &= I^f(T_1, T_2, v) - I^f(T_1, T_2, u) - I^f(T_1, t_1, v) \\ &\quad - I^f(t_2, T_2, v) + I^f(T_1, t_1, u) + I^f(t_2, T_2, u) \\ &\ge 2\Delta - 4(\Delta/64) > 3\Delta/2. \end{aligned}$$

In view of (2.13) and (2.1)

$$(2.15)\qquad \begin{aligned} I^f(t_1, t_2, \tilde{u}) - I^f(t_1, t_2, u) &= I^f(t_1, T_1+\gamma, \tilde{u}) - I^f(t_1, T_1+\gamma, u) \\ &\quad + I^f(T_2-\gamma, t_2, \tilde{u}) - I^f(T_2-\gamma, t_2, u) \\ &\le U^f(t_1, T_1+\gamma, v(t_1), u(T_1+\gamma)) + \Delta/128 \\ &\quad - U^f(t_1, T_1+\gamma, u(t_1), u(T_1+\gamma)) \\ &\quad + U^f(t_2, T_2-\gamma, u(T_2-\gamma), v(t_2)) + \Delta/128 \\ &\quad - U^f(T_2-\gamma, t_2, u(T_2-\gamma), u(t_2)). \end{aligned}$$

It follows from (2.12), (2.7), (2.10) and the choice of δ (see (2.8), (2.9)) that

$$(2.16)\qquad \begin{aligned} &|U^f(t_1, T_1+\gamma, u(t_1), u(T_1+\gamma)) - U^f(t_1, T_1+\gamma, v(t_1), u(T_1+\gamma))| \le \Delta/64, \\ &|U^f(T_2-\gamma, t_2, u(T_2-\gamma), u(t_2)) - U^f(T_2-\gamma, t_2, u(T_2-\gamma), v(t_2))| \le \Delta/64. \end{aligned}$$

Relations (2.15) and (2.16) imply that

$$(2.17)\qquad I^f(t_1, t_2, \tilde{u}) - I^f(t_1, t_2, u) \le \Delta/64 + \Delta/64 + \Delta/64 < \Delta/16.$$

By (2.17) and (2.14),

$$\begin{aligned} &I^f(t_1, t_2, v) - I^f(t_1, t_2, \tilde{u}) \\ &\quad = I^f(t_1, t_2, v) - I^f(t_1, t_2, u) + I^f(t_1, t_2, u) - I^f(t_1, t_2, \tilde{u}) \\ &\quad \ge 3\Delta/2 - \Delta/16 > 0, \end{aligned}$$

a contradiction (see (2.1)). The obtained contradiction proves the proposition. □

In the sequel we also need the next two propositions proved in [13].

PROPOSITION 2.8 ([13, Proposition 2.8]). *Let $f \in \mathcal{M}$ and let $0 < c_1 < c_2 < \infty$, $c_3, \epsilon > 0$. Then there exists a neighborhood V of f in $\mathcal{M}$ such that for each $g \in V$, each $T_1, T_2 \ge 0$ satisfying $T_2 - T_1 \in [c_1, c_2]$ and each $y, z \in \mathbb{R}^n$ satisfying $|y|, |z| \le c_3$ the relation $|U^f(T_1, T_2, y, z) - U^g(T_1, T_2, y, z)| \le \epsilon$ holds.*

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PROPOSITION 2.9 ([13, Proposition 2.7]). *Let $f \in \mathcal{M}$, $0 < c_1 < c_2 < \infty$ and $D, \epsilon > 0$. Then there exists a neighborhood V of f in $\mathcal{M}$ such that for each $g \in V$, each $T_1, T_2 \geq 0$ satisfying $T_2 - T_1 \in [c_1, c_2]$ and each a.c. function $x : [T_1, T_2] \to \mathbb{R}^n$ satisfying $\min\{I^f(T_1, T_2, x), \ I^g(T_1, T_2, x)\} \leq D$ the relation $|I^f(T_1, T_2, x) - I^g(T_1, T_2, x)| \leq \epsilon$ holds.*

3. Proof of Proposition 1.2

For each $h \in \mathcal{M}$, $\delta \in (0, 1)$ and each $z \in \mathbb{R}^n$, let an a.c. function $Z_\delta^h : [0, \infty) \to \mathbb{R}^n$ be as guaranteed by Proposition 2.2.

Assume that $z \in \mathbb{R}^n$, $f \in \mathcal{M}$ and that for each $(t, x) \in [0, \infty) \times \mathbb{R}^n$ the function $f(t, x, \cdot) : \mathbb{R}^n \to \mathbb{R}^1$ is convex.

For each integer $i \geq 0$, set

$$z_i^* = Z_\epsilon^f(i) \quad \text{with} \quad \epsilon \in (0, 1). \tag{3.1}$$

In view of Assertion (5) of Proposition 2.2, z_i^* $(i \geq 0)$ does not depend on ϵ. By Proposition 2.1, there exists an a.c. function $Z^* : [0, \infty) \to \mathbb{R}^n$ such that for each integer $i \geq 0$,

$$Z^*(i) = z_i^*, \quad I^f(i, i+1, Z^*) = U^f(i, i+1, Z^*(i), Z^*(i+1)). \tag{3.2}$$

It follows from (3.2), (3.1) and Assertion (4) of Proposition 2.2 that for each integer $k \geq 1$ and each $\epsilon \in (0, 1)$

$$\begin{aligned} I^f(0, k, Z^*) &= \sum_{i=0}^{k-1} I^f(i, i+1, Z^*) = \sum_{i=0}^{k-1} U^f(i, i+1, z_i^*, z_{i+1}^*) \\ &= \sum_{i=0}^{k-1} U^f(i, i+1, Z_\epsilon^f(i), Z_\epsilon^f(i+1)) \leq I^f(0, k, , Z_\epsilon^f) \\ &\leq U^f(0, k, Z_\epsilon^f(0), Z_\epsilon^f(k)) + \epsilon = U^f(0, k, Z^*(0), Z^*(k)) + \epsilon. \end{aligned}$$

Since ϵ is an arbitrary element of $(0, 1)$ we conclude that

$$I^f(0, k, Z^*) = U^f(0, k, Z^*(0), Z^*(k))$$

for any integer $k \geq 0$. This implies that $I^f(0, T, Z^*) = U^f(0, T, Z^*(0), Z^*(T))$ for any $T > 0$. By Assertion (1) of Proposition 2.2 and Proposition 1.1 the function Z^* is bounded and (f)-good. Proposition 1.2 is proved. □

4. Overtaking optimal trajectories

PROPOSITION 4.1. *Let $f \in \mathcal{M}$ and property* (P1) *hold* (*see Theorem* 1.3). *Assume that $x : [0, \infty) \to \mathbb{R}^n$ is a bounded a.c. function such that for each $T > 0$*

$$I^f(0, T, x) = U^f(0, T, x(0), x(T)). \tag{4.1}$$

Then x is an (f)-overtaking optimal function.

