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ISOCHRONOUS CENTERS AND FLAT FINSLER METRICS (I)

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ABSTRACT. The local structure of rotationally symmetric Finsler surfaces with vanishing flag curvature is completed determined in this paper. A geometric method for constructing such surfaces is introduced. The construction begins with a planar vector field X that depends on two functions of one variable. It is shown that the flow of X could be used to generate a generalized Finsler surface with zero flag curvature. Moreover, this generalized structure reduces to a regular Finsler metric if and only if X has an isochronous center. By relating X to a Liénard system, we obtain the isochronicity condition and discover numerous new examples of complete flat Finsler surfaces, depending on an odd function and an even function.

1. Introduction

In many situations, the role of flag curvature in Finsler geometry is analogous to that of sectional curvature in Riemannian geometry. Understanding the geometric significance of flag curvature is a central theme in the study of Finsler geometry. As a first step in this direction, the study of manifolds with constant flag curvature has always been popular [20, 8, 5, 17].

In B. Riemann's famous speech, which gave birth to both Riemannian geometry and Finsler geometry, contains only one displayed equation (see [21]). The equation provides the local normal form of a Riemannian manifold with constant sectional curvature. It shows that, for each constant K, the local structure of Riemannian space forms with sectional curvature K is unique up to isometry. In Finsler geometry, the local structure of metrics with constant flag curvature is more complicated. There are many non-isometric local structures that share the same constant flag curvature K. For example, on $\mathbb{R}^n$, all Minkowski metrics have vanishing flag curvature; all Hilbert metrics [1] have constant flag curvature $K = -1$ (for more details, refer to textbooks like [6]). R. Bryant [8] constructed several non-isometric Finsler metrics on S^n with constant flag curvature $+1$.

Thus, there are three natural questions concerning constant flag curvature (CFC, for short). (A) Given a constant number K, how many non-isometric CFC local structures are there? (B) Can we find a way to explicitly describe these structures? (C) Which local structures can be made global, i.e., complete?

These problems can be studied in both the generalized and classical senses. In 2002, R. Bryant proved a celebrating result that provides an answer to question (A) when $K = +1$.

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[1]Traditionally, a Hilbert metric is defined on a strongly convex domain Ω in $\mathbb{R}^n$. Since Ω is diffeomorphic to $\mathbb{R}^n$, the Hilbert metric can also be thought of as defined on $\mathbb{R}^n$.

Theorem 1.1. [8] *The space of local isometry classes of* generalized *Finsler structures in dimension n that have constant flag curvature $K = +1$ depends on $n(n-1)$ functions of n variables (in the sense of Cartan-Kähler).*

We conjecture that the theorem also applies to other constants. Moreover, as indicated by the results in the current paper, the distinction between generalized Finsler structure and classical Finsler structure is subtle. It is expected that *classical* Finsler structures of constant flag curvature also depend on $n(n-1)$ functions of n variables.

When $n = 2$, Bryant provides descriptions of metrics with $K = 1$ that depend on two functions of two variables (refer to [7] for details, and for an alternative explanation, see [8]), this addresses question (B) for *generalized* Finsler structures when $n = 2$ and $K = 1$.

The above questions are sometimes studied with symmetry conditions. Let $\mathrm{Iso}(M)$ be the isometry group of the n-dimensional Finsler manifold (M, F). When F is Riemannian, it is well known that $\dim \mathrm{Iso}(M) \leq n(n+1)/2$, and the equality holds only if (M, F) has constant sectional curvature (see [10]). Moreover, S. Kobayashi proved that when $n \neq 4$, the isometry group $\mathrm{Iso}(M)$ does not contain any closed subgroup whose dimension strictly lies between $1 + n(n-1)/2$ and $n(n+1)/2$ (see [15, Theorem 3.2]). Later, Yano [24] classified all Riemannian manifolds M with $\dim \mathrm{Iso}(M) = 1 + n(n-1)/2$ (For a systematic treatment of results of this type, refer to [15]). It is easy to see that the same classification holds true if the metric is assumed to be Finslerian. Hence, the maximal possible dimension of $\mathrm{Iso}(M)$, which could produce interesting non-Riemannian examples in Finsler geometry, is $n(n-1)/2$. There are indeed such metrics with *constant flag curvature*, as the following classical examples show.

(1) The Funk metric on $B^n(1)$ can be written as
$$F(x,y) = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x,y\rangle^2}}{1-|x|^2} + \frac{\langle x,y\rangle}{1-|x|^2}.$$
It has constant flag curvature $K = -1/4$.

(2) The Hilbert metric on $B^n(1)$ is given by
$$F(x,y) = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x,y\rangle^2}}{1-|x|^2}.$$
It has constant flag curvature $K = -1$.

(3) Berwald's metric on $B^n(1)$ is given by
$$F(x,y) = \frac{(\sqrt{(1-|x|^2)|y|^2 + \langle x,y\rangle^2} + \langle x,y\rangle)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2 + \langle x,y\rangle^2}}.$$
It has constant flag curvature $K = 0$.

(4) The very special Bryant's metric on S^n can be locally written as
$$F(x,y) = \sqrt{\frac{\sqrt{A}+B}{D} + \Big(\frac{C}{D}\Big)^2} + \frac{C}{D},$$
where $C = \sqrt{1-\varepsilon^2}\langle x,y\rangle$, $D = |x|^4 + 2\varepsilon|x|^2 + 1$, $B = (\varepsilon + |x|^2)|y|^2 - \langle x,y\rangle^2$, $A = B^2 + (1-\varepsilon^2)|y|^4$, $\varepsilon \in (-1,1)$. This metric has constant flag curvature $K = +1$.

A common feature of the above examples is that they admit $O(n)$ or $SO(n)$ as the isometry group. Thus, it is interesting to know if there are other Finsler n-manifolds of constant flag curvature that admit $O(n)$ or $SO(n)$ symmetry. The $O(n)$ invariant metrics are also referred to as spherically symmetric by some researchers (see [25, 12]).

When studying spherically symmetric metrics, there is a significant difference between the two-dimensional case and the higher-dimensional case. For example, the CFC condition is a single PDE in two dimensions, while it consists of a system of two PDEs in higher dimensions (see [25, 12, 19] for related discussions). It is easily seen that when $n \geq 3$, an $O(n)$-invariant Finsler manifold has plenty of totally geodesic submanifolds of dimension two. Moreover, such a manifold has constant flag curvature if and only if it has scalar flag curvature and any one of the totally geodesic surfaces has constant Gauss curvature. Thus, it is desirable to understand the two-dimensional case before studying the higher-dimensional case. For this reason, we shall only address the two-dimensional case in this paper; the treatment of the higher-dimensional case will be addressed in a subsequent paper.

To be more focused, we will concentrate on the case where $K = 0$. Notice that the $SO(2)$ symmetry implies the existence of a Killing field. A prior result is as follows.

Proposition 1.2. *Let M be a smooth surface that admits a generalized Finsler structure with vanishing flag curvature. If in addition M possesses a Killing field, then at every point of M there is a local coordinate system (x, y) such that (i) the Killing field X is given by $X = (yP(x) + Q(x))\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ for some smooth functions $P(x) > 0$ and $Q(x)$; (ii) the vector field $Y = \frac{\partial}{\partial y}$ has constant length 1.*

Conversely, we have the following local construction.

