AN APPLICATION OF A THEOREM OF SINGER

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The authors have recently treated (**2**) the problem of finding subsets E of the real line $\mathbb{R}$, of type F_σ, such that $E-E$ contains an interval and the k-fold vector sum $(k)E$ is of measure zero. Positive results can be obtained, for all k, on the basis of a recent theorem of J. A. Haight (**3**), following earlier partial results (**1**), (**4**) for $k \leqq 7$; and indeed in these cases the problem has a solution with E a perfect set. An analogous problem, apparently in most respects subtler than the first, is the following. Do there exist finite regular Borel measures μ on $\mathbb{R}$ such that $\mu * \tilde{\mu}$ is absolutely continuous (where $\tilde{\mu}$ is the adjoint of μ) and the kth convolution power μ^k is singular? Both problems are of interest in the general context of elucidating the properties of the measure algebra $M(\mathbb{R})$ or, more generally, $M(G)$ for locally compact abelian G. The second problem may be regarded as an attempt to provide (at least one aspect of) a multiplicity theory for the first.

The purpose of the present note is to point out that the necessary machinery for a positive solution of this second problem in the simplest case, $k = 2$, has been available for many years, in a classical theorem of Singer (**7**, pp. 380-381; see also **6**, Theorem 6.1); we give the solution in this case. It appears that the construction has inherent limitations and necessarily fails to provide a solution for large k. We show by an elementary argument that it must break down for $k \geqq 6$. This does not, of course, imply that the result is false in such cases. It would seem that for $k \geqq 3$ a subtler approach than ours is needed to determine the truth of the matter.

We begin with some considerations involving measures on $\mathbb{R}$. We shall produce measures with the required properties by taking suitable infinite convolution products of simple probability measures. We shall say that a sequence ν_n of measures tends weakly to a limit ν if $\nu_n(I) \to \nu(I)$ for each interval of continuity of ν (i.e. I is to be open and its end-points are not atoms for ν). In the present context this mode of convergence has various equivalent formulations.

Lemma 1. *Let* (ν_n) *be a sequence of non-negative measures on* $\mathbb{R}$ *with the properties*

(i) $\nu_n \to \nu$ *weakly,*

(ii) *there is a constant* $c > 0$ *such that for any interval* I, *of length* $l(I)$,

$$\limsup_{n\to\infty} \nu_n(I) \leqq cl(I).$$

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Then ν is absolutely continuous: in fact

$$\nu(A) = \int_A f(x)dx$$

where $0 \leqq f(x) \leqq c$ for all x.

Proof. This is a straightforward verification.

Suppose next that $\boldsymbol{q} = (q_1, q_2, \ldots,)$ is a sequence of integers with $q_r \geqq 2$ for all $r \in \mathbb{N}$. Write $d_r = (q_1 q_2 \ldots q_r)^{-1}$. For each $r \in \mathbb{N}$ let F_r be a set of integers, $F_r = \{n_1^{(r)}, \ldots, n_{\sigma_r}^{(r)}\}$, say. Let μ_r be a probability measure (that is, non-negative and of total mass 1) supported on the points $n^{(r)}d_r$ $(n^{(r)} \in F_r)$. We shall suppose, to simplify the situation, that each $n^{(r)} \in F_r$ is essential, that is, $\mu_r(n^{(r)}d_r) > 0$ for each $n^{(r)}$. It is not always true that $\mu = \mu_1 * \mu_2 * \ldots$ converges; and even if it does its total mass may be strictly less than 1. However, if

there is a constant C, independent of r, with $| n^{(r)} | \leqq Cq_r$ for all $n^{(r)} \in F_r$ and all r (1)

then $\mu_1 * \mu_2 \ldots$ converges weakly to a probability measure μ. The support of μ is the set of all points that admit a development of the form $\sum_{r=1}^{\infty} n^{(r)}d_r$, with $n^{(r)} \in F_r$. Condition (1) is satisfied in all the cases with which we are concerned.