PROOF. By (4.1) and Proposition 1.1, x is (f)-good. Assume that x is not an (f)-overtaking optimal function. Then there is an a.c. function $y : [0, \infty) \to \mathbb{R}^n$ such that

$$y(0) = x(0), \quad \limsup_{T\to\infty}[I^f(0, T, x) - I^f(0, T, y)] \geq 2\epsilon \tag{4.2}$$

with some positive number ϵ. By Proposition 2.2, there is a bounded (f)-good function $Z : [0, \infty) \to \mathbb{R}^n$ such that $Z(0) = x(0)$ and that for each a.c. function $v : [0, \infty) \to \mathbb{R}^n$ either

$$\lim_{T\to\infty}[I^f(0, T, v) - I^f(0, T, Z)] = \infty \tag{4.3}$$

or

$$\begin{aligned} \sup\{|I^f(0, T, v) - I^f(0, T, Z)| : T \in (0, \infty)\} < \infty, \\ \sup\{|v(t)| : t \in [0, \infty)\} < \infty. \end{aligned} \tag{4.4}$$

Since the function x is (f)-good we conclude that

$$\sup\{|I^f(0, T, x) - I^f(0, T, Z)| : T \in (0, \infty)\} < \infty. \tag{4.5}$$

Relations (4.2) and (4.5) imply that (4.3) is not valid with $v = y$. Thus (4.4) is true with $v = y$. This implies that y is a bounded (f)-good function. In view of property (P1)

$$\lim_{t\to\infty}|x(t) - y(t)| = 0. \tag{4.6}$$

Since x, y are bounded functions we can choose a number

$$\Delta > \sup\{|x(t)| + |y(t)| : t \in [0, \infty)\} + 2. \tag{4.7}$$

In view of Proposition 2.3, there exists $\delta > 0$ such that for each $T \geq 0$ and each $z_i \in \mathbb{R}^n$, $i = 1, \dots, 4$ satisfying

$$|z_i| \leq \Delta, \quad i = 1, \dots, 4, \quad |z_1 - z_3|, |z_2 - z_4| \leq \delta$$

the following inequality holds:

$$|U^f(T, T+1, z_1, z_2) - U^f(T, T+1, z_3, z_4)| \le \epsilon/8. \tag{4.8}$$

It follows from (4.2) that there exists a sequence $\{T_i\}_{i=1}^{\infty} \subset (0, \infty)$ such that, for $i = 1, 2, \dots,$

$$T_{i+1} \ge T_i + 8, \quad I^f(0, T_i, x) - I^f(0, T_i, y) > 3\epsilon/2. \tag{4.9}$$

Equality (4.6) implies that there exists a natural number j such that

$$|x(T_j) - y(T_j)| \le \delta. \tag{4.10}$$

Consider an a.c. function $\tilde{x} : [0, T_j + 1] \to \mathbb{R}^n$ such that

$$\begin{gathered} \tilde{x}(t) = y(t), \quad t \in [0, T_j], \quad \tilde{x}(T_{j+1}) = x(T_j + 1), \\ I^f(T_j, T_j + 1, \tilde{x}) \le U^f(T_j, T_j + 1, y(T_j), x(T_j + 1)) + \epsilon/8. \end{gathered} \tag{4.11}$$

Relations (4.2) and (4.11) imply that

$$\tilde{x}(0) = x(0), \quad \tilde{x}(T_j + 1) = x(T_j + 1). \tag{4.12}$$

It follows from (4.11) and (4.9) that

$$\begin{aligned} I^f(0, T_j + 1, \tilde{x}) &- I^f(0, T_j + 1, x) \\ &= I^f(0, T_j, \tilde{x}) + I^f(T_j, T_j + 1, \tilde{x}) \\ &\quad - I^f(0, T_j, x) - I^f(T_j, T_j + 1, x) \\ &\le I^f(0, T_j, y) - I^f(0, T_j, x) + U^f(T_j, T_j + 1, y(T_j), x(T_j + 1)) \\ &\quad + \epsilon/8 - U^f(T_j, T_j + 1, x(T_j), x(T_j + 1)) \\ &< -3\epsilon/2 + \epsilon/8 + U^f(T_j, T_j + 1, y(T_j), x(T_j + 1)) \\ &\quad - U^f(T_j, T_j + 1, x(T_j), x(T_j + 1)). \end{aligned} \tag{4.13}$$

By (4.7), (4.10) and the choice of δ (see (4.8))

$$|U^f(T_j, T_j + 1, y(T_j), x(T_j + 1)) - U^f(T_j, T_j + 1, x(T_j), x(T_j + 1))| \le \epsilon/8.$$

Combined with (4.13) this inequality implies that

$$I^f(0, T_j + 1, \tilde{x}) - I^f(0, T_j + 1, x) < -3\epsilon/2 + \epsilon/8 + \epsilon/8 < 0.$$

This contradicts (4.1). The contradiction we have reached proves the proposition. □

Proposition 1.2 and Proposition 4.1 imply the following result.

PROPOSITION 4.2. *Assume that* $f \in \mathscr{M}$, *for each* $(t, x) \in [0, \infty) \times \mathbb{R}^n$ *the function* $f(t, x, \cdot) : \mathbb{R}^n \to \mathbb{R}^1$ *is convex and that property* (P1) *holds* (*see Theorem* 1.3). *Then for each* $z \in \mathbb{R}^n$ *there exists a bounded* (f)*-overtaking optimal function* $Z : [0, \infty) \to \mathbb{R}^n$ *satisfying* $Z(0) = z$.

PROPOSITION 4.3. *Let* $f \in \mathscr{M}$ *and assume that property* (P1) *holds* (*Theorem* 1.3). *Assume that* $v_1, v_2 : [0, \infty) \to \mathbb{R}^n$ *are bounded a.c. functions,* v_1 *is* (f)*-overtaking optimal,* $T_0 > 0$,

$$v_1(t) = v_2(t), \quad t \in [0, T_0] \tag{4.14}$$

and

$$I^f(T_0, \tau, v_2) = U^f(T_0, \tau, v_2(T_0), v_2(\tau)) \quad \textit{for each } \tau > T_0. \tag{4.15}$$

Then v_2 *is an* (f)*-overtaking optimal function.*

PROOF. Clearly v_1 is an (f)-good function. We will show that v_2 is an (f)-good function. Choose a number

$$M_0 > \sup\{|v_i(t)| : t \in [0, \infty),\ i = 1, 2\}. \tag{4.16}$$

By Proposition 2.6 there is $M_1 > 0$ such that

$$M_1 > \sup\left\{|U^f(t_1, t_2, y, z)| \;\middle|\; \begin{array}{l} t_1 \geq 0,\ t_2 \in [t_1 + 1/8, t_1 + 8], \\ y, z \in \mathbb{R}^n,\ |y|, |z| \leq M_0 + 2 \end{array}\right\}. \tag{4.17}$$