Theorem 1.3. *Let P and Q be smooth functions of x, and let $X = \big(yP(x) + Q(x)\big)\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ be a vector field defined on a plane region M. Let φ_t be the flow generated by X, and let Σ be the flow domain. Define a map $\iota : \Sigma \to TM$ as follows.*

$$\iota(p, t) = \big(\varphi_{t*}\frac{\partial}{\partial y}\big)_p, \quad \forall p \in M, t \in \mathbb{R}.$$

If $P(x) > 0$, then ι is a generalized Finsler structure on M with vanishing flag curvature, and X is a Killing field. Moreover, the generalized Finsler structure is complete if $P(x)$ and $Q(x)$ are defined for all $x \in \mathbb{R}$, $P(x)$ is bounded, and $Q(x)$ grows sublinearly.

It is natural to ask, when will this generalized structure reduce to a classical Finsler structure? This is answered in the following.

Theorem 1.4. *Define the generalized Finsler structure as stated in Theorem 1.3. Then the generalized structure is a classical Finsler structure if and only if the vector field X generates an $SO(2)$ action on M, if and only if X admits an isochronous center and M is in the isochronous period annulus. In this case, the Finsler structure possesses rotational symmetry.*

To obtain concrete examples, we need to identify the isochronicity condition. Notice that without loss of generality, we may assume $Q(0) = 0$, so X has a unique singular point at $(0, 0)$.

Theorem 1.5. *Suppose $P(x)$ and $Q(x)$ are analytic functions near 0 and $Q(0) = 0$. The vector field $X = (yP(x) + Q(x))\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ has an isochronous center at $(0, 0)$ if and only if there exists a function α and an odd function b such that the following relations hold:*

$$\begin{aligned} x &= -A(u), \\ P(x) &= A'(u), \\ Q(x) &= -P(x)\int_0^u B(s)\,\mathrm{d}\,s, \\ A(u) &= \alpha'(u)\Big(\alpha(u) + \frac{1}{\alpha^3(u)}\Big(\int_0^{\alpha(u)} zb(z)\,\mathrm{d}\,z\Big)^2\Big), \\ B(u) &= \alpha'(u)b(\alpha(u)), \end{aligned}$$

where the function α is invertible near 0 and its inverse function α^{-1} satisfies that $\alpha^{-1}(x) - x$ is even.

A few comments on the organization of this paper are in order. Section 2 reviews the basics of Finsler geometry, focusing on generalized Finsler structure and the dynamical approach to flag curvature as outlined in [11] and [14]. In Section 3 we prove Theorem 1.3 using knowledge from previous works by Bryant, Huang, and Mo [9]. In Section 4, Theorem 1.4 will be proved after closely examining the relationships between the shape of indicatrices and the isochronicity properties of the phase flow. We analyze the isochronicity condition and prove Theorem 1.5 in Section 5. Several examples of flat Finsler surfaces are also provided there.

2. Generalized and Classical Finsler Structures

This section mainly serves to review some fundamental concepts in Finsler geometry and establish notation.

Let M be a smooth manifold of dimension n. A Finsler metric on M is a smooth assignment of Minkowski norms to the tangent spaces of M. Each Minkowski norm is uniquely determined by its indicatrix, i.e., the set of vectors of unit length. By the definition of a Minkowski norm, each indicatrix must be strongly convex (toward the origin) and diffeomorphic to S^{n-1} [2]. To generalize the concept of a Finsler structure, we recall the following definition.

Definition 1. Let $\pi : TM \to M$ be the tangent bundle of the manifold M. Let $\iota : \Sigma \to TM$ be an immersed hypersurface in TM. If $\pi \circ \iota : \Sigma \to M$ is a submersion with connected fibers, and for each point $p \in M$, the immersion $\iota_p : (\pi \circ \iota)^{-1}(p) \to T_pM$ is strongly convex toward the origin, then the triple (M, Σ, ι) is called a *generalized Finsler structure* on M (sometimes we simply say that ι is a generalized Finsler structure on M). If all the ι_p are embeddings and their images are diffeomorphic to S^{n-1}, then ι is called a *classical Finsler structure*, a *regular Finsler structure*, or simply a *Finsler structure*.

Remark 1. R. Bryant's definition of a generalized Finsler structure differs slightly from the one presented here; for more details, refer to [7, 8]. In [8], a generalized Finsler structure is defined as the manifold Σ along with an adapted dual coframe field that satisfies some structure equations (refer to (1) for the two-dimensional scenario).

Now, let $\iota : \Sigma \to TM$ be a generalized Finsler structure. At each point $p \in M$, the tangent vectors to the fiber $\Sigma_p := (\pi \circ \iota)^{-1}(p)$ will be referred to as vertical. All the vertical vectors constitute the vertical subbundle of $T\Sigma$. We shall denote by $V\Sigma$ the smooth sections of the vertical subbundle. The immersed hypersurface $\iota(\Sigma_p)$ will be referred to as the indicatrix at point p. Sometimes we will not distinguish between Σ_p and $\iota(\Sigma_p)$, so we will also refer to the tangent vectors of $\iota(\Sigma_p)$ as vertical.

One can construct a globally defined contact form ω on Σ as follows. For each $u \in \Sigma$, let $\ell = \iota(u)$ be the corresponding point on the indicatrix at $p = \pi(\ell)$. There is a unique 1-form α_u in T_p^*M that annihilates $T_\ell \iota(\Sigma_p) \subset T_pM$, and such that $\alpha_u(\ell) = 1$. Define the 1-form ω on Σ as follows:

$$\omega|_u = (\pi \circ \iota)^*(\alpha_u).$$

This form is known in the literature as the *Hilbert form*.

The *geodesic spray* (or *Reeb field* in another terminoledgy) is the unique vector field on Σ determined by the following relations.

$$\omega(\xi) = 1, \quad \mathrm{d}\,\omega(\xi, \,\cdot\,) = 0.$$

If γ is an integral curve of ξ, then $\pi \circ \iota(\gamma)$ is a unit speed geodesic on M. Thus, if ξ is a complete vector field on Σ, then every unit geodesic on M is defined on $\mathbb{R}$. In this case, we say that the manifold M is complete.

2.1. **Dynamical approach to flag curvature.** To define the concept of flag curvature in a generalized Finsler structure, it is preferable to introduce the dynamical approach developed by P. Foulon [11], see also [13].

The *vertical endomorphism* $\mathcal{V}$ is the unique $(1,1)$ tensor on Σ that satisfies the following equations.

$$\mathcal{V}\left([\xi, v]\right) = -v, \quad \mathcal{V}(\xi) = \mathcal{V}\left(v\right) = 0, \quad \forall v \in V\Sigma.$$

The *horizontal endomorphism* $\mathcal{H}$ is the unique $(1,1)$ tensor on Σ that satisfies the following equations.

$$\begin{aligned} \mathcal{H}\left(v\right) &= -\left[\xi, v\right] - \frac{1}{2}\mathcal{V}\left[\xi, [\xi, v]\right], \\ \mathcal{H}\left(\mathcal{H}(v)\right) &= \mathcal{H}(\xi) = 0, \quad \forall v \in V\Sigma. \end{aligned}$$

The image of $\mathcal{H}$ is the horizontal subbundle of $T\Sigma$. So $T\Sigma$ can be decomposed into the direct sum of the horizontal subbundle, the vertical subbundle, and the line bundle spanned by ξ. We shall denote by $H\Sigma$ the set of smooth sections of the horizontal subbundle.

