

ON THE DISTRIBUTION OF αp^k MODULO 1

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1. Introduction. The fractional part of the sequence $\{\alpha n^k\}$, where α is an irrational real number and k is an integer, was first studied early this century, initiated by the work of Hardy, Littlewood and Weyl. It seems very natural to consider the subsequence $\{\alpha p^k\}$, where p denotes a prime variable. The pioneering work in this direction was conducted by Vinogradov [13, 14]. Improvements have since been made by Vaughan [12], Ghosh [4], Harman [6, 7, 8] and Jia [11]. The best results to date have been obtained by Harman for $k = 1$ [9], by Baker and Harman for $2 \leq k \leq 12$ [1], and by Harman for larger k [8]. In the following work, we shall adopt a sieve technique developed by Harman in [6] to show the following.

THEOREM. *Let α be an irrational real number and write $\|x\| = \min_{n \in \mathbb{N}} |x - n|$. Then, for every real β , and every $\varepsilon > 0$, there are infinitely many primes p such that:*

$$\|\alpha p^k + \beta\| < p^{-\rho+\varepsilon}$$

where

$$\rho = \rho(k) = \begin{cases} 5/56, & \text{if } k = 3, \\ 1/21, & \text{if } k = 4, \\ (0.815)2^{-k}, & \text{if } k \geq 5. \end{cases}$$

The above result is an improvement for $3 \leq k \leq 12$ of that of Baker and Harman in [1], who obtained $(2/3)2^{-k}$. The constant 0.815 comes from a calculation which will be explained later in Section 4.

2. Notations and outline of method. Let ε be a small positive constant and η be another small positive constant, which is small compared to ε . The letter p will always denote a prime. The following notations will be used:

$$L = 2N^{\rho-\varepsilon/2}, \quad \delta = N^{-\rho+\varepsilon}/2, \quad \sigma = 2^k \rho.$$

Let a/q be a convergent to the continued fraction expansion of α such that $(LN^k)^{1/2} \leq q \leq (L(N+1)^k)^{1/2}$. Following [1], it suffices to prove that $\|\alpha p^k/q + \beta\| < \delta$ with $p \leq N$. Write:

$$B = \{n \in \mathbb{N} : n \leq N\}; \quad A = \{n \in B : (n, q) = 1, \|\alpha n^k/q + \beta\| < \delta\},$$

$$V(z) = \prod_{p < z} (1 - 1/p), \quad P(z) = \prod_{p < z} p, \quad V_1(z) = \prod'_{p < z} (1 - 1/p),$$

where \prod' (or \sum') here and hereafter means that the product (or sum) is taken over primes that are coprime to q . For a subset E in \mathbb{N} , the sieve function $S(E, z)$ and E_d are defined as:

$$E_d = \{n : dn \in E\};$$

$$S(E, z) = |\{n \in E : (n, P(z)) = 1\}|;$$

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\ll and \gg are the usual Vinogradov's symbols and $m \sim M$ will mean $M \leq m < 2M$. As usual, write $e(x) = e^{2\pi i x}$. We shall, using estimates for certain exponential sums, establish asymptotic formulae of the form:

$$\sum_{m \sim M} a_m S(A_m, z) = 2\delta \sum_{m \sim M} a_m S(B_m, z)(1 + R) \tag{1}$$

where R is an error of suitable size, for various ranges of M and z . Using Buchstab's identity, we may decompose the sum $S(A, N^{1/2})$ into single sums and double sums. It will turn out that we may give asymptotic formulae for each of the single sums and certain subsums of the double sums. The remaining double sums, being nonnegative, are discarded. Provided we have not discarded too much, this gives a positive lower bound for $S(A, N^{1/2})$. This establishes the theorem.