Let $\tau \geq T_0 + 2$. Consider an a.c. function $u : [0, \infty) \to \mathbb{R}^n$ such that

$$\begin{gathered} u(t) = v_1(t), \quad t \in [0, \tau - 1], \quad u(\tau) = v_2(\tau), \\ I^f(\tau - 1, \tau, u) \leq U^f(\tau - 1, \tau, v_1(\tau - 1), v_2(\tau)) + 1. \end{gathered} \tag{4.18}$$

Relations (4.18) and (4.14) imply that

$$u(T_0) = v_2(T_0), \quad u(\tau) = v_2(\tau). \tag{4.19}$$

By (4.14), (4.15), (4.18) and (4.19),

$$I^f(0, \tau, u) - I^f(0, \tau, v_2) = I^f(T_0, \tau, u) - I^f(T_0, \tau, v_2) \geq 0. \tag{4.20}$$

In view of (4.16)–(4.18),

$$\begin{aligned} I^f(0, \tau, u) - I^f(0, \tau, v_1) &= I^f(\tau - 1, \tau, u) - I^f(\tau - 1, \tau, v_1) \\ &\leq U^f(\tau - 1, \tau, v_1(\tau - 1), v_2(\tau)) + 1 \\ &\quad - U^f(\tau - 1, \tau, v_1(\tau - 1), v_1(\tau)) \\ &\leq 2M_1. \end{aligned}$$

Combined with (4.20) this relation implies that

$$I^f(0, \tau, v_2) \le I^f(0, \tau, u) \le I^f(0, \tau, v_1) + 2M_1$$

and

$$I^f(0, \tau, v_2) \le I^f(0, \tau, v_1) + 2M_1 \quad \text{for any } \tau > T_0 + 2.$$

Since v_1 is (f)-good we conclude that v_2 is an (f)-good function. By property (P1)

$$\lim_{t\to\infty} |v_2(t) - v_1(t)| = 0. \tag{4.21}$$

Since v_1 is (f)-overtaking optimal we have

$$\limsup_{T\to\infty} [I^f(0, T, v_1) - I^f(0, T, v_2)] \le 0. \tag{4.22}$$

We show that

$$\limsup_{T\to\infty} [I^f(0, T, v_2) - I^f(0, T, v_1)] \le 0. \tag{4.23}$$

Let $\epsilon > 0$. By Proposition 2.3 there is $\delta > 0$ such that for each $t \ge 0$, each $y_i, z_i \in \mathbb{R}^n$, $i = 1, 2$, satisfying

$$|y_i|, |z_i| \le M_0 + 1, \quad i = 1, 2, \quad |y_i - z_i| \le \delta, \quad i = 1, 2, \tag{4.24}$$

the following inequality holds:

$$|U^f(t, t+1, y_1, y_2) - U^f(t, t+1, z_1, z_2)| \le \epsilon/8. \tag{4.25}$$

In view of (4.21), there is $T_1 > T_0 + 4$ such that

$$|v_2(t) - v_1(t)| \le \delta \quad \text{for all } t \in [T_1, \infty). \tag{4.26}$$

Let $T > T_1$ and consider an a.c. function $w : [0, T+1] \to \mathbb{R}^n$ such that

$$\begin{aligned} &w(t) = v_1(t), \quad t \in [0, T], \quad w(T+1) = v_2(T+1), \\ &I^f(T, T+1, w) \le U^f(T, T+1, v_1(T), v_2(T+1)) + \epsilon/8. \end{aligned} \tag{4.27}$$

Relations (4.14) and (4.27) imply that

$$w(t) = v_2(t), \quad t \in [0, T_0], \quad w(T+1) = v_2(T+1). \tag{4.28}$$

By (4.15) and (4.28),

$$\begin{aligned} &I^f(0, T+1, w) - I^f(0, T+1, v_2) \\ &\qquad = I^f(T_0, T+1, w) - I^f(T_0, T+1, v_2) \ge 0. \end{aligned} \tag{4.29}$$

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It follows from (4.16), (4.26), (4.27) and the choice of δ (see (4.24), (4.25)) that

$$(4.30)\qquad \begin{aligned} &I^f(0, T+1, w) - I^f(0, T+1, v_1)\\ &= I^f(T, T+1, w) - I^f(T, T+1, v_1)\\ &\le U^f(T, T+1, v_1(T), v_2(T+1)) + \epsilon/8\\ &\quad - U^f(T, T+1, v_1(T), v_1(T+1))\\ &\le \epsilon/8 + \epsilon/8 = \epsilon/4. \end{aligned}$$

By (4.29)–(4.30), $I^f(0, T+1, v_2) \le I^f(0, T+1, w) \le I^f(0, T+1, v_1) + \epsilon/4$ for any $T > T_1$. This implies (4.23). In view of (4.22) and (4.23),

$$(4.31)\qquad \lim_{T\to\infty} [I^f(0, T, v_1) - I^f(0, T, v_2)] = 0.$$

Since v_1 is (f)-overtaking optimal we conclude that v_2 is (f)-overtaking optimal. The proposition is proved. □

5. (STP) implies (P1), (P2) and (P3)

Assume that $f \in \mathcal{M}$, for each $(t, x) \in [0, \infty) \times \mathbb{R}^n$ the function $f(t, x, \cdot) : \mathbb{R}^n \to \mathbb{R}^1$ is convex, the function f has (STP) and that a bounded a.c. function $X_f : [0, \infty) \to \mathbb{R}^n$ is the turnpike of f. In [14, Section 4] we showed that properties (P1) and (P3) hold. Now we show that (P2) holds.

By Proposition 4.2 there exists an (f)-overtaking optimal function $v : [0, \infty) \to \mathbb{R}^n$ such that $v(0) = X_f(0)$. Let $v : [0, \infty) \to \mathbb{R}^n$ be any (f)-overtaking optimal function satisfying $v(0) = X_f(0)$. In view of Proposition 2.2, v is bounded. Then it follows from (STP) that $v(t) = X_f(t)$, $t \in [0, \infty)$.

6. Basic lemma

Assume that $f \in \mathcal{M}$, for each $(t, x) \in [0, \infty) \times \mathbb{R}^n$ the function $f(t, x, \cdot) : \mathbb{R}^n \to \mathbb{R}^1$ is convex, $X_f : [0, \infty) \to \mathbb{R}^n$ is a bounded a.c. function and assume that properties (P1), (P2) and (P3) hold. In [14, Lemma 5.1] we proved the following important lemma.

