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TWO NEW GENERALISED HYPERSTABILITY RESULTS FOR THE DRYGAS FUNCTIONAL EQUATION

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Abstract

Let X be a nonempty subset of a normed space such that $0 \notin X$ and X is symmetric with respect to 0 and let Y be a Banach space. We study the generalised hyperstability of the Drygas functional equation

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y),$$

where f maps X into Y and $x, y \in X$ with $x+y, x-y \in X$. Our first main result improves the results of Piszczek and Szczawińska ['Hyperstability of the Drygas functional equation', *J. Funct. Space Appl.* **2013** (2013), Article ID 912718, 4 pages]. Hyperstability results for the inhomogeneous Drygas functional equation can be derived from our results.

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1. Introduction and preliminaries

In the sequel, $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{R}_+$, $\mathbb{R}$ and $\mathbb{N}_{n_0}$ denote the set of nonnegative integers, the set of positive integers, the set of nonnegative real numbers, the set of real numbers and the set of all integers greater than or equal to n_0, respectively. Also, B^A denotes the set of all functions from a set $A \neq \emptyset$ to a set $B \neq \emptyset$.

We say that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the Drygas functional equation if and only if

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y) \tag{1.1}$$

for all $x, y \in \mathbb{R}$. For example, the functions $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = cx$ and $f(x) = cx^2$ for all $x \in \mathbb{R}$, where c is a fixed real number, satisfy the Drygas functional equation. Also note that if $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the Drygas functional equation, then $f \pm g$ also satisfy the Drygas functional equation.

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First, we will recall some history. The equation (1.1) was first considered in 1987 by Drygas [7] in order to obtain a characterisation of quasi-inner-product spaces. The general solution of (1.1) was derived by Ebanks *et al.* [8] as

$$f(x) = A(x) + Q(x),$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function and $Q : \mathbb{R} \to \mathbb{R}$ is a quadratic function, that is, A satisfies the additive functional equation

$$A(x + y) = A(x) + A(y)$$

for all $x, y \in \mathbb{R}$ and Q satisfies the quadratic functional equation

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$$

for all $x, y \in \mathbb{R}$.

The stability of the Drygas functional equation has been studied by many authors under various conditions (see, for example, [9–11, 15] and references therein). In 2013, results on the hyperstability of the Drygas functional equation were obtained by Piszczek and Szczawińska [14]. In fact, a hyperstability result first appeared in [2], but it seems that the term 'hyperstability' was first used in [12]. For more details on the notion of hyperstability, we refer the reader to [6].

The key tool for proving our results is a fixed point theorem derived from a result of Brzdęk *et al.* [5].

Theorem 1.1 [5]. *Let U be a nonempty set, Y a Banach space and $f_1, \dots, f_k : U \to U$ and $L_1, \dots, L_k : U \to \mathbb{R}_+$ given mappings, where k is a positive integer. Suppose that $\mathcal{T} : Y^U \to Y^U$ is an operator satisfying the inequality*

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \le \sum_{i=1}^{k} L_i(x)\|\xi(f_i(x)) - \mu(f_i(x))\|$$

for all $\xi, \mu \in Y^U$ and $x \in U$. Assume that there are functions $\varepsilon : U \to \mathbb{R}_+$ and $\varphi : U \to Y$ which satisfy the following conditions for each $x \in U$:

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \le \varepsilon(x) \quad \text{and} \quad \varepsilon^*(x) := \sum_{n=0}^{\infty}(\Lambda^n\varepsilon)(x) < \infty,$$

where $\Lambda : \mathbb{R}_+^U \to \mathbb{R}_+^U$ is defined by

$$(\Lambda\delta)(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x)) \tag{1.2}$$

for all $\delta \in \mathbb{R}_+^U$ and $x \in U$. Then there exists a unique fixed point ψ of $\mathcal{T}$ with

$$\|\varphi(x) - \psi(x)\| \le \varepsilon^*(x)$$

for all $x \in U$. Moreover,

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n\varphi(x)$$

for all $x \in U$.

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This interesting fixed point theorem has been applied in the proof of hyperstability results for various functional equations, some of which are noted in [1, 16, 17].

The purpose of this work is to prove two new generalised hyperstability results for the Drygas functional equation using Theorem 1.1 and a modification of the method of Brzdęk [3]. The first main result generalises the results of Piszczek and Szczawińska [14]. We also derive the corresponding hyperstability results for the inhomogeneous Drygas functional equation.

2. The main results

In this section, we give two generalised hyperstability results for the Drygas functional equation under certain conditions on the domain and codomain of the unknown function.