3. Lemmas. In this section, we shall establish some lemmas which are essential in proving the theorem.

LEMMA 1. Let $\{a_m\}$ be a sequence of real numbers for $m \in B$, all of the same sign. Then

$$\sum_{m \in A} a_m = 2\delta \sum_{m \in B} a_m + O\left(\delta \sum_{n \leq L} \left| \sum_{m \in B} a_m e\left(\frac{an}{q}(m^k)\right) \right|\right) + O\left(\frac{1}{L} \sum_{m \in B} |a_m|\right).$$

Proof. This may be established in the same way as [6, identity (4)].

LEMMA 2. Suppose that $\{a_d\}$ is a sequence of real numbers, all of the same sign and $a_d \ll N^\eta$. Then:

$$\sum_{d < D} a_d |A_d| = 2\delta \sum_{d < D} a_d |B_d| + O\left(\delta \sum_{d < D} \left| \sum_{n \leq L} a_d \sum_{md \in B} e\left(\frac{an}{q}(md)^k\right) \right|\right) + O(\delta N^{1-2\eta})$$

provided $D \ll N^{1-\epsilon}$.

Proof. This follows from Lemma 1 by noting that:

$$\sum_{d < D} a_d |A_d| = \sum_{m \in A} \left(\sum_{d < D, d|m} a_d \right).$$

In bounding the error terms arising in the asymptotic formulae, we shall require bounds on certain exponential sums, which are given in [1]. The bounds given there remain valid for the increased value of ρ we consider here.

LEMMA 3. Let $Y \ll N^\vartheta$, where $\vartheta = (k - (2k + 1)\rho)/(2k - 2)$. Then

$$\sum_{n \leq L} \sum_{y \sim Y} \max_M \left| \sum_{x \leq M, x \leq N/ly} e\left(\frac{an}{q}(xy)^k\right) \right| \ll N^{1-4\eta}.$$

Proof. This is [1, Lemma 10]. The removal of the condition of coprimality does not affect the result.

LEMMA 4. For $N^{2\rho} \ll M \ll N^{1-\sigma}$, we have

$$\sum_{n \leq L} \sum_{x \leq N/M} \left| \sum_{y \sim M} b_y e\left(\frac{an}{q}(xy)^k\right) \right| \ll N^{1-4\eta}$$

where $\{b_y\}$ is any sequence of complex numbers with $|b_y| \leq 1$.

Proof. This is [1, Lemma 11].

We now reach the stage where asymptotic formulae of the form (1) are to be established. Let θ be as in Lemma 3 above.

LEMMA 5. Suppose that $k \geq 4$ and $M \ll N^{1/2+2\rho}$, $b_m \ll N^\eta$. Then

$$\sum_{m \sim M} b_m S(A_m, N^\epsilon) = \sum_{m \sim M} b_m S(B_m, N^\epsilon) 2\delta(1 + R) \tag{2}$$

whenever the right hand side is $\gg \delta N^{1-\eta}$. Here $R = O(Q(s)) + O(\exp(-(\log N)^{1/2}))$ where $s = \log(N^\theta/M)/\log(N^\epsilon) > (\vartheta - (0.5 + 2\rho))/\epsilon$ and $Q(s) < \exp(-s \log s + s \log \log 3s + O(s))$.

REMARK. For $k = 4$, from now on, we shall take $\rho(4) = 1/21 - \epsilon_2$, where ϵ_2 is a small positive constant. Of course, ϵ_2 can be absorbed into ϵ in the theorem and so it would not affect the result.