LEMMA 6.1. *For each $\epsilon > 0$, there exist $T_0 > 0$, $\delta_0 > 0$ such that the following property holds: If $T_1 \ge T_0$, $T_2 \ge T_1 + 1$ and if an a.c. function $u : [T_1, T_2] \to \mathbb{R}^n$ satisfies, for $i = 1, 2$,*

$$|u(T_i) - X_f(T_i)| \le \delta_0 \quad \text{and} \quad I^f(T_1, T_2, u) \le U^f(T_1, T_2, u(T_1), u(T_2)) + \delta_0,$$

then $|u(t) - X_f(t)| \le \epsilon$, for $t \in [T_1, T_2]$.

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Now we prove our basic lemma.

LEMMA 6.2 (Basic Lemma). *Let* $\epsilon > 0$. *Then there exists* $\delta > 0$ *such that for each* $T_1 \geq 0$, $T_2 \geq T_1 + 1$ *and each a.c. function* $v : [T_1, T_2] \to \mathbb{R}^n$ *satisfying*

$$(6.1)\quad |v(T_i) - X_f(T_i)| \leq \delta \quad \text{and} \quad I^f(T_1, T_2, v) \leq U^f(T_1, T_2, v(T_1), v(T_2)) + \delta,$$

for $i = 1, 2$, *the following inequality holds*:

$$(6.2)\qquad |X_f(t) - v(t)| \leq \epsilon, \quad t \in [T_1, T_2].$$

PROOF. By Lemma 6.1, there exist $\tau_0, \delta_0 \in (0, \epsilon/16)$ such that the following property holds:

(P4) If $T_1 \geq \tau_0$, $T_2 \geq T_1 + 1$ and an a.c. function $v : [T_1, T_2] \to \mathbb{R}^n$ satisfies $|v(T_i) - X_f(T_i)| \leq \delta_0$, $i = 1, 2$, and $I^f(T_1, T_2, v) \leq U^f(T_1, T_2, v(T_1), v(T_2)) + \delta_0$, then $|v(t) - X_f(t)| \leq \epsilon$, $t \in [T_1, T_2]$.

We may assume without loss of generality that $\delta_0 < 1$. Choose

$$(6.3)\qquad M_0 > 4 + \sup\{|X_f(t)| : t \in [0, \infty)\}.$$

By Proposition 2.4 there exists a number $M_1 > 1$ such that for each $T_1 \geq 0$, $T_2 \geq T_1 + 8^{-1}$ and each a.c. function $v : [T_1, T_2] \to \mathbb{R}^n$ satisfying $(i = 1, 2)$

$$(6.4)\qquad |v(T_i)| \leq M_0 + 4 \quad \text{and} \quad I^f(T_1, T_2, v) \leq U^f(T_1, T_2, v(T_1), v(T_2)) + 4$$

the following inequality holds:

$$(6.5)\qquad |v(t)| \leq M_1, \quad t \in [T_1, T_2].$$

In view of property (P3), there exist $\delta_1 \in (0, \min\{1, \delta_0\})$ and $L_1 > 0$ such that the following property holds:

(P5) For each $T \geq 0$ and each a.c. function $v : [T, T + L_1] \to \mathbb{R}^n$ which satisfies

$$(6.6)\qquad \begin{aligned} &|v(T)|, |v(T + L_1)| \leq M_1 + 4, \\ &I^f(T, T + L_1, v) \leq U^f(T, T + L_1, v(T), v(T + L_1)) + \delta_1, \end{aligned}$$

there is $\tau \in [T, T + L_1]$ for which

$$(6.7)\qquad |X_f(\tau) - v(\tau)| \leq \delta_0.$$

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Consider a sequence $\{\delta_i\}_{i=1}^{\infty} \subset (0, 1)$ such that

(6.8) $$\delta_i < 2^{-1}\delta_{i-1}, \quad i = 2, 3, \ldots$$

Assume that the lemma is wrong. Then for each natural number i there exist $T_{i1} \geq 0$, $T_{i2} \geq T_{i1} + 1$, an a.c. function $v_i : [T_{i1}, T_{i2}] \to \mathbb{R}^n$ such that

(6.9) $$|X_f(T_{ij}) - v_i(T_{ij})| \leq \delta_i, \quad j = 1, 2,$$
$$I^f(T_{i1}, T_{i2}, v_i) \leq U^f(T_{i1}, T_{i2}, v_i(T_{i1}), v_i(T_{i2})) + \delta_i$$

and $t_i \in [T_{i1}, T_{i2}]$ for which

(6.10) $$|X_f(t_i) - v_i(t_i)| > \epsilon.$$

Let i be a natural number. It follows from property (P4), (6.9), (6.10), (6.8) and (6.5) that

(6.11) $$T_{i1} < \tau_0.$$

By (6.9), (6.8), (6.3) and the choice of M_1 (see (6.4), (6.5)),

(6.12) $$|v_i(T)| \leq M_1, \quad t \in [T_{i1}, T_{i2}].$$

We show that $t_i \leq \tau_0 + L_1 + 2$. Assume the contrary. Then

(6.13) $$t_i > \tau_0 + L_1 + 2.$$

Consider the restriction of v_i to the interval

(6.14) $$[t_i - L_1 - 1, t_i - 1] \subset (\tau_0 + 1, \infty).$$

Property (P5), (6.14), (6.12), (6.9) and (6.8) imply that there is

(6.15) $$\hat{t} \in [t_i - L_1 - 1, t_i - 1]$$

such that

(6.16) $$|X_f(\hat{t}) - v_i(\hat{t})| \leq \delta_0.$$

By (6.15) and (6.13), $\hat{t} > \tau_0 + 1$, $T_{i2} - \hat{t} \geq 1$. It follows from these inequalities, (6.16), (6.9), (6.8) and property (P4) that $|v_i(t) - X_f(t)| \leq \epsilon$, $t \in [\hat{t}, T_{i2}]$. Combined with (6.15) this inequality implies that $|v_i(t_i) - X_f(t_i)| \leq \epsilon$, a contradiction. The contradiction we have reached proves that

(6.17) $$t_i \leq \tau_0 + L_1 + 2.$$

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Extracting if it is necessary a subsequence and re-indexing we may assume without loss of generality that there exist

$$\tilde{T}_1 = \lim_{i\to\infty} T_{i1} \in [0, \tau_0], \quad \tilde{t} = \lim_{i\to\infty} t_i \in [\tilde{T}_1, \tau_0 + L_1 + 2], \tag{6.18}$$
$$\tilde{T}_2 = \lim_{i\to\infty} T_{i2} \in [\tilde{t}, \infty]$$

(see (6.11), (6.17)).

It follows from (A.ii), (6.9), (6.12) and Proposition 2.6 that for each $\tau_1 \in (\tilde{T}_1, \tilde{T}_2)$, $\tau_2 \in (\tau_1, \tilde{T}_2)$ the sequence $\{I^f(\tau_1, \tau_2, v_i)\}_{i=1}^{\infty}$ is bounded.