The *Riemann curvature tensor* or *Jacobi endomorphism* $\mathcal{R}$ is defined on the vertical subbundle as follows.

$$\mathcal{R}(w) = \mathcal{V}\mathcal{H}\left[\xi, \mathcal{H}(w)\right], \quad \forall w \in V\Sigma.$$

Notice that in [11], the Jacobi endomorphism $\mathcal{R}$ is also defined for horizontal vectors, and $\mathcal{R}(\xi) = 0$. However, only the vertical component is essential for us to define flag curvature.

The following $(0,2)$ tensor h is a Riemannian metric on the vertical subbundle, known as the *angular metric*.

$$h(u, v) = \mathrm{d}\,\omega([\xi, u], v) = \mathrm{d}\,\omega(u, \mathcal{H}(v)), \quad \forall u, v \in V\Sigma.$$

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Finally, the flag curvature K is given by

$$K(v) = \frac{h(\mathcal{R}(v), v)}{h(v, v)}, \quad \forall v \in V\Sigma \setminus \{0\}.$$

Notice that $\mathcal{R}$ is self-adjoint with respect to h, so the flag curvature of a generalized Finsler structure is identically zero if and only if $\mathcal{R} \equiv 0$.

Remark 2. If $\iota : \Sigma \to TM$ is a classical Finsler structure, then the definition of K above actually corresponds to a function defined for flags. Recall that a flag (P, y) in T_pM, is a two-dimensional subspace P in T_pM together with a nonzero vector y in P, where y is referred to as the *flagpole*. We may replace y with a unit vector ℓ in y direction, so ℓ represents a point on the indicatrix $\iota(\Sigma_p)$. The subspace P is spanned by ℓ and a vertical vector v that is tangent to the indicatrix at ℓ. The flag curvature $K(P, y)$, in traditional notation, is equivalent to our $K(v)$.

2.2. **Two dimensional case.** In two dimensions, the above scenario is significantly simplified. First, for a generalized Finsler structure on a surface, the indicatrix at each point is simply a strongly convex curve towards the origin. Moreover, the generalized Finsler structure reduces to a classical one if all the indicatrices are simple closed curves. Let us recall a simple criterion for strong convexity in [2, Chapter 4].

Lemma 2.1. *An immersed curve $\ell : \mathbb{R} \to \mathbb{R}^2$ is strongly convex toward the origin if it never passes through the origin and satisfies the following condition.*

$$\frac{\det(\ell', \ell'')}{\det(\ell, \ell')} > 0, \quad \forall t \in \mathbb{R}.$$

Now, the vertical subbundle is of rank one, so there is a globally defined vertical vector field e_3 satisfying $h(e_3, e_3) = 1$. Put $e_2 = \mathcal{H}(e_3)$; then e_2 is a globally defined horizontal vector field. In this way, $\{e_1 = \xi, e_2, e_3\}$ becomes a global frame field on Σ, known as the *Berwald frame*. Notice that e_3 is unique up to a minus sign. The dual coframe field $\{\omega_1 = \omega, \omega_2, \omega_3\}$ satisfies the following structure equations (see [7, 9]).

$$\begin{aligned} \mathrm{d}\,\omega_1 &= -\omega_2 \wedge \omega_3, \\ \mathrm{d}\,\omega_2 &= -\omega_3 \wedge \omega_1 - I\omega_2 \wedge \omega_3, \\ \mathrm{d}\,\omega_3 &= -K\omega_1 \wedge \omega_2 - J\omega_2 \wedge \omega_3, \end{aligned} \tag{1}$$

where I, J, and K are known as the *Cartan scalar*, the *Landsberg curvature*, and the *Gauss curvature*, respectively. In two dimensions, the flag (P, y) can only be (T_xM, y), so the flag curvature $K(P, y)$ can be written as $K(y)$, and it is referred to as the *Gauss curvature*.

2.3. **Flat Finsler structures.** Since we are mainly interested in the $K = 0$ (flat) case, we will now review some relevant results in this context. A basic model of flat Finsler space is $\mathbb{R}^n$ equipped with a Minkowski norm; it is called a Minkowski space. A Finsler manifold is called locally Minkowski if every point has a neighborhood that is isometric to an open subset of a Minkowski space. For a locally Minkowski space, the Cartan scalar is bounded, and the Landsberg curvature vanishes. A classical theorem by Akbar-Zadeh [2] states that if the flat Finsler structure is complete and the Cartan scalar is bounded, then it is a locally Minkowski space.

When $n = 2$, the Bryant normal form provides the local model of a flat Finsler structure (see [7]). It expresses the Berwald coframe using two arbitrary functions of two variables.

$$\begin{aligned} \omega_1 &= \mathrm{d}\, y - x \,\mathrm{d}\, z, \\ \omega_2 &= \hat{P}^{-1} \,\mathrm{d}\, x + (\hat{P}y + \hat{Q}) \,\mathrm{d}\, z, \\ \omega_3 &= \hat{P} \,\mathrm{d}\, z, \end{aligned} \tag{2}$$

where (x, y, z) is an adapted local coordinate system on Σ, $\hat{P}$ and $\hat{Q}$ are arbitrary functions of x and z, and $\hat{P} \neq 0$. Notice that when $\hat{P}$ and $\hat{Q}$ are only functions of x, the corresponding Finsler structure admits a Killing vector field $\frac{\partial}{\partial z}$, since the transformations $(x, y, z) \mapsto (x, y, z + c)$ leave the above coframe field unchanged.

Under the assumption that the Finsler structure admits a Killing vector field, Bryant, Huang, and Mo [9] derived another local normal form.

$$\begin{aligned} \omega_1 &= \mathrm{d}\, \tilde{y} + v(\tilde{x}) \,\mathrm{d}\, \tilde{x} + \tilde{x} \,\mathrm{d}\, \tilde{z}, \\ \omega_2 &= -u(\tilde{x})^{-1} \,\mathrm{d}\, \tilde{x} + \tilde{y} u(\tilde{x}) \,\mathrm{d}\, \tilde{z}, \\ \omega_3 &= u(\tilde{x}) \,\mathrm{d}\, \tilde{z}, \end{aligned} \tag{3}$$

where (t, a, b) is a local coordinate system on Σ, u and v are functions of a, and $u \neq 0$.

The above two normal forms are related as follows. Let $V(a) = \int v(a) \,\mathrm{d}\, a$; then the change of coordinates

$$\tilde{z} = z, \quad \tilde{x} = -x, \quad \tilde{y} = y - V(-x)$$

transforms the Bryant-Huang-Mo normal form into the Bryant normal form, with the constraint that $\hat{P}$ and $\hat{Q}$ depend only on x. Thus, it is fair to say that these two normal forms are equivalent when the Finsler structure admits a Killing field.