Lemma 2. *Suppose that* (1) *holds, and also*

the residues of the $n^{(r)}$ mod q_r form a complete set. (2)

Let μ_r be a probability measure on the points $n^{(r)}d_r$ such that for each integer a, $0 \leqq a \leqq q_r - 1$, the total mass of μ_r on the $n^{(r)}d_r$ with $n^{(r)} \equiv a \bmod q_r$ is exactly q_r^{-1}. Then $\mu = \mu_1 * \mu_2 * \ldots$ is absolutely continuous.

Proof. Write $\nu_r = \mu_1 * \mu_2 * \ldots * \mu_r$. Let E_r be the set of numbers of the form $\sum_{s=1}^{r} n^{(s)}d_s$, with $n^{(s)} \in F_s$ $(1 \leqq s \leqq r)$ and E the set of all numbers of the form $\sum_{s=1}^{\infty} n^{(s)}d_s$ with $n^{(s)} \in F_s$ $(s \geqq 1)$. The measure ν_r is concentrated on E_r, and the total mass concentrated on any one point of E_r does not exceed d_r, as is easily verified. Moreover, the points of E_r are distant at least d_r from each other. The measures ν_r converge weakly to a limit μ, whose support is E, and it is immediate from Lemma 1 that μ is absolutely continuous (with $c = 1$).

Lemma 3. *If* (1) *and* (2) *hold, and if μ_r is a probability measure on the $n^{(r)}d_r$ with $\mu_r = \mu_r' + \mu_r''$, where μ_r' satisfies the conditions specified for μ_r in Lemma* 2, *and*

$$\sum_{r=1}^{\infty} \| \mu_r'' \| < \infty$$

then $\mu = \mu_1 * \mu_2 * \ldots$ is absolutely continuous.

Proof. It is clear from Lemma 2 that $\mu' = \mu_1' * \mu_2' * \ldots$ is absolutely continuous;

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the present Lemma is now a consequence of the classical theorem of Jessen and Wintner (**5**, Theorem 35).

The explicit reference to Jessen and Wintner's theorem can be avoided in the present case by a simple direct argument. For each $r \in \mathbb{N}$ we can write

$$\mu = \mathop{\ast}_{s=1}^{r} \mu_s * \mathop{\ast}_{s=r+1}^{\infty} \mu'_s + \left(\mu - \mathop{\ast}_{s=1}^{r} \mu_s * \mathop{\ast}_{s=r+1}^{\infty} \mu'_s\right).$$

The first term is clearly absolutely continuous and the norm of the second is $1 - \prod_{s=r+1}^{\infty} (1 - \| \mu''_s \|)$; this$\to 0$ as $r \to \infty$. The measure μ is then the limit in norm of absolutely continuous measures, and is thus itself absolutely continuous.

Lemma 4. *If* (1) *holds, and now* $F_r = \{n_1^{(r)}, \ldots, n_{\sigma_r}^{(r)}\}$ *is a set of integers with*

$$\sigma_r \leqq (1-\omega_r)q_r, \textit{ where } \prod_{r=1}^{\infty} (1-\omega_r) = 0$$

and μ_r *is a probability measure on the points* $n^{(r)}d_r$ *then* $\mu = \mu_1 * \mu_2 * \ldots$ *is singular.*

Proof. If as before $\nu_r = \mu_1 * \mu_2 * \ldots * \mu_r$ then the support E_r of ν_r contains at most

$$d_r^{-1} \prod_{s=1}^{r} (1-\omega_s)$$

points, all of the form nd_r. For any given a, b the proportion of the points of the form nd_r that are in $E_r \cap [a, b]$, as compared with $[a, b]$, tends to 0 as $r \to \infty$ because $\Pi(1-\omega_s)$ diverges to 0. It follows that the support of $\mu = \lim \nu_r$ is of measure 0, and this in turn implies that μ is singular ($\mu = 0$ is excluded by condition (1)).