By lower semicontinuity results [1], we may assume that there exists a function $\hat{v} : (\tilde{T}_1, \tilde{T}_2) \to \mathbb{R}^n$ such that the following property holds:

(P6) For each $\tau_1 \in (\tilde{T}_1, \tilde{T}_2)$, $\tau_2 \in (\tau_1, \tilde{T}_2)$, the function $\hat{v}$ is a.c. on $[\tau_1, \tau_2]$, $v_i(t) \to \hat{v}(t)$ as $i \to \infty$ uniformly in $t \in [\tau_1, \tau_2]$, $v_i' \to \hat{v}'$ as $i \to \infty$ weakly in $L^1([\tau_1, \tau_2]; \mathbb{R}^n)$, and $I^f(\tau_1, \tau_2, \hat{v}) \le \liminf_{i\to\infty} I^f(\tau_1, \tau_2, v_i)$.

We show that $X_f(\tilde{T}_1) = \lim_{t\to\tilde{T}_1^+} \hat{v}(t)$. Let $\Delta > 0$. By Proposition 2.5, Proposition 2.6, (6.9) and (6.12) there is $\gamma \in (0, 1/8)$ such that the following properties hold: For each integer $i \ge 1$ and each $t_1, t_2 \in [T_{i1}, T_{i2}]$ satisfying $|t_1 - t_2| \le 4\gamma$ we have

$$|v_i(t_1) - v_i(t_2)| \le \Delta. \tag{6.19}$$

For each $t_1, t_2 \in [0, \infty)$ satisfying $|t_1 - t_2| \le 4\gamma$, we have

$$|X_f(t_1) - X_f(t_2)| \le \Delta. \tag{6.20}$$

Let $\tau \in (\tilde{T}_1, \tilde{T}_1 + \gamma)$. Then for all sufficiently large natural numbers i

$$T_{i1} < \tau < \tilde{T}_1 + \gamma < T_{i1} + 2\gamma \tag{6.21}$$

and in view of the choice of γ

$$|v_i(\tau) - v_i(T_{i1})| \le \Delta. \tag{6.22}$$

It follows from (6.21), (6.22) and (6.9) that for all sufficiently large natural numbers i

$$|v_i(\tau) - X_f(T_{i1})| \le |v_i(\tau) - v_i(T_{i1})| + |v_i(T_{i1}) - X_f(T_{i1})| \le \Delta + \delta_i. \tag{6.23}$$

By the choice of γ, (6.20) and (6.18) for all sufficiently large natural numbers i,

$$|X_f(\tilde{T}_1) - X_f(T_{i1})| \le \Delta.$$

Combined with (6.23) this inequality implies that for all sufficiently large natural numbers i

$$|v_i(\tau) - X_f(\tilde{T}_1)| \le |v_i(\tau) - X_f(T_{i1}| + |X_f(T_{i1}) - X_f(\tilde{T}_1)| \le \Delta + \delta_i + \Delta.$$

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Thus by (P6) and (6.8)

$$|\hat{v}(\tau) - X_f(\tilde{T}_1)| = \lim_{i\to\infty} |v_i(\tau) - X_f(\tilde{T}_1)| \le \lim_{i\to\infty} 2\Delta + \delta_i = 2\Delta.$$

We have shown that for each $\tau \in (\tilde{T}_1, \tilde{T}_1 + \gamma)$, $|\hat{v}(\tau) - X_f(\tilde{T}_1)| \le 2\Delta$. Since Δ is an arbitrary positive number we conclude that

$$X_f(\tilde{T}_1) = \lim_{\tau\to\tilde{T}_1^+} \hat{v}(\tau). \tag{6.24}$$

Analogously we can show that if $\tilde{T}_2 < \infty$, then

$$X_f(\tilde{T}_2) = \lim_{\tau\to\tilde{T}_2^-} \hat{v}(t). \tag{6.25}$$

We set $\hat{v}(\tilde{T}_1) = X_f(\tilde{T}_1)$ and if $\tilde{T}_2 < \infty$, then $\hat{v}(\tilde{T}_2) = X_f(\tilde{T}_2)$. It follows from (P6), (6.9), (6.8) and Proposition 2.3 that for each $S_1, S_2 \in (\tilde{T}_1, \tilde{T}_2)$ satisfying $S_1 < S_2$

$$\begin{aligned} I^f(S_1, S_2, \hat{v}) &\le \liminf_{i\to\infty} I^f(S_1, S_2, v_i) \\ &\le \liminf_{i\to\infty} [U^f(S_1, S_2, v_i(S_1), v_i(S_2)) + \delta_i] \\ &= \liminf_{i\to\infty} U^f(S_1, S_2, v_i(S_1), v_i(S_2)) = U^f(S_1, S_2, \hat{v}(S_1), \hat{v}(S_2)) \end{aligned}$$

and

$$I^f(S_1, S_2, \hat{v}) = U^f(S_1, S_2, \hat{v}(S_1), \hat{v}(S_2)). \tag{6.26}$$

By (P6), (6.12), (6.24) and (6.25), $\hat{v}$ is bounded. It follows from Proposition 2.7, (P6) and (6.24)–(6.26) that $\hat{v}$ is a.c. function on $[\tilde{T}_1, \tau)$ for each real $\tau \le \tilde{T}_2$ and that the following properties hold:

$$I^f(\tilde{T}_1, \tau, \hat{v}) = U^f(\tilde{T}_1, \tau, \hat{v}(T_1), \hat{v}(\tau)) \tag{6.27}$$

for each $\tau \in (\tilde{T}_1, \tilde{T}_2]$ if $\tilde{T}_2 < \infty$; and equality (6.27) holds for each $\tau > \tilde{T}_1$ if $\tilde{T}_2 = \infty$.

We will show that $\hat{v}(\tilde{t}) \ne X_f(\tilde{t})$. It follows from Proposition 2.6 that there is $M_2 > 0$ such that

$$\sup\left\{ |U^f(s_1, s_2, x, y)| \;\middle|\; \begin{array}{l} s_1 \ge 0,\ s_2 \in [s_1 + 8^{-1}, s_1 + 8], \\ x, y \in \mathbb{R}^n,\ |x|, |y| \le M_1 + 2 \end{array} \right\} + 4 < M_2. \tag{6.28}$$

Relations (6.28), (6.12) and (6.9) imply that the following property holds:

(P7) For each integer $i \ge 1$, each $s_1, s_2 \in [T_{i1}, T_{i2}]$ satisfying $s_2 \in [s_1 + 8^{-1}, s_1 + 8]$, $I^f(s_1, s_2, v_i) < M_2$.