THEOREM 2.1. *Let X be a nonempty subset of a normed space such that $0 \notin X$ and X is symmetric with respect to 0, that is, $x \in X$ implies that $-x \in X$, and let Y be a Banach space. Suppose that there exist $n_0 \in \mathbb{N}$ with $nx \in X$ for all $x \in X$, $n \in \mathbb{N}_{n_0}$, and a function $h : X \to \mathbb{R}_+$ such that*

$$M_0 := \{n \in \mathbb{N}_{n_0} : 2s(n+1) + s(n) + s(-n) + s(2n+1) < 1\}$$

is an infinite set, where

$$s(n) := \inf\{t \in \mathbb{R}_+ : h(nx) \le th(x) \text{ for all } x \in X\}$$

and $s(n)$ satisfies the following conditions for $n \in \mathbb{N}$:

$$\lim_{n\to\infty} s(n) = 0 \quad \text{and} \quad \lim_{n\to\infty} s(-n) = 0.$$

If $f : X \to Y$ satisfies the inequality

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \le h(x) + h(y) \tag{2.1}$$

for all $x, y \in X$ with $x+y, x-y \in X$, then f satisfies the equation

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y) \tag{2.2}$$

for all $x, y \in X$.

PROOF. Replacing x by $(m+1)x$ and y by mx for $m \in M_0$ in (2.1) gives

$$\|2f((m+1)x) + f(mx) + f(-mx) - f((2m+1)x) - f(x)\| \le h((m+1)x) + h(mx) \tag{2.3}$$

for all $x \in X$. For each $m \in M_0$, define the operator $\mathcal{T}_m : Y^X \to Y^X$ by

$$(\mathcal{T}_m\xi)(x) := 2\xi((m+1)x) + \xi(mx) + \xi(-mx) - \xi((2m+1)x), \quad x \in X, \xi \in Y^X.$$

Further, observe that

$$\varepsilon_m(x) := h((m+1)x) + h(mx) \le [s(m+1) + s(m)]h(x), \quad x \in X. \tag{2.4}$$

Then the inequality (2.3) takes the form

$$\|\mathcal{T}_m f(x) - f(x)\| \le \varepsilon_m(x), \quad x \in X.$$

For each $m \in M_0$, the operator $\Lambda_m : \mathbb{R}_+^X \to \mathbb{R}_+^X$ defined by

$$(\Lambda_m \eta)(x) := 2\eta((m+1)x) + \eta(mx) + \eta(-mx) + \eta((2m+1)x), \quad \eta \in \mathbb{R}_+^X, x \in X$$

has the form (1.2) with $k = 4$ and $f_1(x) = (m+1)x$, $f_2(x) = mx$, $f_3(x) = -mx$ and $f_4(x) = (2m+1)x$, $L_1(x) = 2$ and $L_2(x) = L_3(x) = L_4(x) = 1$ for $x \in X$. Further, for each $\xi, \mu \in Y^X$, $x \in X$,

$$\begin{aligned}\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\| &= \|2\xi((m+1)x) + \xi(mx) + \xi(-mx) - \xi((2m+1)x) \\ &\qquad - 2\mu((m+1)x) - \mu(mx) - \mu(-mx) + \mu((2m+1)x)\| \\ &\le 2\|(\xi-\mu)((m+1)x)\| + \|(\xi-\mu)(mx)\| \\ &\qquad + \|(\xi-\mu)(-mx)\| + \|(\xi-\mu)((2m+1)x)\| \\ &= \sum_{i=1}^{4} L_i(x)\|(\xi-\mu)(f_i(x))\|.\end{aligned}$$

By mathematical induction, we will show that, for each $x \in X$,

$$\Lambda_m^n \varepsilon_m(x) \le [s(m+1) + s(m)][2s(m+1) + s(m) + s(-m) + s(2m+1)]^n h(x) \tag{2.5}$$

for all $n \in \mathbb{N}_0$. From (2.4), the inequality (2.5) holds for $n = 0$. Assume that (2.5) holds for $n = k$, where $k \in \mathbb{N}_0$. Then

$$\begin{aligned}\Lambda_m^{k+1}\varepsilon_m(x) &= \Lambda_m(\Lambda_m^k \varepsilon_m(x)) \\ &= 2\Lambda_m^k\varepsilon_m((m+1)x) + \Lambda_m^k\varepsilon_m(mx) + \Lambda_m^k\varepsilon_m(-mx) + \Lambda_m^k\varepsilon_m((2m+1)x) \\ &\le [s(m+1) + s(m)][2s(m+1) + s(m) + s(-m) + s(2m+1)]^k \\ &\qquad \times [2h((m+1)x) + h(mx) + h(-mx) + h((2m+1)x)] \\ &\le [s(m+1) + s(m)][2s(m+1) + s(m) + s(-m) + s(2m+1)]^{k+1} h(x).\end{aligned}$$