3. Construction of Flat Surfaces

In this section, we shall demonstrate how the above normal forms can be utilized to construct flat Finsler surfaces that admit a Killing field. To begin with, we shall gather some information about such a surface M. Recall that the indicatrix bundle $\hat{\Sigma}$ of M has the following Bryant normal form.

$$\begin{aligned} \omega_1 &= \mathrm{d}\, y - x \,\mathrm{d}\, z, \\ \omega_2 &= \hat{P}^{-1} \,\mathrm{d}\, x + (y\hat{P} + \hat{Q}) \,\mathrm{d}\, z, \\ \omega_3 &= \hat{P} \,\mathrm{d}\, z, \end{aligned}$$

where (x, y, z) is a local coordinate system on $\hat{\Sigma}$, $\hat{P}0$ and $\hat{Q}$ are functions of x, and $\hat{P} \neq 0$. The dual frame field is given by the following.

$$\begin{aligned} e_1 &= \frac{\partial}{\partial y}, \\ e_2 &= \hat{P} \frac{\partial}{\partial x}, \\ e_3 &= \hat{P}^{-1} \frac{\partial}{\partial z} - (y\hat{P} + \hat{Q}) \frac{\partial}{\partial x} + x\hat{P}^{-1} \frac{\partial}{\partial y}. \end{aligned}$$

By definition, $e_1 = \frac{\partial}{\partial y}$ is the geodesic spray, e_2 spans the horizontal subbundle, and e_3 spans the vertical bundle.

The leaves of the foliation defined by $\omega_1 = \omega_2 = 0$ are simply integral curves of e_3; they represent the indicatrices of the Finsler structure. Thus, M is locally diffeomorphic to a submanifold of $\hat{\Sigma}$ that is transverse to the integral curves of e_3. Since $\hat{P} \neq 0$, the integral curves of $\hat{P} \cdot e_3$ are the same as those of e_3 (as point sets) and they are always transverse to the slice $z = 0$ in $\hat{\Sigma}$. For this reason, we may identify M as this slice, i.e., $M = \{(x, y, z) \in \hat{\Sigma} \mid z = 0\}$. Thus (x, y) forms a local coordinate system on M.

The integral curve of $\hat{P} \cdot e_3$ passing through a point $(x, y, 0)$ in $\hat{\Sigma}$ is the indicatrix at that point. Therefore, the projection $\pi \circ \iota : \hat{\Sigma} \to M$ is given by

$$(\pi \circ \iota)(x, y, z) = \hat{\varphi}_{-z}(x, y, z),$$

where $\hat{\varphi}_t$ is the (local) flow generated by $\hat{P} \cdot e_3$. As such, the Killing field $\frac{\partial}{\partial z}$ on $\hat{\Sigma}$ is projected to the following vector field on M,

$$\begin{aligned} X &= \frac{\mathrm{d}}{\mathrm{d}\, t} \hat{\varphi}_{-z+t}(x, y, z+t)\Big|_{t=0} \\ &= \varphi_{-z*} \frac{\partial}{\partial z} - \hat{P} \cdot e_3 \\ &= (y\hat{P}^2 + \hat{P}\hat{Q}) \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}. \end{aligned}$$

Thus, we have proved that the Killing field X on M has a local expression

$$X = (yP + Q) \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, \tag{4}$$

where $P = \hat{P}^2$, $Q = \hat{P}\hat{Q}$. It is easy to see that the flow φ_t of X is related to $\hat{\varphi}_t$ via

$$\hat{\varphi}_t(x, y, z) = (\varphi_{-t}(x, y), t + z). \tag{5}$$

Moreover, since the integral curves of $e_1 = \frac{\partial}{\partial y}$ are of the form $t \mapsto (x, y+t, z)$, the projected curves $t \mapsto (\pi \circ \iota)(x, y+t, z)$ are unit speed geodesics on M. Consequently, the tangent vector $\hat{Y}$ of the curve $t \mapsto (\pi \circ \iota)(x, y+t, z)$ has a unit length. We have

$$\begin{aligned} \hat{Y} &= \frac{\mathrm{d}}{\mathrm{d}\, t} (\pi \circ \iota)(x, y+t, z)\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\, t} \hat{\varphi}_{-z}(x, y+t, z)\Big|_{t=0} \\ &= \varphi_{z*} \frac{\partial}{\partial y}. \end{aligned}$$

Since X is a Killing field, its flow φ_t consists of isometries. Thus, the tangent vector $\frac{\partial}{\partial y} = \varphi_{-z*} \hat{Y}$ also has unit length. Proposition 1.2 has been proven.

3.1. **A motivating example.** In the proof of Proposition 1.2, we have seen that for each fixed t, the vector field

$$\hat{Y} = \varphi_{t*} \frac{\partial}{\partial y}$$

has a unit length on M. Thus, at any point $p \in M$, the curve

$$t \mapsto \gamma(t) := \left(\varphi_{t*} \frac{\partial}{\partial y} \right)_p$$

traces a portion of the indicatrix at point p.

For example, when $P = 1$ and $Q = 0$, the vector field X has the following expression.

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

It is defined on the entire plane $M = \mathbb{R}^2$ and it generates the S^1 action φ_t on M, where

$$\varphi_t(x, y) = (x\cos t + y\sin t, -x\sin t + y\cos t), \quad \forall (x, y) \in M.$$

Now let $Y = \frac{\partial}{\partial y}$, then it is easy to see that

$$\gamma(t) = (\varphi_{t*}Y)_p = \sin t\, \frac{\partial}{\partial x}\bigg|_p + \cos t\, \frac{\partial}{\partial y}\bigg|_p.$$

Thus, at each point $p \in \mathbb{R}^2$, the curve $\gamma(t) = (\varphi_{t*}Y)|_p$ is a circle of radius one, known as the indicatrix of the Euclidean metric.

Motivated by the above example, we restate Theorem 1.3 as follows.

Theorem 3.1. *Let P and Q be two smooth functions defined near 0 in $\mathbb{R}$ such that $P(x) > 0$ for all x. Let M be the maximal plane region where the vector field $X = (yP(x) + Q(x))\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ is defined. Let φ_t be the local flow generated by X with a flow domain $\Sigma \subset M \times \mathbb{R}$. Define a map $\iota : \Sigma \to TM$ as follows.*

$$\iota(p, t) := \left(\varphi_{t*}\frac{\partial}{\partial y}\right)_p, \quad \forall (p, t) \in \Sigma. \tag{6}$$

Then ι is a generalized Finsler structure on M with vanishing flag curvature, and X is a Killing field. Moreover, if P and Q are defined on $\mathbb{R}$, P is bounded, and Q grows sublinearly, then M covers the entire $\mathbb{R}^2$, and the generalized Finsler structure is complete.

The proof of this theorem shall consists the rest of this section and it is divided into three subsections.

3.2. **The strong convexity of the indicatrices.** To show that ι is a generalized Finsler structure, we only need to prove that $\gamma(t) := \iota(p, t)$ is a strongly convex curve towards the origin in T_pM for each $p \in M$. Let $Y := \frac{\partial}{\partial y}$; then direct computation shows that

$$-[X, Y] = P\frac{\partial}{\partial x}, \quad [X, [X, Y]] = (PQ' - QP'))\frac{\partial}{\partial x} - P\frac{\partial}{\partial y}.$$

Thus, it is easy to obtain

$$\det(Y, -[X, Y]) = \begin{vmatrix} 0 & P \\ 1 & 0 \end{vmatrix} = -P,$$
$$\det(-[X, Y], [X, [X, Y]]) = \begin{vmatrix} P & PQ' - QP' \\ 0 & -P \end{vmatrix} = -P^2.$$

Notice that these two determinants are negative at any point.