Proof. Write $r(A_m, d) = (\varphi(q)/q)(2\delta N/md) - |A_{md}|$. By the fundamental lemma of Rosser's sieve [10], we have:

$$S(A_m, N^\epsilon) = \frac{2\delta N}{m} V_1(N^\epsilon) \frac{\varphi(q)}{q} [1 + O(Q(s))] + O\left(\sum_{d < N^\theta/M} a_d r(A_m, d)\right).$$

Also, by the Fundamental Lemma ([5, Theorem 2.5]),

$$S(B_m, N^\epsilon) = \frac{N}{m} V(N^\epsilon) (1 + O(Q(s)) + O(\exp(-\sqrt{\log N}))).$$

It suffices therefore to show that

$$\sum_{m \sim M} \left| b_m \sum_{d < N^\theta/M} a_d r(A_m, d) \right| \ll \delta N^{1-2\eta}.$$

Note that for some $c_m \in \mathbb{C}$, we have

$$\begin{aligned} \sum_{m \sim M} \left| b_m \sum_{d < N^\theta/M} a_d r(A_m, d) \right| &= \sum_{m \sim M} c_m \sum_{d < N^\theta/M} a_d \left(\frac{2\delta N}{md} - |A_{md}| \right) \\ &= \sum_j \lambda_j \left(2\delta \frac{N}{j} - |A_j| \right) \\ &\ll \delta \sum_{n \leq L} \left| \sum_j \lambda_j \sum_{jr \in B} e\left(\frac{an}{q}(jr)^k\right) \right| + O(\delta N^{1-2\eta}) \end{aligned}$$

by Lemma 2. The λ_j are complex numbers with $|\lambda_j| \ll N^\eta$. The result now follows from Lemma 3.

LEMMA 6. Suppose that $N^{2\rho} \leq M \leq N^{1-\sigma}$, $a_m, b_n, c_r \ll N^\eta$ with $c_r = 0$ whenever $r < N^\varepsilon$. Then

$$\sum_{mr \sim M} a_m c_r \sum_{n \sim X} b_n S(A_{mnr}, r) = 2\delta \sum_{mr \sim M} a_m c_r \sum_{n \sim X} b_n S(B_{mnr}, r)(1 + R) \tag{3}$$

whenever the right hand side is $\gg \delta N^{1-\eta}$. Here $R = O(1/\log N)$.

Proof. By Lemma 1, we have

$$\begin{aligned} \sum_{mr \sim M} a_m c_r \sum_{n \sim X} b_n S(A_{mnr}, r) &= \sum_{mr \sim M} a_m c_r \sum_{n \sim X} b_n \sum_{\substack{kmnr \in A, \\ p|k=p \geq r}} 1 = \\ &2\delta(1 + R) \sum_{mr \sim M} a_m c_r \sum_{n \sim X} b_n S(B_{mnr}, r) + O\left(\delta \sum_{l \leq L} \left| \sum_{mr \sim M} a_m c_r \sum_{n \sim X} b_n \sum_{\substack{wmnr \in B \\ p|w \Rightarrow p \geq r}} e\left(\frac{aj}{q}(wmnr)^k\right) \right.\right) \end{aligned}$$

where $R = O(1/\log N)$.

It remains to show that the exponential sum is $\ll N^{1-2\eta}$. To achieve this, we have recourse to the following result ([3, p. 165]):

$$\int_{\Gamma} e^{ita} \frac{\sin bt}{\pi t} dt = \Delta(a, b) + O\left(\frac{1}{T(b - |a|)}\right), \quad \text{where } \Delta(a, b) = 1 \text{ if } |a| \leq b, \\ \Delta(a, b) = 0 \text{ if } |a| > b. \tag{4}$$

Here Γ is $[-T, T]$.

The variable w in the inner sum is a product of at most s distinct primes where $s \ll 1/\varepsilon$. Write

$$\sum_{mr \sim M} a_m c_r \sum_{n \sim X} b_n \sum_{wmnr \in B, p|w=p \geq r} e\left(\frac{aj}{q}(wmnr)^k\right) = \sum_1 + \sum_2 + \dots + \sum_s$$

where, for $t = 1, 2, \dots, s$,

$$\sum_t = \sum_{mr \sim M} a_m c_r \sum_{n \sim X} b_n \sum_{\substack{p_1 \dots p_t nmr \in B \\ r < p_1 \leq p_2 \leq \dots \leq p_t}} e\left(\frac{aj}{q}(p_1 \dots p_t nmr)^k\right).$$