This shows that (2.5) holds for $n = k+1$, so we can conclude that the inequality (2.5) holds for all $n \in \mathbb{N}_0$. From (2.5),

$$\begin{aligned}\varepsilon^*(x) &= \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \\ &\le \sum_{n=0}^{\infty} [s(m+1) + s(m)][2s(m+1) + s(m) + s(-m) + s(2m+1)]^n h(x) \\ &= \frac{[s(m+1) + s(m)]h(x)}{1 - 2s(m+1) - s(m) - s(-m) - s(2m+1)}\end{aligned}$$

for all $x \in X$ and $m \in M_0$. Thus, according to Theorem 1.1, for each $m \in M_0$ there exists a unique solution $F_m : X \to Y$ of the equation

$$F_m(x) = 2F_m((m+1)x) + F_m(mx) + F_m(-mx) + F_m((2m+1)x)$$

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such that

$$\|f(x) - F_m(x)\| \le \frac{[s(m+1) + s(m)]h(x)}{1 - 2s(m+1) - s(m) - s(-m) - s(2m+1)}, \quad x \in X.$$

We now show that

$$\begin{aligned} &\|\mathcal{T}_m^n f(x+y) + \mathcal{T}_m^n f(x-y) - 2\mathcal{T}_m^n f(x) - \mathcal{T}_m^n f(y) - \mathcal{T}_m^n f(-y)\| \\ &\quad \le [2s(m+1) + s(m) + s(-m) + s(2m+1)]^n (h(x) + h(y)) \end{aligned} \tag{2.6}$$

for every $x, y \in X$ with $x+y, x-y \in X$ and $n \in \mathbb{N}_0$. If $n = 0$, then (2.6) is simply (2.1). So, take $r \in \mathbb{N}_0$ and suppose that (2.6) holds for $n = r$ and all $x, y \in X$ such that $x+y, x-y \in X$. Then

$$\begin{aligned} &\|\mathcal{T}_m^{r+1} f(x+y) + \mathcal{T}_m^{r+1} f(x-y) - 2\mathcal{T}_m^{r+1} f(x) - \mathcal{T}_m^{r+1} f(y) - \mathcal{T}_m^{r+1} f(-y)\| \\ &\quad = \|2\mathcal{T}_m^r f((m+1)(x+y)) + \mathcal{T}_m^r f(m(x+y)) + \mathcal{T}_m^r f(-m(x+y)) \\ &\qquad - \mathcal{T}_m^r f((2m+1)(x+y)) + 2\mathcal{T}_m^r f((m+1)(x-y)) + \mathcal{T}_m^r f(m(x-y)) \\ &\qquad + \mathcal{T}_m^r f(-m(x-y)) - \mathcal{T}_m^r f((2m+1)(x-y)) \\ &\qquad - 2(2\mathcal{T}_m^r f((m+1)x) + \mathcal{T}_m^r f(mx) + \mathcal{T}_m^r f(-mx) - \mathcal{T}_m^r f((2m+1)x)) \\ &\qquad - 2\mathcal{T}_m^r f((m+1)y) - \mathcal{T}_m^r f(my) - \mathcal{T}_m^r f(-my) + \mathcal{T}_m^r f((2m+1)y) \\ &\qquad - 2\mathcal{T}_m^r f((m+1)(-y)) - \mathcal{T}_m^r f(m(-y)) - \mathcal{T}_m^r f(-m(-y)) + \mathcal{T}_m^r f((2m+1)(-y))\| \\ &\quad \le [2s(m+1) + s(m) + s(-m) + s(2m+1)]^r \\ &\qquad \times [2h((m+1)x) + 2h((m+1)y) + h(mx) + h(my) + h(-mx) + h(-my) \\ &\qquad\qquad + h((2m+1)x) + h((2m+1)y)] \\ &\quad = [2s(m+1) + s(m) + s(-m) + s(2m+1)]^{r+1} (h(x) + h(y)). \end{aligned}$$

This implies that (2.6) holds for all $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (2.6) yields

$$F_m(x+y) + F_m(x-y) = 2F_m(x) + F_m(y) + F_m(-y)$$

for all $x, y \in X$ with $x+y, x-y \in X$. So, we have a sequence $\{F_m\}_{m \in M_0}$ of functions satisfying (2.2) such that

$$\|f(x) - F_m(x)\| \le \frac{[s(m+1) + s(m)]h(x)}{1 - 2s(m+1) - s(m) - s(-m) - s(2m+1)}, \quad x \in X.$$

It follows, by letting $m \to \infty$, that f also satisfies (2.2) for $x, y \in X$. This completes the proof. □

The idea of the next theorem derives from [13], where Piszczek studied the hyperstability of the general linear functional equation.