By successively using [16, Corollary 1.10] we have

$$\frac{\mathrm{d}}{\mathrm{d}\,t}(\varphi_{t*}Y) = -\varphi_{t*}[X, Y], \quad \frac{\mathrm{d}^2}{\mathrm{d}\,t^2}(\varphi_{t*}Y) = \varphi_{t*}[X, [X, Y]].$$

Restricting to the point p, we have

$$\det\left(\gamma(t), \gamma'(t)\right) = det\left((\varphi_{t*}Y)_p, -(\varphi_{t*}[X, Y])_p\right) = \det(\varphi_{t*})\det(Y, -[X, Y]),$$

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where the last two determinants are evaluated at the point $\varphi_{-t}(p)$. Similarly we have

$$\det(\gamma'(t), \gamma''(t)) = \det(\varphi_{t*}) \det(-[X, Y], [X, [X, Y]]).$$

Thus

$$\frac{\det(\gamma(t), \gamma'(t))}{\det(\gamma'(t), \gamma''(t))} = \frac{\det(Y, [Y, X])}{\det([Y, X], [X, [X, Y]])} = \frac{1}{P},$$

where the function P is evaluated at the x-coordinate of the point $\varphi_{-t}(p)$. Since $P > 0$, we have proved that the curve $\gamma(t)$ is strongly convex toward the origin (see Lemma 2.1).

Remark 3. Since φ_t is a local diffeomorphism, $\det(\varphi_{t*})$ never vanishes. Together with the fact that $\det(\varphi_{0*}) = 1$, we have $\det(\varphi_{t*}) > 0$. Thus we can actually prove $\det(\gamma(t), \gamma'(t)) < 0$, i.e., $\gamma(t)$ travels clockwise around the origin.

3.3. **Computation of flag curvature.** Now we compute the flag curvature of the above generalized Finsler structure.

3.3.1. *Hilbert form.* As above, let (x, y, t) be the natural coordinate system on the flow domain Σ. We define another coordinate system (u, v, s) on Σ as follows.

$$(u, v) = \varphi_{-t}(x, y), \quad s = t.$$

By the definition of the flow, we know that

$$u' = -vP(u) - Q(u), \quad v' = u, \tag{7}$$

where $'$ denotes derivative with respect to t.

Let $\pi : TM \to M$ be the natural projection, then $\pi \circ \iota(x, y, t) = (x, y)$. Recall that at each point $p = (x, y)$, the indicatrix is parametrized by the curve

$$\gamma(t) = (\varphi_{t*}Y)_p,$$

and it satisfies

$$\gamma'(t) = -(\varphi_{t*}[X, Y])_p.$$

Now, since the 1-form $\mathrm{d}\, y$ satisfies

$$(\mathrm{d}\, y)(Y) = 1, \quad (\mathrm{d}\, y)[-X, Y] = 0,$$

we find that the 1-form $\ell^* = (\varphi^*_{-t}\, \mathrm{d}\, y)_p \in T^*_p M$ satisfies

$$\ell^*(\gamma) = 1, \quad \ell^*(\gamma') = 0.$$

As a result, the Hilbert form ω is given by

$$\omega = (\pi \circ \iota)^*(\ell^*) = \mathrm{d}\, v - v_t\, \mathrm{d}\, t = \mathrm{d}\, v - u\, \mathrm{d}\, s. \tag{8}$$

3.3.2. *Riemann curvature tensor.* The geodesic spray ξ is the unique vector field on Σ determined by

$$\omega(\xi) = 1, \quad \mathrm{d}\, \omega(\xi, \cdot\,) = 0.$$

Since $\omega = \mathrm{d}\, v - u\, \mathrm{d}\, s$, it is easy to see that $\xi = \frac{\partial}{\partial v}$.

Within the coordinate system (u, v, s), the vertical vector field $\frac{\partial}{\partial t}$ is expressed as

$$V := \frac{\partial}{\partial t} = vP(u)\frac{\partial}{\partial u} - u\frac{\partial}{\partial v} + \frac{\partial}{\partial s}.$$

Direct calculation shows that

$$[\xi, V] = P(u)\frac{\partial}{\partial u}, \quad [\xi, [\xi, V]] = 0.$$

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Consequently we have

$$\mathcal{H}(V) = -[\xi, V] - \frac{1}{2}\mathcal{V}[\xi, [\xi, V]] = -P(u)\frac{\partial}{\partial u}.$$

Therefore, the Riemann curvature tensor $\mathcal{R}$ is given by

$$\mathcal{R}(V) = \mathcal{V}\mathcal{H}[\xi, \mathcal{H}(V)] = \mathcal{V}\mathcal{H}\left[\frac{\partial}{\partial v}, -P(u)\frac{\partial}{\partial u}\right] = 0,$$

meaning that the generalized Finsler surface we constructed is indeed a flat one.

3.4. **Completeness.** Recall that a generalized Finsler structure (M, Σ, ι) is called *complete* if the geodesic spray vector field on Σ is complete.

Lemma 3.2. *If the vector field X is complete on the entire plane $M = \mathbb{R}^2$, i.e., its flow domain Σ is $M \times \mathbb{R} = \mathbb{R}^3$, then the corresponding generalized Finsler structure $(\mathbb{R}^2, \mathbb{R}^3, \iota)$ is complete.*

Proof. As the above computation shows, within the (u, v, s) coordinate system, the geodesic spray is given by $\frac{\partial}{\partial v}$. Hence, the geodesic spray is complete if and only if the function v can take all real values. Now, the vector field X is complete on the entire plane $M = \mathbb{R}^2$, so its flow φ_t is defined for all $t \in \mathbb{R}$. By the relation $(u, v) = \varphi_{-t}(x, y)$, it is readily seen that v can take any real values, thus the generalized Finsler structure is complete. □

Lemma 3.3. *Let P and Q be smooth functions on $\mathbb{R}$. If P is bounded and Q grows sublinearly, i.e., there exist positive real numbers C_1 and C_2 such that*

$$|P(x)| \leq C_1, \quad |Q(x)| \leq C_2|x|, \quad \forall x \in \mathbb{R},$$

then the vector field $X = (yP(x) + Q(x))\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ is complete on $\mathbb{R}^2$.

Proof. To prove that the vector field X is complete on $\mathbb{R}^2$, we need to show that for any $(x_0, y_0) \in \mathbb{R}^2$, the solution of the initial value problem

$$(x, y)' = (yP(x) + Q(x), -x), \quad (x, y)|_{t=0} = (x_0, y_0)$$

is defined for all $t \in \mathbb{R}$. Let $C_3 = (2C_1^2 + 2C_2^2 + 1)^{1/2}$, then one can easily verify that

$$\left((yP(x) + Q(x))^2 + (-x)^2\right)^{1/2} \leq C_3 \cdot (x^2 + y^2)^{1/2}.$$

By using Theorem 2.17 in [22, pp. 53], the above estimate guarantees that the solution of the above initial value problem is defined for all $t \in \mathbb{R}$. □

Together the above two lemmas, we have proved the last assertion in Theorem 1.3. This finishes the proof of Theorem 1.3.

4. Isochronicity and rotational symmetry

Given the generalized Finsler surface constructed above, our main goal in this section is to determine the conditions under which the generalized Finsler structure reduces to a classical one.