To get rid of the inter-dependence of the variables, we apply (4) t times, with T being a suitable power of N . Each application incurs a factor of $O(\log N)$ due to the integration. For example, to get rid of the condition $p_i \leq p_j$, we let $a = p_i$ and $b = p_j + 1/2$.

$$\sum_t \ll_{\eta} N^\eta \left| \sum_{mr \sim M} a_m c_r \sum_{n \sim X} b_n \sum_{p_1 \dots p_t nmr \in B} e\left(\frac{aj}{q}(p_1 \dots p_t nmr)^k\right) \lambda(p_1) \dots \lambda(p_t) \right|$$

where $|\lambda(p_k)| \leq 1$. Therefore:

$$\sum_{mr \sim M} a_m c_r \sum_{n \sim X} b_n \sum_{\substack{wmnr \in B \\ p|w \Rightarrow p > r}} e\left(\frac{aj}{q} (wmnr)^k\right) \ll_{\eta, \epsilon} N^\eta \left| \sum_{mr \sim M} a_m c_r \sum_{n \sim X} b_n \sum_{wmnr \in B} u_w e\left(\frac{aj}{q} (wmnr)^k\right) \right|$$

where $|u_w| \leq 1$. The sum on the right hand side is

$$\ll_{\eta, \epsilon} N^\eta \left| \sum_{n \leq N/M} d_n \sum_{m \sim M, mn \in B} \alpha_m e\left(\frac{al}{q} (mn)^k\right) \right|$$

for some d_n and α_m whose magnitudes are $O_\eta(N^\eta)$ for each n and m . We apply (4) again to eliminate the dependence between the variables m and n , at the cost of a factor of $O(\log N)$. Bringing in the summation over l , the result now follows from Lemma 4.

The following lemma allows us to control the error that accumulates as we iterate using the fundamental Buchstab identity, and which was not dealt with accurately in [6].

LEMMA 7. *Let E be any subset of B . Let $t = [1/\epsilon]$, where $[x]$ denotes the integer part of x . Then*

$$\sum_{1 \leq h \leq t} \sum_{N^\epsilon \leq p_1 \leq \dots \leq p_h} S(E_{p_1, \dots, p_h}, N^\epsilon) \leq 2^t S(E, N^\epsilon).$$

Proof. This is Lemma 10 in [2].

We are now in a position to prove our main versions of (1). To simplify the notation, we write $\xi = N^{2\rho}$ and $\zeta = N^{1-\sigma}$.

LEMMA 8. *Suppose that $M \leq \zeta$, $a_m \ll N^\eta$, $a_m \geq 0$, and $a_m = 0$ if $p | m \Rightarrow p < \zeta/\xi$. Then*

$$\sum_{m \sim M} a_m S(A_m, \zeta/\xi) = \sum_{m \sim M} a_m S(B_m, \zeta/\xi) 2\delta(1 + R)$$

whenever the R.H.S. $\gg \delta N^{1-\eta}$. Here:

$$R = O_\epsilon\left(\frac{1}{\log N}\right) + O\left(2^{1/\epsilon} Q\left(\frac{\theta - 0.5 + 2\rho}{\epsilon}\right)\right).$$

Proof. If $M \geq N^{2\rho}$, this follows immediately from Lemma 6, so we may assume $M < N^{2\rho}$. Apply Buchstab's identity to obtain

$$\sum_{m \sim M} a_m S(A_m, \zeta/\xi) = \sum_{m \sim M} a_m S(A_m, N^\epsilon) - \sum_{m \sim M} a_m \sum_{N^\epsilon \leq p < \zeta/\xi} S(A_{mp}, p).$$