In view of property (P2), Proposition 2.6 and (6.3),

$$\sup\{I^f(s_1, s_2, X_f) : s_1 \geq 0,\ s_2 \in [s_1 + 8^{-1}, s_1 + 8]\} < \infty. \tag{6.29}$$

By (P7), (6.29) and Proposition 2.5, there exists a positive number

$$\gamma < \min\{1, \tilde{T}_2 - \tilde{T}_1\}/32$$

such that the following properties hold:

(P8) For each $s_1, s_2 \geq 0$ satisfying $|s_1 - s_2| \leq \gamma$, $|X_f(s_1) - X_f(s_2)| \leq \epsilon/64$.

(P9) For each integer $i \geq 1$, each $s_1, s_2 \in [T_{i1}, T_{i2}]$ satisfying $|s_1 - s_2| \leq \gamma$,

$$|v_i(s_1) - v_i(s_2)| \leq \epsilon/64.$$

Let i be a natural number. We show that $t_i - T_{i1} > \gamma$. Assume the contrary. Then $t_i - T_{i1} \leq \gamma$ and by properties (P8) and (P9)

$$|X_f(t_i) - X_f(T_{i1})|,\ |v_i(t_i) - v_i(T_{i1})| \leq \epsilon/64.$$

Combining with (6.9), these inequalities imply that

$$\begin{aligned} |X_f(t_i) - v_i(t_i)| &\leq |X_f(t_i) - X_f(T_{i1})| + |X_f(T_{i1}) - v_i(T_{i1})| + |v_i(T_{i1}) - v_i(t_i)| \\ &\leq \epsilon/64 + \delta_i + \epsilon/64 \leq \epsilon/32 + \delta_0 < \epsilon/32 + \epsilon/16 \\ &< \epsilon/2, \end{aligned}$$

This inequality contradicts (6.10). The obtained contradiction proves that

$$t_i - T_{i1} > \gamma. \tag{6.30}$$

Analogously we can show that

$$T_{i2} - t_i > \gamma. \tag{6.31}$$

It follows from (6.30), (6.31), (6.17), (6.18) and property (P6) that

$$\lim_{i \to \infty} |v_i(t_i) - \hat{v}(t_i)| = 0.$$

Combined with (6.18) this equality implies that

$$\lim_{i \to \infty} v_i(t_i) = \hat{v}(\tilde{t}). \tag{6.32}$$

By (6.18), $\lim_{i \to \infty} X_f(t_i) = X_f(\tilde{t})$. Combined with (6.32) and (6.10) this equality implies that

$$|\hat{v}(\tilde{t}) - X_f(\tilde{t})| = \lim_{i \to \infty} |v_i(t_i) - X_f(t_i)| \geq \epsilon. \tag{6.33}$$

Thus

$$\hat{v}(\tilde{t}) \neq X_f(\tilde{t}). \tag{6.34}$$

There are two cases: (1) $\tilde{T}_2 = \infty$; (2) $\tilde{T}_2 < \infty$. Assume that $\tilde{T}_2 < \infty$. Then (6.27) holds for each $\tau \in (\tilde{T}_1, \tilde{T}_2)$. By Proposition 2.7, $\hat{v} : [\tilde{T}_1, \tilde{T}_2] \to \mathbb{R}^n$ is an a.c. function and

$$I^f(\tilde{T}_1, \tilde{T}_2, \hat{v}) = U^f(\tilde{T}_1, \tilde{T}_2, \hat{v}(\tilde{T}_1), \hat{v}(\tilde{T}_2)). \tag{6.35}$$

We have $X_f(\tilde{T}_i) = \hat{v}(\tilde{T}_i)$, $i = 1, 2$. Define an a.c. function $u : [0, \infty) \to \mathbb{R}^n$ by

$$u(t) = \begin{cases} X_f(t), & t \in [0, \infty) \setminus (\tilde{T}_1, \tilde{T}_2), \\ \hat{v}(t), & t \in (T_1, T_2). \end{cases}$$

Clearly u is well defined. By property (P2) and (6.35), $\hat{v}$ is (f)-overtaking optimal. On the other hand, $u(0) = X_f(0)$ and $u(\tilde{t}) = \hat{v}(\tilde{t}) \neq X_f(\tilde{t})$. This contradicts property (P2). Thus case (2) does not hold and $\tilde{T}_2 = \infty$.

For each $t \geq 0$ satisfying $t < \tilde{T}_1$, set $\hat{v}(t) = X_f(t)$. Now (6.27) holds for each $\tau > \tilde{T}_1$. It follows from this fact, the boundedness of $\hat{v}$, the equality $\hat{v}(T_1) = X_f(\tilde{T}_1)$ and Proposition 4.3 that $\hat{v}$ is (f)-overtaking optimal. Now (6.34) contradicts property (P2). The obtained contradiction proves the lemma. □

7. Proof of Theorem 1.3

In this section we prove the following theorem which is an extension of Theorem 1.3.

THEOREM 7.1. *Let $f \in \mathcal{M}$, for each $(t, x) \in [0, \infty) \times \mathbb{R}^n$ the function $f(t, x, \cdot) : \mathbb{R}^n \to \mathbb{R}^1$ is convex and let $X_f : [0, \infty) \to \mathbb{R}^n$ be a bounded a.c. function. Assume that properties* (P1)–(P3) *from Theorem* 1.3 *hold.*

Then for each $K, \epsilon > 0$ there exist $\delta, L > 0$ and a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$ such that the following property holds: For each $g \in \mathcal{U}$, each $T_1 \geq 0$, $T_2 \geq T_1 + 2L$ and each a.c. function $v : [T_1, T_2] \to \mathbb{R}^n$ which satisfies

$$|v(T_1)|, |v(T_2)| \leq K, \;\; I^g(T_1, T_2, v) \leq U^g(T_1, T_2, v(T_1), v(T_2)) + \delta, \tag{7.1}$$

there exist $\tau_1 \in [T_1, T_1 + L]$, $\tau_2 \in [T_2 - L, T_2]$ such that $|v(t) - X_f(t)| \leq \epsilon$, $t \in [\tau_1, \tau_2]$. Moreover, if $|v(T_1) - X_f(T_1)| \leq \delta$, then $\tau_1 = T_1$, and if $|v(T_2) - X_f(T_2)| \leq \delta$, then $T_2 = \tau_2$, PROOF. Let $K, \epsilon > 0$. By Lemma 6.2 there exist $\delta_0 \in (0, 1)$ such that the following property holds:

(C1) If $T_1 \geq 0$, $T_2 \geq T_1 + 1$ and if an a.c. function $v : [T_1, T_2] \to \mathbb{R}^n$ satisfies

$$|v(T_i) - X_f(T_i)| \leq \delta_0, \quad i = 1, 2, \quad I^f(T_1, T_2, v) \leq U^f(T_1, T_2, v(T_1), v(T_2)) + \delta_0,$$

then $|v(t) - X_f(t)| \leq \epsilon$, $t \in [T_1, T_2]$.