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4.1. **A sufficient condition.** Let P and Q be smooth functions near 0 in $\mathbb{R}$ and $P > 0$. The vector field $X = (yP(x) + Q(x))\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ has a unique singular point at $(0, -Q(0)/P(0))$. By suitably translating the coordinates along the y-axis, we may assume the singular point is $(0,0)$, i.e., $Q(0) = 0$.

Recall that for a vector field X, the singular point $(0,0)$ is called an *isochronous center* if all the integral curves near $(0,0)$ are closed and have a constant period. The maximal domain enclosing all such integral curves is called the *period annulus*.

Lemma 4.1. *If $(0,0)$ is an isochronous center of the vector field $X = (yP(x) + Q(x))\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$, then for any point p in the period annulus, the curve $\gamma(t) = \left(\varphi_{t*}\frac{\partial}{\partial y}\right)_p$ is smooth and closed.*

Proof. Let $Y = \frac{\partial}{\partial y}$. Suppose that T is the period constant of the isochronous center. Then, we would have $\varphi_t = \varphi_{t+T}$, $\forall t$. Hence $\varphi_{0*}Y = \varphi_{T*}Y$ and $\gamma(0) = \gamma(T)$. Moreover, since $\gamma'(t) = (\varphi_{t*}[Y,X])_p$, we have $\gamma'(0) = \gamma'(T)$. In a similar manner, we can prove that $\gamma^{(n)}(0) = \gamma^{(n)}(T)$, for all $n \in \mathbb{N}$. □

Now we are ready to derive the following proposition.

Proposition 4.2. *Suppose $(0,0)$ is an isochronous center of the vector field $X = (yP(x) + Q(x))\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ and M is the isochronicity period annulus. Define the generalized Finsler structure (M, Σ, ι) as in Theorem 1.3. Then this generalized Finsler structure is actually a classical Finsler structure.*

Proof. Let p be a point in the period annulus M. Let $\gamma(t) = \left(\varphi_{t*}Y\right)_p = \varphi_{t*}Y_{\varphi_{-t}(p)}$ as before. We already know from the proof of Theorem 1.3 that $\gamma(t)$ travels clockwise around the origin (see Remark 3 at the end of §3.2). Lemma 4.1 ensures that $\gamma(t)$ is smooth and closed. Therefore, we only need to prove now that $\gamma(t)$ is a simple curve in T_pM. This amounts to showing that the curve $\gamma(t)$, $0 \le t \le T$, has a winding number of -1.

First, consider the case $p = (0,0)$. In this situation, we have $\varphi_{-t}(p) = p$ and $\gamma(t) = \varphi_{t*}Y_p$. By the definition of flow, $\varphi_{t+s} = \varphi_t \circ \varphi_s$, we have $\varphi_{(t+s)*} = \varphi_{t*}\varphi_{s*}$. Since the maps $\varphi_{t*} : T_pM \to T_pM$ are linear, we find that $t \mapsto \varphi_{t*}$ is a representation of $S^1 = \mathbb{R}/T\mathbb{Z}$ on T_pM. Consequently, as the orbit through Y_p, the curve $\gamma(t)$ must be an ellipse. Moreover, since T is the least period of φ_t, it must also be the least period of $\gamma(t)$. Thus, we have proved that $\gamma(t)$ has a winding number of -1 when $p = (0,0)$.

Next, we consider a general point p in the period annulus. Suppose $\gamma(t) = h_1(t)\left.\frac{\partial}{\partial x}\right|_p + h_2(t)\left.\frac{\partial}{\partial y}\right|_p$, then we have the following formula for the winding number (see [3]).

$$W = \frac{1}{2\pi}\int_0^T \frac{h_1h_2' - h_2h_1'}{h_1^2 + h_2^2}\,\mathrm{d}\,t.$$

Since h_1 and h_2 in the integrand depend continuously on p, the integral W must also be a continuous function on M; but we know that the winding number must be an integer for closed curves, so $W \equiv -1$. □

4.2. **A necessary condition.** The following proposition suggests that $(0,0)$ being an isochronous center is also the necessary condition for the above generalized Finsler structure to become a classical Finsler structure, at least in a neighborhood of $(0,0)$.

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Proposition 4.3. *If the singular point* $(0,0)$ *is not an isochronous center of* $X = (yP(x)+Q(x))\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$, *then for any open subset* $U \subset M$, *there exists a point* $p \in U$ *such that the curve* $\gamma(t) = (\varphi_{t*}Y)_p$ *is not closed.*

Proof. It is easy to verify that the conclusion holds if $(0,0)$ is not even a center of X.

Meanwhile, if $(0,0)$ is a non-isochronous center of X, then the function $T : M \to \mathbb{R}$, where $T(p)$ is the least period of the orbit containing p, is bound to be differentiable and non-constant on any open set U and its open subset $U_0 := U \setminus \{(x,y) \in U \,|\, yP(x)+Q(x) = 0\}$. As the directional derivative of T along the vector X is already zero, we must have $\frac{\partial T}{\partial y} \neq 0$ on U_0.

Let $p = (x,y) \in U_0$. Consider a curve $c(t)$ in M satisfying $c(0) = p$ and $c'(0) = Y_p$ as well as its image under $\varphi_{T(p)}$. Expanding near $t = 0$, we have $T(c(t)) = T(p) + t\frac{\partial T}{\partial y}(p) + o(t)$. Thus

$$\begin{aligned}\varphi_{T(p)}\,(c(t)) &= \varphi_{T(p)-T(c(t))}(c(t))\\ &= \varphi_{-t\frac{\partial T}{\partial y}(p)-o(t)}(c(t))\\ &= c(t) - t\frac{\partial T}{\partial y}(p)\cdot X_p + o(t).\end{aligned}$$

Consequently

$$(\varphi_{T(p)*}Y)_p = \varphi_{T(p)*}c'(0) = Y_p - \frac{\partial T}{\partial y}(p)\cdot X_p.$$

By using a similar argument, we have

$$(\varphi_{kT(p)*}Y)_p = Y_p - k\frac{\partial T}{\partial y}(p)X_p, \quad k \in \mathbb{N}.$$

Thus, since $\gamma(0) \neq \gamma(kT(p))$, the curve $\gamma(t)$ cannot be closed. □

Even if the vector field X admits an isochronous center, one can similarly prove that for any open set U outside the isochronous period annulus of X, there exists a point $p \in U$ such that the indicatrix at p is not closed. Thus, the above two propositions imply that, for the generalized Finsler structure on M to be a classical one, X must have an isochronous center and M must be included in the isochronicity period annulus. Moreover, combining this result with the last assertion of Theorem 1.3, we find that the constructed generalized structure is classical and complete if and only if $(0,0)$ is a global isochronous center on $\mathbb{R}^2$, i.e., the entire $\mathbb{R}^2$ is the period annulus.

4.3. **Rotational symmetry.** At this point, we have almost finished the proof of Theorem 1.4. It remains to show that X admits an isochronous center if and only if it generates an $SO(2)$ action, but this is indeed a conceptual transition between dynamical systems and differential geometry.

If X admits an isochronous center, then its flow φ_t has a constant period T, i.e., $\varphi_{t+T} = \varphi_t$ holds for all $t \in \mathbb{R}$. This means that the action of the flow on M reduces to $\mathbb{R}/T\mathbb{Z} = S^1$.