We can deal with the first sum using Lemma 5. For the subsum in the second sum with $mp \geq \xi$, we could apply Lemma 6. For the remaining parts, we apply Buchstab's identity:

$$\sum_{m \sim M} a_m \sum_{\substack{N^\epsilon \leq p < \zeta/\xi \\ mp < \xi}} S(A_{mp}, p) = \sum_{m \sim M} a_m \sum_{\substack{N^\epsilon \leq p < \zeta/\xi \\ mp < \xi}} \left(S(A_{mp}, N^\epsilon) - \sum_{N^\epsilon \leq r < p} S(A_{mpr}, r) \right).$$

The first term can again be handled by Lemma 5. The parts in the second term with

$mpr \geq \xi$ can be dealt with by Lemma 6. As before, we apply Buchstab's identity to the remaining parts.

Continuing in this way, we have the result above after $O(1/\varepsilon)$ steps. The error can be at most

$$\sum_{1 \leq h \leq t} \delta \sum_{m \sim M} a_m \sum_{N^\varepsilon \leq p_1 \dots < p_h} S(B_{mp_1 \dots p_h}, N^\varepsilon) \left(K_1 Q(s) + K_2 \left(\frac{1}{\log N} \right) \right)$$

where $t = [1/\varepsilon]$ and $s = (\theta - (0.5 + \rho))/\varepsilon$. K_1 is an absolute constant while K_2 may depend on ε . By Lemma 7, this is

$$\begin{aligned} &<< \delta \sum_{m \sim M} a_m 2^t S(B_m, N^\varepsilon) \left(K_1 Q(s) + K_2 \left(\frac{1}{\log N} \right) \right) \\ &<< \delta \sum_{m \sim M} a_m \frac{2^t}{\varepsilon} S(B_m, \zeta/\xi) \left(K_1 Q(s) + K_2 \left(\frac{1}{\log N} \right) \right). \end{aligned}$$

Note that $2^t Q(s)/\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. This establishes the result.

Similarly, we may prove the following.

LEMMA 9. For $k \geq 4$, and $\zeta < M \leq N^{1/2}$, $0 \leq a_m \leq N^\eta$, and $a_m \neq 0 \Rightarrow p \geq \xi$ for all $p \mid m$. Then

$$\sum_{m \sim M} a_m S(A_m, \xi) = \sum_{m \sim M} a_m S(B_m, \xi) 2\delta(1 + R)$$

whenever the R.H.S. is $\gg \delta N^{1-\eta}$. Here $R = O(2^{1/\varepsilon} \varepsilon^{-1} Q(1/\varepsilon)) + O_\varepsilon(1/\log N)$.

Proof. This is similar to that of Lemma 8. The reason for the value ξ is that it is the maximum value for which $M\xi \leq N^\theta$ with equality possible for $k = 4$, a condition required for the application of Lemma 5.

4. Proof of Theorem for $k \geq 4$. We decompose $S(A, N^{1/2})$ as follows:

$$S(A, N^{1/2}) = S(A, \zeta/\xi) - \sum_{\zeta/\xi \leq p \leq \zeta} S(A_p, p) - \sum_{\zeta < p < N^{1/2}} S(A_p, \xi) + \sum_{\zeta < p < N^{1/2}} \sum_{\xi \leq r < p} S(A_{pr}, r).$$

Note that the condition $pr^2 < N$ is implicit in the last sum. By Lemmas 8 and 9, we can give asymptotic formulae for the first three terms. In the double sum, we can give an asymptotic formula for the part with $\xi \leq r \leq \zeta$ by Lemma 6. We discard the rest. This gives an overall lower bound for $S(A, N^{1/2})$:

$$S(A, N^{1/2}) \geq 2\delta \frac{N}{\log N} \left(1 - \frac{\log N}{N} \sum_{\zeta < p < N^{1/2}} \sum_{\zeta < r < p} S(B_{pr}, r) \right).$$