By Proposition 2.4, there exist a number

$$M_0 > K + 2 + \sup\{|X_f(t)| : t \in [0, \infty)\} \tag{7.2}$$

and a neighborhood $\mathcal{U}_0$ of f in $\mathcal{M}$ such that the following property holds:

(C2) For each $g \in \mathcal{U}_0$, each $T_1 \geq 0$, $T_2 \geq T_1 + 1$ and each a.c. function $v : [T_1, T_2] \to \mathbb{R}^n$ which satisfies

$$|v(T_i)| \leq K + 2 + \sup\{|X_f(t)| : t \in [0, \infty)\}, \quad i = 1, 2,$$
$$I^g(T_1, T_2, v) \leq U^g(T_1, T_2, v(T_1), v(T_2)) + 4$$

the inequality $|v(t)| \leq M_0$ holds for all $t \in [T_1, T_2]$.

In view of property (P3), there exist $\delta_1 \in (0, \delta_0)$, $L_1 > 0$ such that the following property holds:

(C3) For each $T \geq 0$ and each a.c. function $w : [T, T + L_1] \to \mathbb{R}^n$ which satisfies

$$|w(T)|, |w(T + L_1)| \leq M_0 + 4,$$
$$I^f(T, T + L_1, w) \leq U^f(T, T + L_1, w(T), w(T + L_1)) + \delta_1,$$

there is $\tau \in [T, T + L_1]$ for which $|X_f(\tau) - w(\tau)| \leq \delta_0$.

Proposition 2.8 implies that there exists a neighborhood $\mathcal{U}_1$ of f in $\mathcal{M}$ such that the following property holds:

(C4) For each $T_1 \geq 0$, $T_2 \in [T_1 + L_1, T_1 + 8(L_1 + 1)]$, each $g \in \mathcal{U}_1$, each $x, y \in \mathbb{R}^n$ satisfying $|x|, |y| \leq M_0 + 4$, $|U^g(T_1, T_2, x, y) - U^f(T_1, T_2, x, y)| \leq \delta_1/32$.

By Proposition 2.6, there exists a number $M_1 > 0$ such that

$$\sup\left\{|U^f(T_1, T_2, x, y)| \;\middle|\; \begin{array}{l} T_1 \geq 0, \;\; T_2 \in [T_1 + 1, T_1 + 8(L_1 + 1)], \\ x, y \in \mathbb{R}^n, \;\; |x|, |y| \leq M_0 + 4 \end{array}\right\} \leq M_1. \tag{7.3}$$

It follows from Proposition 2.9 that there exists a neighborhood $\mathcal{U}_2$ of f in $\mathcal{M}$ such that the following property holds:

(C5) For each $T_1 \geq 0$, $T_2 \in [T_1 + L_1, T_1 + 8(L_1 + 1)]$, each $g \in \mathcal{U}_2$ and each a.c. function $v : [T_1, T_2] \to \mathbb{R}^n$ satisfying

$$\min\{I^f(T_1, T_2, v), \;\; I^g(T_1, T_2, v)\} \leq M_1 + 8,$$

the inequality $|I^f(T_1, T_2, v) - I^g(T_1, T_2, v)| \le \delta_1/32$ holds.

Define

$$\mathcal{U} = \mathcal{U}_0 \cap \mathcal{U}_1 \cap \mathcal{U}_2, \tag{7.4}$$

choose a positive number $\delta < \min\{\epsilon, \delta_0, \delta_1\}/32$ and set

$$L = 8 + 6L_1. \tag{7.5}$$

Assume that $g \in \mathcal{U}$, $T_1 \ge 0$, $T_2 \ge T_1 + 2L$ and an a.c. function $v : [T_1, T_2] \to \mathbb{R}^n$ satisfies ($i = 1, 2$)

$$|v(T_i)| \le K \quad \text{and} \quad I^g(T_1, T_2, v) \le U^g(T_1, T_2, v(T_1), v(T_2)) + \delta. \tag{7.6}$$

By (7.6), (7.4) and property (C2),

$$|v(t)| \le M_0, \quad t \in [T_1, T_2]. \tag{7.7}$$

Let

$$s_1, s_2 \in [T_1, T_2], \quad s_2 - s_1 \in [L_1, 8(L_1 + 1)]. \tag{7.8}$$

It follows from (7.7), (7.4) and property (C4) that

$$|U^g(s_1, s_2, v(s_1), v(s_2)) - U^f(s_1, s_2, v(s_1), v(s_2))| \le \delta_1/32. \tag{7.9}$$

Relations (7.3), (7.7) and (7.8) imply that $U^f(s_1, s_2, v(s_1), v(s_2)) \le M_1$. Combined with (7.9) this inequality implies that $U^g(s_1, s_2, v(s_1), v(s_2)) \le M_1 + \delta_1/32$. In view of this inequality and (7.6),

$$I^g(s_1, s_2, v) \le U^g(s_1, s_2, v(s_1), v(s_2)) + \delta \le M_1 + \delta_1/32 + \delta. \tag{7.10}$$

By (7.10), (7.8), (7.4) and property (C5), $|I^f(s_1, s_2, v) - I^g(s_1, s_2, v)| \le \delta_1/32$. It follows from this inequality, (7.10), (7.9) and the choice of δ that

$$\begin{aligned} I^f(s_1, s_2, v) &\le I^g(s_1, s_2, v) + \delta_1/32 \le U^g(s_1, s_2, v(s_1), v(s_2)) + \delta + \delta_1/32 \\ &\le U^f(s_1, s_2, v(s_1), v(s_2)) + \delta_1/32 + \delta + \delta_1/32 \end{aligned}$$

and

$$I^f(s_1, s_2, v) \le U^f(s_1, s_2, v(s_1), v(s_2)) + 3\delta_1/32. \tag{7.11}$$

We have shown that the following property holds:

(C6) Inequality (7.11) is valid for each s_1, s_2 satisfying (7.8).

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Assume that

$$\tau \in [T_1 + L_1 + 1, T_2 - L_1 - 1]. \tag{7.12}$$

Relations (7.12) and (7.5) imply that $\tau - 1 - L_1, \tau + 1 + L_1 \in [T_1, T_2]$. By property (C6)

$$\begin{aligned} &I^f(\tau - 1 - L_1, \tau - 1, v) \\ &\quad \le U^f(\tau - 1 - L_1, \tau - 1, v(\tau - 1 - L_1), v(\tau - 1)) + 3\delta_1/32, \end{aligned} \tag{7.13}$$

$$\begin{aligned} &I^f(\tau + 1, \tau + 1 + L_1, v) \\ &\quad \le U^f(\tau + 1, \tau + 1 + L_1, v(\tau + 1), v(\tau + 1 + L_1)) + 3\delta_1/32. \end{aligned} \tag{7.14}$$