Conversely, if X generates an $SO(2) = S^1$ action on M, then every orbit has a constant period T. Together with the fact that X has a unique singular point, we find that X must have an isochronous center. The proof of Theorem 1.4 ends here.

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Corollary 4.4. *If M is a regular Finsler surface with vanishing flag curvature and its isometry group G is one-dimensional, then necessarily $G = S^1$.*

In the following, we shall discuss the difference between rotational symmetry and spherical symmetry. Let us begin with the following definition.

Definition 2. A generalized Finsler structure (M^n, Σ, ι) is called *rotationally symmetric* (resp. *spherically symmetric*) if M admits an effective $SO(n)$ (resp. $O(n)$) action that maps the indicatrices into indicatrices.

Remark 4. In general, if a Finsler structure is spherically symmetric, then the Finsler metric can be expressed as $F = |y| \cdot \phi(|x|, \langle x, y\rangle/|y|)$ in some well-chosen local coordinate system (see [25, 12]); but for a rotationally symmetric Finsler structure, this is not the case in dimension two, as the following example shows.

The difference between rotational symmetry and spherical symmetry is better illustrated by the following two dimensional example: Let us consider the usual $SO(2)$ or $O(2)$ action on $\mathbb{R}^2$. Let

$$F(x_1, x_2; y_1, y_2) = |y|\phi(s),$$

where $|y| = \sqrt{(y_1)^2 + (y_2)^2}$ is the Euclidean norm, and $s = (x_1y_2 - x_2y_1)/|y|$. If the one-variable function ϕ satisfies some open conditions, then F is a Finsler metric defined in a neighborhood of $(0, 0)$. It is easy to check that the corresponding Finsler structure is rotationally symmetric.

$$F(Ax, Ay) = F(x, y), \quad \forall A \in SO(2),\ x \in \mathbb{R}^2, y \in T_x\mathbb{R}^2 \simeq \mathbb{R}^2,$$

but in general it is not spherically symmetric because the above relation does not hold for any $A \in O(n)$. Actually, this metric is spherically symmetric only if ϕ is an even function. Indeed, a generalized Finsler structure on $\mathbb{R}^2$ is just an assignment of indicatrices (strongly convex curves) to the tangent spaces. So along the (positive) x_1-axis, there is a family of such strongly convex curves. If the Finsler structure is rotationally symmetric, then this family of curves can be freely assigned, and it completely determines the Finsler structure since the indicatrices at any point (x_1, x_2) can be obtained by rotating the corresponding curve at $(|x|, 0)$. However, to make the Finsler structure spherically symmetric, this family of curves has to be symmetric about the x_1-axis because the reflection on the x_1-axis is an element of $O(2)$.

However, in dimensions ≥ 3, one cannot distinguish between rotational symmetry and spherical symmetry by looking at the expression of the Finsler metric F, because in both cases, the Finsler metric can be expressed as $F = |y| \cdot \phi(|x|, \langle x, y\rangle/|y|)$ in some local coordinate system. For a proof of this fact, one may consult [12], in which the proof of Proposition 3.1 also works for rotationally symmetric Finsler metrics.

5. Isochronoucity conditions

In this section, we will try to find appropriate conditions on P and Q to make the vector field X isochronous. Although the general case can be settled, we shall treat the $Q = 0$ case first, not only because the result is more elegant, but also because the method is more elementary.

5.1. **Isochronicity conditions when $Q = 0$.** When $Q = 0$, the integral curves of X are given by solutions of the dynamical system

$$x' = yP(x), \quad y' = -x. \tag{9}$$

Let $\kappa = P(0)^{-1/2}$, and let $b(x)$ be the solution to the following initial value problem

$$\kappa + b(x) - xb'(x) = P\Big(\frac{x}{\kappa + b(x)}\Big) \cdot (\kappa + b(x))^3, \quad b(0) = 0. \tag{10}$$

By the following change of variables

$$x = \frac{\tilde{x}}{\kappa + b(\tilde{x})}, \quad y = \tilde{y},$$

we can rewrite the above system (9) as

$$\tilde{x}' = \frac{\tilde{y}}{\kappa + b(\tilde{x})}, \quad \tilde{y}' = \frac{-\tilde{x}}{\kappa + b(\tilde{x})}. \tag{11}$$

The isochronicity condition of this system will follow in the next two lemmas.

Lemma 5.1. *Let λ be a smooth function defined on the open interval $(-\epsilon, \epsilon)$. If $\int_0^{2\pi} \lambda(r\cos\theta)\,\mathrm{d}\,\theta$ is a constant for any $r \in [0, \epsilon)$, then $\lambda(x) - \lambda(0)$ is an odd function.*

Proof. Let $f(x) = \lambda(x) + \lambda(-x) - 2\lambda(0)$, then $f(x)$ is even and $f(0) = 0$. Moreover, the condition implies that

$$\int_0^\pi f(r\cos\theta)\,\mathrm{d}\,\theta = 0.$$

By changing variables $x = r\cos\theta$, one can rewrite the above equation as

$$\int_{-r}^r f(x)(r^2 - x^2)^{-1/2}\,\mathrm{d}\,x = 0.$$

Since f is even, we also have

$$\int_0^r f(x)(r^2 - x^2)^{-1/2}\,\mathrm{d}\,x = 0.$$

Now we use induction to show that $\int_0^r f(x)x^{2m}(r^2 - x^2)^{-1/2}\,\mathrm{d}\,x = 0$ holds for any nonpositive integer m. The base case $m = 0$ is already established above. Suppose it holds for some $m \geq 0$, then we have

$$\begin{aligned}
&\int_0^r f(x)x^{2m+2}(r^2 - x^2)^{-1/2}\,\mathrm{d}\,x \\
=& -\int_0^r f(x)x^{2m}(r^2 - x^2)(r^2 - x^2)^{-1/2}\,\mathrm{d}\,x \\
=& -\int_0^r f(x)x^{2m}(r^2 - x^2)^{1/2}\,\mathrm{d}\,x \\
=& -\int_0^r \mathrm{d}\,x \int_x^r sf(x)x^{2m}(s^2 - x^2)^{-1/2}\,\mathrm{d}\,s \\
=& -\int_0^r \mathrm{d}\,s \int_0^s sf(x)x^{2m}(s^2 - x^2)^{-1/2}\,\mathrm{d}\,x = 0.
\end{aligned}$$

Thus, the equality holds for all non-positive integers m. From the above deduction, we also have

$$\int_0^r f(x)x^{2m}(r^2 - x^2)^{1/2}\,\mathrm{d}\,x = 0.$$

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Since any continuous even function can be approximated by linear combinations of 1, x^2, x^4, $\cdots$, we conclude from the above equation that

$$\int_0^r f(x)(r^2 - x^2)^{1/2} g(x) \, \mathrm{d}\, x = 0$$

holds for any even function $g(x)$. Taking $g(x) = f(x)(r^2 - x^2)^{1/2}$ then shows that $f(x) = 0$ on $[0, r]$. □

Remark 5. If λ is analytic, then one can use its power series expansion to get a simple proof of this lemma.

Lemma 5.2. *Suppose λ is a smooth function satisfying $\lambda(0) > 0$, then the neccessary and sufficient condition for $(0,0)$ to be an isochronous center of the system*

$$x' = -\frac{y}{\lambda(x)}, \quad y' = \frac{x}{\lambda(x)}$$

is that $\lambda(x) - \lambda(0)$ is an odd function.