We shall estimate the amount discarded using the prime number theorem and the asymptotic formula:

$$S(B_n, z) = \frac{N}{n \log N} \omega \left(\frac{\log(N/n)}{\log z} \right) \left(1 + O \left(\frac{1}{\log N} \right) \right).$$

Here, $\omega(u)$ is Buchstab's function which satisfies:

$$\begin{aligned} \omega(u) &= 1/u && \text{for } 1 \leq u \leq 2, \\ \omega(u) &= (1 + \log(u - 1))/u && \text{for } 2 \leq u \leq 3, \\ \omega(u) &< 0.5644 && \text{for } u > 3. \end{aligned}$$

We can replace the double sums by the double integral (with a negligible error)

$$\iint_{\Delta} \omega\left(\frac{1 - \alpha - \beta}{\beta}\right) \frac{d\alpha d\beta}{\alpha\beta^2}$$

where $\Delta := \{(\alpha\beta) : 1 - \sigma \leq \alpha \leq 1/2, 1 - \sigma \leq \beta \leq \min(\alpha, (1 - \alpha)/2)\}$.

By a computer calculation, one can show that the amount discarded is < 1 . Hence the result is proved. The limit of the value for $\rho(4)$ is set by the condition that $1/2 + 2\rho \leq \vartheta$, which is required in establishing Lemma 5 and Lemma 9. The limit for the other values is set by the fact that we cannot decompose the double sums into manageable parts with discardable positive sums.

5. The case of $k = 3$. For the above to work, we require at least $\vartheta \geq 1/2 + 2\rho$. For $k = 3$, this leads to $\rho \leq 1/15$. Thus, for an improvement, a trick is required to circumvent this difficulty. With the introduction of this new idea, the restriction can now be relaxed to $\vartheta \geq \rho + 1/2$. This gives $\rho \leq 1/11$. The following is the consequence of Lemmas 5 and 6 with $k = 3$. To simplify the notation, write $\tau = N^{1/2-\rho}$, $\nu = N^{1/2+\rho}$.

LEMMA 10. For $M \leq \tau$, $a_m \ll N^\eta$, we have

$$\sum_{m \sim M} a_m S(A_m, \zeta/\xi) = \sum_{m \sim M} a_m S(B_m, \zeta/\xi) 2\delta(1 + R)$$

where R is as in Lemma 8.

It is clear that simply using the lemmas obtained so far, we cannot give an asymptotic formula for the sum $\sum S(A_p, N^c)$ for the range $\tau < p < N^{1/2}$, for some suitable value c as before. We decompose the sum $S(A, N^{1/2})$ as follows:

$$\begin{aligned} S(A, N^{1/2}) &= S(A, \zeta/\xi) - \sum_{\zeta/\xi \leq p \leq \zeta} S(A_p, p) - \sum_{\zeta < p \leq \tau} S(A_p, p) - \sum_{\tau < p < N^{1/2}} S(A_p, p) \\ &= S(A, \zeta/\xi) - \sum_{\zeta/\xi \leq p \leq \xi} S(A_p, \zeta/\xi) - \sum_{\xi \leq p \leq \zeta} S(A_p, p) - \sum_{\zeta < p \leq \tau} S(A_p, \zeta/\xi) \\ &\quad - \sum_{\tau < p < N^{1/2}} S(A_p, p) + \sum_{\zeta/\xi \leq p < \xi} \sum_{\zeta/\xi \leq q < p} S(A_{pq}, q) + \sum_{\zeta < p \leq \tau} \sum_{\zeta/\xi \leq q < p} S(A_{pq}, q) \\ &= S_1 - S_2 - S_3 - S_4 - S_5 + S_6 + S_7 \end{aligned}$$

say. We may give asymptotic formulae for S_1, S_2, S_3, S_4 by Lemmas 6 and 10. We may give asymptotic formulae for the subsums of S_6 where $pq < \zeta$. We may also give an asymptotic formula for the subsums of S_7 with $\zeta/\xi \leq q < \zeta^{1/2}$ using similar methods in establishing Lemma 8: here, if $\zeta^{1/2} \geq p_1 \geq p_2 \geq p_3 \geq \dots \geq p_h$, $p_1 \dots p_{h-1} \leq \xi$, $p_1 \dots p_h \geq \xi$, then