It follows from (7.13), (7.14), (7.7) and property (C3) that there exist

$$t_1 \in [\tau - 1 - L_1, \tau - 1], \ t_2 \in [\tau + 1, \tau + 1 + L_1] \tag{7.15}$$

such that

$$|X_f(t_i) - v(t_i)| \le \delta_0, \ i = 1, 2. \tag{7.16}$$

Property (C6) implies that

$$\begin{aligned} &I^f(\tau - 1 - L_1, \tau + 1 + L_1, v) \\ &\quad \le U^f(\tau - 1 - L_1, \tau + 1 + L_1, v(\tau - 1 - L_1), v(\tau + 1 + L_1)) + 3\delta_1/32. \end{aligned}$$

Together with (7.15) this inequality implies that

$$I^f(t_1, t_2, v) \le U^f(t_1, t_2, v(t_1), v(t_2)) + 3\delta_1/32. \tag{7.17}$$

It follows from (7.15)–(7.17) and property (C1) that $|v(t) - X_f(t)| \le \epsilon$, $t \in [t_1, t_2]$ and

$$|v(\tau) - X_f(\tau)| \le \epsilon. \tag{7.18}$$

We have shown that the following property holds:

(C7) Inequality (7.18) is true for each $\tau \in [T_1 + L_1 + 1, T_2 - L_1 - 1]$. (Note that $[T_1 + L, T_2 - L] \subset [T_1 + L_1 + 1, T_2 - L_1 - 1]$.)

Assume that

$$|v(T_1) - X_f(T_1)| \le \delta. \tag{7.19}$$

Let $\tau = T_1 + L_1 + 1$. We have shown that there is

$$t_2 \in [T_1 + L_1 + 1, T_1 + L_1 + 1 + 1 + L_1] \tag{7.20}$$

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such that

$$(7.21)\qquad |X_f(t_2) - v(t_2)| \le \delta_0$$

(see (7.15), (7.16)).

By property (C6),

$$I^f(T_1, T_1 + 2L_1 + 2, v) \le U^f(T_1, T_1 + 2L_1 + 2, v(T_1), v(T_1 + 2L_1 + 2)) + 3\delta_1/2.$$

Together with (7.20) this inequality implies that

$$(7.22)\qquad I^f(T_1, t_2, v) \le U^f(T_1, t_2, v(T_1), v(t_2)) + 3\delta_1/2.$$

It follows from (7.19)–(7.22) and property (C1) that $|v(t) - X_f(t)| \le \epsilon$, $t \in [T_1, t_2]$ (note that $[T_1, T_1 + L_1 + 1] \subset [T_1, t_2]$). Together with property (C7) this implies that (7.18) is true for each τ belonging to the interval $[T_1, T_2 - L_1 - 1]$ which contains $[T_1, T_2 - L]$.

Assume that

$$(7.23)\qquad |v(T_2) - X_f(T_2)| \le \delta.$$

Let $\tau = T_2 - L_1 - 1$. We have shown (see (7.15), (7.16)) that there is

$$(7.24)\qquad t_1 \in [T_2 - L_1 - 2 - L_1, T_2 - L_1 - 2]$$

such that

$$(7.25)\qquad |X_v(t_1) - v(t_1)| \le \delta_0.$$

By (7.24) and Property (C6),

$$(7.26)\qquad I^f(t_1, T_2, v) \le U^f(t_1, T_2, v(t_1), v(T_2)) + 3\delta_1/2.$$

It follows from (7.23)–(7.26) and property (C1) that $|v(t) - X_f(t)| \le \epsilon$ for any t in the interval $[t_1, T_2]$ which contains $[T_2 - L_1 - 2, T_2]$. Together with property (C7) this implies that (7.18) is true for each τ in the interval $[T_1 + L_1 + 1, T_2]$ which contains $[T_1 + L, T_2]$. This completes the proof of theorem. □

References

[1] L. D. Berkovitz, 'Lower semicontinuity of integral functionals', *Trans. Amer. Math. Soc.* **192** (1974), 51–57.

[2] Z. Dzalilov, A. F. Ivanov and A. M. Rubinov, 'Difference inclusions with delay of economic growth', *Dynam. Systems Appl.* **10** (2001), 283–293.
[3] Z. Dzalilov, A. M. Rubinov and P. E. Kloeden, 'Lyapunov sequences and a turnpike theorem without convexity', *Set-Valued Analysis* **6** (1998), 277–302.
[4] D. Gale, 'On optimal development in a multisector economy', *Rev. Econom. Stud.* **34** (1967), 1–19.
[5] V. L. Makarov, M. J. Levin and A. M. Rubinov, *Mathematical economic theory: pure and mixed types of economic mechanisms* (North-Holland, Amsterdam, 1995).
[6] V. L. Makarov and A. M. Rubinov, *Mathematical theory of economic dynamics and equilibria* (Nauka, Moscow, 1973); English translation (Springer, New York, 1977).
[7] M. A. Mamedov and S. Pehlivan, 'Statistical convergence of optimal paths', *Math. Japon.* **52** (2000), 51–55.
[8] ———, 'Statistical cluster points and turnpike theorem in nonconvex problems', *J. Math. Anal. Appl.* **256** (2001), 686–693.
[9] L. W. McKenzie, 'Turnpike theory', *Econometrica* **44** (1976), 841–866.
[10] R. Radner, 'Path of economic growth that are optimal with regard only to final states; a turnpike theorem', *Rev. Econom. Stud.* **28** (1961), 98–104.
[11] A. M. Rubinov, 'Economic dynamics', *J. Soviet Math.* **26** (1984), 1975–2012.
[12] P. A. Samuelson, 'A catenary turnpike theorem involving consumption and the golden rule', *American Economic Review* **55** (1965), 486–496.
[13] A. J. Zaslavski, 'Existence and uniform boundedness of approximate solutions of variational problems without convexity assumptions', *Dynam. Systems Appl.* **13** (2004), 161–178.
[14] ———, 'The structure of approximate solutions of variational problems without convexity', *J. Math. Anal. Appl.* **296** (2004), 578–593.
[15] ———, 'The turnpike property for approximate solutions of variational problems without convexity', *Nonlinear Analysis* **58** (2004), 547–569.
[16] ———, 'Existence and uniform boundedness of optimal solutions of variational problems', *Abstract and Appl. Analysis* **3** (1998), 265–292.
[17] ———, 'The turnpike property for extremals of nonautonomous variational problems with vector-valued functions', *Nonlinear Analysis* **42** (2000), 1465–1498.
[18] A. J. Zaslavski and A. Leizarowitz, 'Optimal solutions of linear control systems with nonperiodic integrands', *Math. Oper. Res.* **22** (1997), 726–746.
[19] ———, 'Optimal solutions of linear periodic control systems with convex integrands', *Appl. Math. Optim.* **37** (1998), 127–150.

Department of Mathematics
Technion-Israel Institute of Technology
32000, Haifa
Israel
e-mail: ajzasl@techunix.technion.ac.il

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