Proof. The integral curves of this system obviously share the same shapes as those of the system $x' = -y$, $y' = x$, so $(0,0)$ is a center of this system. By changing to polar coordinates, i.e., $x = r\cos\theta$, $y = r\sin\theta$, this system can be written as

$$r' = 0, \quad \theta' = \frac{1}{\lambda(r\cos\theta)}.$$

Thus, the period of an orbit passing through $(r, 0)$ is given by

$$T(r) = \int_0^{T(r)} \mathrm{d}\, t = \int_0^{2\pi} \lambda(r\cos\theta) \, \mathrm{d}\, \theta.$$

The point $(0,0)$ is an isochronous center if and only if $T(r)$ is independent of r. By Lemma 5.1, this happens if and only if $\lambda(x) - \lambda(0)$ is an odd function. □

Using this lemma, we know that the vector field $X = yP(x)\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ possesses an isochronous center if and only if the function b is odd. What's more, after picking a positive constant κ and an odd function b, we can easily solve for P from the relations

$$P(x) = (\kappa + b(\tilde{x}) - \tilde{x}b'(\tilde{x}))(\kappa + b(\tilde{x}))^{-3}, \quad x = \tilde{x}/(\kappa + b(\tilde{x})). \tag{12}$$

Example 1. If we take $\kappa = 1$ and $b(x) = -\varepsilon x$, then one can solve the above equations to get

$$P(x) = (1 + \varepsilon x)^3.$$

In this case, we have the following well-known polynomial Abel system (see [23, Theorem 8], see also [4, Theorem 6.2]).

$$x' = y(1 + \varepsilon x)^3, \quad y' = -x,$$

where $\varepsilon \in (-1, 1)$.

Example 2. Set $b(x) = \varepsilon x/(1 + x^2)$, then we have

$$P(x) = \frac{(z^2 + 1)^2}{(z^2 + \varepsilon z + 1)^2},$$

where z is determined by the relation $x = z(z^2 + 1)/(z^2 + \varepsilon z + 1)$. From the above expression, it is easy to see that $P(x)$ is bounded on R when $\varepsilon \in (-2, 2)$. Thus, by Theorem 1.3 and Theorem 1.4, the corresponding Finsler structure is defined on the entire $\mathbb{R}^2$, and it is complete.

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5.2. **Isochronicity conditions when $Q \neq 0$.** In the general case, integral curves of X are given by solutions of

$$x' = yP(x) + Q(x), \quad y' = -x,$$

where, without loss of generality, we can assume that $P(0) = 1$ and $Q(0) = 0$.

This time, using a change of variables

$$u = -\int_0^x \frac{1}{P(s)} ds, \quad v = y + \frac{Q(x)}{P(x)},$$

we can turn this system into a Liénard system

$$u' = -v, \quad v' = A(u) + vB(u), \tag{13}$$

where the functions A and B are determined by the relations

$$A(u) = -x, \quad B(u) = \frac{Q'(x)P(x) - P'(x)Q(x)}{P(x)}. \tag{14}$$

The necessary and sufficient conditions for Liénard systems to have isochronous centers were studied by Amel'kin and Rudenok, who, in 2018, proved the following theorem (see [1, Theorem 18] or [18, Theorem 6]).

Theorem 5.3. [1, 18] *If A and B are analytic near 0, then the Liénard system (13) possesses an isochronous center at $(0,0)$ if and only if there exists a function α and an odd function b, such that*

$$A(u) = \alpha'(u)\Big(\alpha(u) + \frac{1}{\alpha^3(u)}\Big(\int_0^{\alpha(u)} zb(z)\,\mathrm{d}\,z\Big)^2\Big),$$
$$B(u) = \alpha'(u)b(\alpha(u)),$$

where the function α is invertible near 0 and its inverse function α^{-1} satisfies that $\alpha^{-1}(x) - x$ is even.

To obtain concrete examples, we present the following proposition.

Proposition 5.4. *Let A, B, P, and Q be defined as above; then we have the following relations.*

$$\begin{aligned} x &= -A(u),\\ P(x) &= A'(u),\\ Q(x)/P(x) &= -\int_0^u B(s)\,\mathrm{d}\,s. \end{aligned} \tag{15}$$

Proof. As $u'(x) = -\frac{1}{P(x)}$, we have $x'(u) = -P(x)$. Together with the relation $x = -A(u)$, we get $P(x) = A'(u)$.

Meanwhile, the expression for $B(u)$ in (14) can be rewritten into the differential equation $(Q/P)' = B(u)/P(x)$. Together with $x'(u) = -P(x)$, we have $\frac{\mathrm{d}(Q/P)}{\mathrm{d}\,u} = -B(u)$. Thus, $Q/P = -\int_0^u B(s)\,\mathrm{d}\,s$. ∎

Combining Theorem 5.3 and Proposition 5.4, we have proved Theorem 1.5. Now we shall present several examples.

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Example 3. Set $\alpha(u) = u$ and $b(z) = z$ in Theorem 5.3, then we get

$$A(u) = u + u^{-3}\Big(\int_0^u z^2 \,\mathrm{d}\, z\Big)^2 = u + u^3/9,$$
$$B(u) = u.$$

From the relations (15) we have

$$P(x) = 1 + u^2/3,$$
$$Q(x) = -\frac{1}{2}u^2(1 + u^2/3),$$

where $x = -A(u) = -u - u^3/9$.

Example 4. Set $\alpha(u) = u$ and $b(z) = \sin z$ in Theorem 5.3, then we have

$$P(x) = 1 + u^{-4}(\sin u - u\cos u)\left(2u^2 \sin u - 3(\sin u - u\cos u)\right),$$
$$Q(x) = -P(x)(1 - \cos u),$$

where u is determined by the relation $x = u + (\sin u - u\cos u)^2 u^{-3}$. One can verify that this relationship is globally invertible. From the above expression, it is seen that $P(x)$ and $Q(x)$ are bounded on $\mathbb{R}$. Therefore, the corresponding Finsler structure is complete on $\mathbb{R}^2$.

5.3. **The Bryant-Huang-Mo normal form.** So far, our discussions are based on the Bryant normal form (2). If we use the Bryant-Huang-Mo normal form (3) instead, similar results can be obtained. Since these two normal forms only differ by a coordinate transformation, the corresponding results are identical in content, but different in expressions. For example, the vector field to be considered has the following form:

$$X = -y\frac{\partial}{\partial x} + \big(xu(x)^{-2} + yv(x)\big)\frac{\partial}{\partial y}, \tag{16}$$

where $u(x)$ and $v(x)$ are the arbitrary functions appeared in (3). We may again use its flow φ_t to push the vector field $Y = \frac{\partial}{\partial y}$ to generate a generalized Finsler structure. This generalized Finsler structure reduces to a classical one, if and only if X admits an isochronous center. However, to find the isochronicity condition, we do not need to do any coordinate transformations, because it is already a Liénard system.

$$x' = -y, \quad y' = xu(x)^{-2} + yv(x).$$

Thus, Theorem 5.3 directly gives the expressions of u and v when the system is isochronous. Both normal forms suggest the following corollary holds.

Corollary 5.5. *The local isometry class of* regular *rotationally symmetric Finsler surfaces with vanishing flag curvature depends on two functions of one variable. One of them is an odd function, and the other is even.*

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