We now come to the main construction. Let a_r be any prime-power; Singer has established (**7**, pp. 380-381) the existence of a set A_r of $a_r + 1$ residues mod $a_r^2 + a_r + 1$, such that $A_r - A_r$ contains all residues, with zero represented $a_r + 1$ times and the other $a_r^2 + a_r$ residues once each. We may suppose without loss of generality that the elements of A_r are integers n with $0 \leqq n < a_r^2 + a_r + 1$. Let $q_r = a_r^2 + a_r + 1$, and let λ_r be the measure with mass $(a_r + 1)^{-1}$ at each point nd_r, $(n \in A_r)$.

Theorem 1. *If* $\sum_{r=1}^{\infty} a_r^{-1} < \infty$ *then* $\lambda = \lambda_1 * \lambda_2 * \ldots$ *is such that* $\lambda * \tilde{\lambda}$ *is absolutely continuous and* λ^2 *is singular.*

Proof. Write $\mu_r = \lambda_r * \tilde{\lambda}_r$: then it is immediate that μ_r has mass $(a_r + 1)^{-1}$ at 0 and $(a_r + 1)^{-2}$ at the other $a_r^2 + a_r$ points of $\{nd_r: n \in A_r - A_r\}$. Let μ'_r have mass $(a_r^2 + a_r + 1)^{-1}$ at each point of $\{nd_r: n \in A_r - A_r\}$; if $\mu_r = \mu'_r + \mu''_r$ then it is clear that

$$\| \mu'' \| = 2a_r^2/(a_r + 1)(a_r^2 + a_r + 1) \sim 2a_r^{-1}$$

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and we are thus in the situation of Lemma 3. It follows from that lemma that $\mu = \mu_1 * \mu_2 * \ldots = \lambda_1 * \tilde{\lambda}_1 * \lambda_2 * \tilde{\lambda}_2 * \ldots = \lambda * \tilde{\lambda}$ is absolutely continuous.

Moreover, if $\nu_r = \lambda_r^2$, the support of ν_r is $\{nd_r: n \in (2)A_r\}$, and this contains at most $a_r + 1 + \frac{1}{2}(a_r^2 + a_r)$ points. With $q_r = a_r^2 + a_r + 1$ this leads to

$$\limsup\,(1-\omega_r) \leqq \tfrac{1}{2}$$

and hence $\Pi(1-\omega_r) = 0$. It follows from Lemma 4 that

$$\nu = \nu_1 * \nu_2 * \ldots = \lambda_1 * \lambda_1 * \lambda_2 * \lambda_2 * \ldots = \lambda^2$$

is singular.

One consequence of Theorem 1 is the following:

Corollary. *There exists on $\mathbb{R}$ a function f (necessarily continuous and tending to zero at $\pm\infty$) that is the Fourier-Stieltjes transform of a singular measure, such that $|f|$ is the Fourier-Stieltjes transform of an absolutely continuous measure.*

Proof. Let λ be as in Theorem 1 and let f be the Fourier-Stieltjes transform of λ^2.

We conclude by showing, as promised, that our method does not seem well suited to discuss the cases of larger values of k.

Let $M(\mathbb{R}, d)$ be the subset (in fact a subalgebra) of $M(\mathbb{R})$ consisting of measures supported on the subgroup $\{nd: n \in \mathbb{Z}\}$, and let q be a positive integer. Defining the map ϕ from $M(\mathbb{R}, d)$ to $M(\mathbb{Z}(q))$ by $(\phi\mu)(a) = \sum_{n \equiv a \bmod q} \mu(nd)$ it is clear that ϕ is a homomorphism. In general $\|\phi\mu\| \leqq \|\mu\|$, and if for each a all the $\mu(nd)$ with $n \equiv a \bmod q$ have the same sign (or the same phase in the complex case) then $\|\phi\mu\| = \|\mu\|$. It will be convenient in the following to look at the images under ϕ of measures on $\mathbb{R}$ rather than directly at the measures themselves.