$pp_1 \dots p_{h-1} \leq N^{1/2+\rho}$, also $p_1 \dots p_h \leq N^{1-8\rho} \leq \zeta$. We now deal with S_5 . By Buchstab's identity, we have

$$S_5 = \sum_{\tau < p < N^{1/2}} S(A_p, N^\epsilon) - \sum_{\tau < p < N^{1/2}} \sum_{N^\epsilon \leq q < p} S(A_{pq}, q),$$

The second sum can be written as a sum of four sums T_1, T_2, T_3, T_4 where $T_1 = \sum S(A_{pq}, q)$ with $\tau < p < N^{1/2}$ and $N^\epsilon \leq q < \zeta/\xi$, $T_2 = \sum S(A_{pq}, q)$ with $\tau < p < N^{1/2}$ and $\zeta/\xi < q < \xi$, $T_3 = \sum S(A_{pq}, q)$ with $\tau < p < N^{1/2}$ and $\xi \leq q < \zeta$, and $T_4 = \sum S(A_{pq}, q)$ with $\tau < p < N^{1/2}$ and $q > \zeta$. We shall discard T_2 and T_4 . T_3 can be dealt with using Lemma 6. For T_1 , since $pq < v$, we may apply Buchstab's identity:

$$T_1 = \sum S(A_{pq}, N^\epsilon) - \sum_{N^\epsilon \leq t < q} S(A_{pqt}, t)$$

By Lemma 5, we may give an asymptotic formula for the first sum. We can also give an asymptotic formula for subsums in the second sum with $qt \geq \xi$ (since it is necessary that $qt < \zeta$). For the remainder, we split the sum again:

$$\sum_{qt < N^{2\rho}} S(A_{pqt}, t) = \sum_{pqt \leq v} S(A_{pqt}, t) + \sum_{pqt > v} S(A_{pqt}, t),$$

We first consider the first term. We apply Buchstab's identity:

$$\sum_{pqt \leq v} S(A_{pqt}, t) = \sum S(A_{pqt}, N^\epsilon) - \sum_{N^\epsilon \leq h < t} S(A_{pqth}, h).$$

It is necessary that $qth \leq \zeta$, so that subsums in the second term are manageable if $qth \geq \xi$. Split the remaining subsum according to whether $pqth > v$ or $pqth \leq v$. Thus, it suffices to consider sums of the form

$$\sum_{pq > v} S(A_{pq}, q), \quad \sum_{pq r_2 > v} \sum_{N^\epsilon \leq q < \zeta/\xi} \sum_{N^\epsilon \leq r_2 < q, q r_2 < \xi} S(A_{pq r_2}, r_2), \dots \text{ etc.}$$

We use \sum^* to denote the sum over the ranges: $\tau < p \leq N^{1/2}$, $N^\epsilon \leq r_m < r_{m-1} < \dots < r_1 \leq \zeta/\xi$, $r_1 r_2 \dots r_m < \xi$, $pr_1 r_2 \dots r_m > v$, $pr_1 r_2 \dots r_{m-1} \leq v$. Thus, the general form of the sum we have to consider is

$$\sum^* S(A_{pr_1 \dots r_m}, r_m),$$

The "different technique" mentioned in the introduction involves writing

$$\sum^*_{pr_1 \dots r_m > v} S(A_{pr_1 \dots r_m}, r_m) = \sum^*_{s|x \Rightarrow s > r_m} S\left(A_{xr_1 \dots r_m}, \left(\frac{N}{xr_1 \dots r_m}\right)^{1/2}\right).$$

