Canad. Math. Bull. Vol. 18 (1), 1975

REGULAR ULTRAFILTERS AND LONG ULTRAPOWERS

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The Ultrapower construction, which builds a new structure $\mathbf{A}^I/D$ from a relational structure $\mathbf{A}$ and an ultrafilter D on a set I, is by now a familiar tool in Model Theory and many other branches of mathematics. In this note we present a result that belongs in the theory of ordered sets, i.e. where the relational structure $\mathbf{A}$ has just a single binary relation $<$, which satisfies the axioms for a strict linear order. We assume that the reader is familiar with the definition and standard notation for ultrapowers, as may be found, for example, in [1]. We differ from [1] only in our eschewing of Gothic capitals.

§ We will use $<$ for both the order on $\mathbf{A}$ and the order on $\mathbf{A}^I/D$. By definition we have for all f/D, $g/D \in \mathbf{A}^I/D$

$$f/D < g/D \quad \text{iff} \quad \{i \in I : f(i) < g(i)\} \in D.$$

In general the order properties of $\mathbf{A}^I/D$ will depend on the structure of the ultrafilter D. We will show in this note how one important property of an ultrafilter is determined by the order behaviour of a certain ultrapower.

We work in ordinary set theory with the axiom of choice and employ standard notation and terminology without comment. Greek letters denote (von Neumann) ordinals. Cardinals are taken to be initial ordinals.

If $\mathbf{B}=(B, <)$ is an ordered set and $A \subseteq B$ we say that A is *bounded* in $\mathbf{B}$ if there is some $b \in B$ such that for all $a \in A$ we have $a<b$, otherwise we say that A is *confinal* in $\mathbf{B}$. We define the *length* of an ordered set $\mathbf{B}$ to be the smallest cardinal of a confinal subset of B.

If I is any set, μ is a cardinal, and U is a family of subsets of I we say that U is a *μ-covering of I* if for all $i \in I$

$$|\{X \in U : i \in X\}| < \mu$$

If D is an ultrafilter on I we say that D is *(μ, λ)-regular* if there exists a μ-covering U of I such that $U \subseteq D$ and $|U|=\lambda$. We say that D is *uniform* if $|X|=|I|$ for all $X \in D$. Also D is *ω-incomplete* if for some countable subset U of D, $\cap U \notin D$.

If α is any cardinal we denote by $\boldsymbol{\alpha}$ the ordered set $(\alpha, <)$ where $<$ is the order of size on the (ordinal) members of α. We can now state our main result.

Received by the editors December 20, 1971 and, in revised form, May 5, 1972.

https://doi.org/10.4153/CMB-1975-008-x Published online by Cambridge University Press

THEOREM. *If λ is a regular cardinal and D is a uniform ultrafilter on λ^+ then D is (λ, λ^+)-regular iff every set of $\leq\lambda^+$ elements of $\boldsymbol{\lambda}^{\lambda^+}/D$ is bounded.* (*i.e., iff the length of $\boldsymbol{\lambda}^{\lambda^+}/D$ is greater than λ^+.*)

From, for example, [1, p. 132] we see that if D is $(\omega, |I|)$-regular on I then

$$|A^I/D| = |A^I|.$$

Also from [2, p. 405] we see that if D is an ω-incomplete ultrafilter on I then the length of $\boldsymbol{\omega}^I/D$ is greater than ω.

Take D to be a uniform ultrafilter on $\omega_1=\omega^+$. Then D is ω-incomplete and by our Theorem if D is not regular then the length of $\boldsymbol{\omega}^{\omega_1}/D$ is $\leq\omega_1$. Hence we have:

COROLLARY. *If D is a uniform ultrafilter on ω_1 then either $|\omega^{\omega_1}/D|=2^{\omega_1}$ or the length of $\boldsymbol{\omega}^{\omega_1}/D$ is ω_1.*

§ **Proof of Main Theorem.** Suppose that D is a uniform ultrafilter on λ^+ and that every subset of $\boldsymbol{\lambda}^{\lambda^+}/D$ with cardinality $\leq\lambda^+$ is bounded. Then for each $i \in \lambda^+$ let h_i be a 1-1 map from i into λ-{0}. Set

$$f_\xi(i) = \begin{cases} h_i(\xi) & \text{if } \; i > \xi \\ 0 & \text{if } \; i \leq \xi \end{cases}$$

Now if g/D is an upper bound in $\boldsymbol{\lambda}^{\lambda^+}/D$ for the set $\{f_\xi/D:\xi<\lambda^+\}$ and we take $X_\xi=\{i \in \lambda^+:0<f_\xi(i)<g(i)\}$ it is easy to check that $\{X_\xi:\xi<\lambda^+\}$ is a λ-covering of λ^+ by members of D.

Conversely, assume that D is (λ, λ^+)-regular and that $\{f_\eta/D:\eta<\lambda^+\}$ is any set of λ^+ elements of $\boldsymbol{\lambda}^{\lambda^+}/D$.

Let $\{X_\eta:\eta<\lambda^+\}$ be a λ-covering of λ^+ by members of D. Define functions g_η by

$$g_\eta(i) = \begin{cases} f_\eta(i) & i \in X_\eta \\ 0 & \text{otherwise} \end{cases}$$

Then $g_\eta/D=f_\eta/D$ and as $|\{\eta:i \in X_\eta\}|<\lambda$ and λ is a regular cardinal we have that

$$g(i) = \sup_{\eta<\lambda^+} g_\eta(i) \quad \text{exists.}$$

If $g \in \lambda^{\lambda^+}$ is defined in this way it is clear that g/D is an upper bound in $\boldsymbol{\lambda}^{\lambda^+}/D$ for the set $\{g_\eta/D:\eta<\lambda^+\}=\{f_\eta/D:\eta<\lambda^+\}$.

The assumption that λ is regular is only needed for the second part of the proof. We can easily generalize the proof to show that for arbitrary infinite cardinals λ we have the following sequence of implications:

"D is $(cf(\lambda), \lambda^+)$-regular"

implies

"every set of $\leq\lambda^+$ elements of $\boldsymbol{\lambda}^{\lambda^+}/D$ is bounded"

implies

"D is (λ, λ^+)-regular".

We are grateful to the referee for effecting a considerable simplification of our proof in the inference from boundedness to regularity. The original form of this part of the proof was a modification of the Ulam Matrix argument used by Prikry in [3]. No new ideas are involved in the other direction of the proof.

References

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https://doi.org/10.4153/CMB-1975-008-x Published online by Cambridge University Press