Theorem 2. *If λ is as in Theorem* 1 *then λ^6 is absolutely continuous.*

Proof. Let q_r, d_r be as before and write $G = \mathbb{Z}(q_r)$. Let ϕ be defined as above relative to d_r, q_r. Let $\phi\lambda_r = \alpha_r$; then $\alpha_r * \tilde{\alpha}_r = \beta_r + \gamma_r$, say with

$$\beta_r = \phi\mu_r', \quad \gamma_r = \phi\mu_r''.$$

It is clear that β_r is Haar measure on G (normalised to have mass 1), and that $\|\gamma_r\| = \|\mu_r''\| = 2a_r^2/(a_r+1)(a_r^2+a_r+1)$. Taking Fourier transforms, defined for any $\sigma \in M(G)$ by $\hat{\sigma}(x) = \sum_{y \in G} \sigma(y)\langle y, x\rangle$, so that

$$\sigma(x) = \frac{1}{|G|} \sum_{y \in \hat{G}} \overline{\hat{\sigma}(y)\langle x, y\rangle},$$

where $\hat{G}$ ($\cong G$, here) is the dual group of G, we have

$$|\hat{\alpha}_r(x)|^2 = \hat{\beta}_r(x) + \hat{\gamma}_r(x)$$

where $\hat{\beta}_r(0) = 1$, $\hat{\beta}_r(x) = 0$ if $x \neq 0$ and (since $\hat{\alpha}_r(0) = 1$) $\hat{\gamma}_r(0) = 0$, $|\hat{\gamma}_r(x)| \leqq \|\gamma_r\|$ everywhere.

We may thus write

$$\hat{\alpha}_r(x) = \hat{\beta}_r(x) + \hat{\varepsilon}_r(x),$$

where $\hat{\varepsilon}_r(0) = 0$ and $|\hat{\varepsilon}_r(x)| \leqq \|\gamma_r\|^{\frac{1}{2}}$ everywhere, and it follows that for any $k \in \mathbb{N}$ we have

$$\hat{\alpha}_r^k(x) = \hat{\beta}_r(x) + \hat{\varepsilon}_r^k(x).$$

Thus $|\hat{\varepsilon}_k^r(x)| \leqq \|\gamma_r\|^{k/2}$ everywhere, whence $|\varepsilon_r^k(y)| \leqq \dfrac{1}{|G|}|G|\,\|\gamma_r\|^{k/2}$, and

$$\|\varepsilon_r^k\| = \sum_{y \in G} |\varepsilon_r^k(y)| \leqq |G|\,\|\gamma_r\|^{k/2}$$
$$= (a_r^2 + a_r + 1)[2a_r^2/(a_r+1)(a_r^2+a_r+1)]^{k/2} \sim 2^{k/2} a_r^{2-k/2}.$$

Let θ_r be chosen so that $\phi\theta_r = \varepsilon_r^k$, $\|\theta_r\| = \|\varepsilon_r^k\|$, and $\lambda_r^k - \theta_r$ is non-negative. Then $\phi(\lambda_r^k - \theta_r)$ is Haar measure on G, so $\lambda_r^k - \theta_r$ satisfies the requirement for μ_r' in Lemma 3, while $\|\theta_r\| = \|\varepsilon_r^k\| \sim 2^{k/2} a_r^{2-k/2}$. Thus, if $k \geqq 6$, $\Sigma\|\theta_r\| < \infty$ (since $\Sigma a_r^{-1} < \infty$) and all the conditions of Lemma 3 hold. Thus

$$\mathop{\ast}_{r=1}^{\infty} \lambda_r^6 = \left(\mathop{\ast}_{r=1}^{\infty} \lambda_r\right)^6 = \lambda^6$$

is absolutely continuous.

It is evident that if $a_r \to \infty$ more rapidly, so that $\Sigma a_r^{-\frac{1}{2}} < \infty$, then λ^5 is already absolutely continuous. It seems likely that these results could be substantially improved by the use of more delicate arguments.

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