We remark that $x \leq \tau$ and $N/(xr_1 \dots r_m)$ has the size of p . Thus, we may apply Buchstab's identity:

$$\sum^*_{s|x \Rightarrow s > r_m} S\left(A_{xr_1 \dots r_m}, \left(\frac{N}{xr_1 \dots r_m}\right)^{1/2}\right) = \sum^*_x S(A_{xr_1 \dots r_m}, N^\epsilon) - \sum_{N^\epsilon \leq r < (N/r_1 \dots r_m x)^{1/2}} S(A_{xr_1 \dots r_m}, r).$$

By Lemma 5, the first term can be dealt with. By Lemma 6, the subsums in the second sum are manageable if $r \geq \xi$ (since it is necessary that $r < \zeta$). For the remainder, we observe that when $\zeta/\xi \leq r \leq \xi$, $rr_1 \dots r_m \geq \zeta/\xi$. $N^\rho \geq \xi$, as it is necessary that $r_1 \dots r_m \geq N^\rho$.

Thus, by Lemma, 6 we may give an asymptotic formula for the subsum of the second sum with $\zeta/\xi \leq r \leq \xi$.

For the part with $r < \zeta/\xi$, we have an iterative process: the sum with $r_1 \dots r_m r \geq \xi$ is manageable by Lemma 6. If $r_1 \dots r_m r < \xi$, we iterate again and so we eventually obtain the desired result after $O(1/\epsilon)$ operations. Thus, we may give an asymptotic formula for T_1 with an error of suitable size, similar to that in Lemma 8 or 9.

We discard T_2, T_4 and the subsums of S_5 and S_7 for which we fail to give asymptotic formulae. We approximate those sums by integrals as before. This produces a lower bound of

$$\frac{\delta N}{\log N} \left[1 - \left(\iint_{D_1} + \iint_{D_2} + \iint_{D_3} \right) \omega \left(\frac{1 - \alpha - \beta}{\beta} \right) \frac{d\alpha d\beta}{\alpha\beta^2} \right]$$

where

$$D_1 = \{(\alpha, \beta) : \alpha + \beta \geq 1 - 8\rho, 1 - 10\rho < \beta < \alpha < 2\rho\},$$

$$D_2 = \{(\alpha, \beta) : \beta \leq \min(\alpha, 1 - 2\alpha), \beta > 1 - 8\rho\},$$

$$D_3 = \{(\alpha, \beta) : 1 - 8\rho < \alpha \leq 1/2 - \rho, (1 - 8\rho)/2 \leq \beta \leq 2\rho\}$$

$$\cup \{(\alpha, \beta) : 1/2 - \rho < \alpha \leq 1/2, 1 - 10\rho \leq \beta \leq 2\rho\}.$$

The function ω is Buchstab's function as given in section 4. The region D_1 corresponds to the subsum of S_6 to be discarded and D_2 corresponds to T_4 and the subsum of S_7 with $q > \zeta$, while D_3 corresponds to the subsum of S_7 with $\zeta/\xi \leq q < \xi$ and T_2 . We evaluate these integrals by machine calculations. With $\rho = 5/56$, we have

$$\iint_{D_1} \omega \left(\frac{1 - \alpha - \beta}{\beta} \right) \frac{d\alpha d\beta}{\alpha\beta^2} < 0.22,$$

$$\iint_{D_2} \omega \left(\frac{1 - \alpha - \beta}{\beta} \right) \frac{d\alpha d\beta}{\alpha\beta^2} < 0.10,$$

$$\iint_{D_3} \omega \left(\frac{1 - \alpha - \beta}{\beta} \right) \frac{d\alpha d\beta}{\alpha\beta^2} < 0.65.$$

Thus, we do have a positive lower bound, so our result is proved. The value of ρ may be improved slightly, but it is noted that, with $\rho = 6/67$, the amount discarded would be greater than 1.

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