THEOREM 2.2. *Let X be a nonempty subset of a normed space such that $0 \notin X$ and X is symmetric with respect to 0 (that is, $x \in X$ implies that $-x \in X$) and let Y be a Banach space. Assume that there exist $n_0 \in \mathbb{N}$ with $nx \in X$ for all $x \in X$, $n \in \mathbb{N}_{n_0}$, and functions $u, v : X \to \mathbb{R}_+$ such that*

$$\begin{aligned} M_0 := \{n \in \mathbb{N}_{n_0} : &\ 2s_1(n+1)s_2(n+1) + s_1(n)s_2(n) \\ &+ s_1(-n)s_2(-n) + s_1(2n+1)s_2(2n+1) < 1\} \end{aligned}$$

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is an infinite set, where

$$s_1(n) := \inf\{t \in \mathbb{R}_+ : u(nx) \le tu(x) \text{ for all } x \in X\}$$

and

$$s_2(n) := \inf\{t \in \mathbb{R}_+ : v(nx) \le tv(x) \text{ for all } x \in X\}$$

and s_1, s_2 satisfy the following conditions for all $n \in \mathbb{N}$:

(W_1) $\lim_{n\to\infty} s_1(\pm n)s_2(\pm n) = 0$;

(W_2) $\lim_{n\to\infty} s_1(n) = 0$ *or* $\lim_{n\to\infty} s_2(n) = 0$.

If $f : X \to Y$ satisfies the inequality

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \le u(x)v(y) \tag{2.7}$$

for all $x, y \in X$ with $x+y, x-y \in X$, then f satisfies the equation

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y) \tag{2.8}$$

for all $x, y \in X$.

Proof. Replacing x by $(m+1)x$ and y by mx for $m \in M_0$ in (2.7) yields

$$\|2f((m+1)x) + f(mx) + f(-mx) - f((2m+1)x) - f(x)\| \le u((m+1)x)v(mx) \tag{2.9}$$

for all $x \in X$. For each $m \in M_0$, define the operator $\mathcal{T}_m : Y^X \to Y^X$ by

$$(\mathcal{T}_m\xi)(x) := 2\xi((m+1)x) + \xi(mx) + \xi(-mx) - \xi((2m+1)x), \quad x \in X, \xi \in Y^X.$$

Further, observe that

$$\varepsilon_m(x) := u((m+1)x)v(mx) \le [s_1(m+1)s_2(m)]u(x)v(x), \quad x \in X. \tag{2.10}$$

Then (2.9) takes the form

$$\|\mathcal{T}_m f(x) - f(x)\| \le \varepsilon_m(x), \quad x \in X.$$

For each $m \in M_0$, the operator $\Lambda_m : \mathbb{R}_+^X \to \mathbb{R}_+^X$ defined by

$$(\Lambda_m\eta)(x) := 2\eta((m+1)x) + \eta(mx) + \eta(-mx) + \eta((2m+1)x), \quad \eta \in \mathbb{R}_+^X, x \in X$$

has the form (1.2) with $k = 4$ and $f_1(x) = (m+1)x$, $f_2(x) = mx$, $f_3(x) = -mx$ and $f_4(x) = (2m+1)x$, $L_1(x) = 2$ and $L_2(x) = L_3(x) = L_4(x) = 1$ for $x \in X$. Further, for each $\xi, \mu \in Y^X$, $x \in X$,

$$\begin{aligned}
\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\| &= \|2\xi((m+1)x) + \xi(mx) + \xi(-mx) - \xi((2m+1)x) \\
&\qquad - 2\mu((m+1)x) - \mu(mx) - \mu(-mx) + \mu((2m+1)x)\| \\
&\le 2\|(\xi-\mu)((m+1)x)\| + \|(\xi-\mu)(mx)\| \\
&\qquad + \|(\xi-\mu)(-mx)\| + \|(\xi-\mu)((2m+1)x)\| \\
&= \sum_{i=1}^{4} L_i(x)\|(\xi-\mu)(f_i(x))\|.
\end{aligned}$$

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By mathematical induction, we will show that, for each $x \in X$,

$$\begin{aligned}\Lambda_m^n \varepsilon_m(x) \le [s_1(m+1)s_2(m)][&2s_1(m+1)s_2(m+1) + s_1(m)s_2(m) + s_1(-m)s_2(-m) \\ &+ s_1(2m+1)s_2(2m+1)]^n u(x)v(x)\end{aligned} \tag{2.11}$$

for all $n \in \mathbb{N}_0$. From (2.10), we see that the inequality (2.11) holds for $n = 0$. Next, suppose that (2.11) holds for $n = k$, where $k \in \mathbb{N}_0$. Then

$$\begin{aligned}\Lambda_m^{k+1}\varepsilon_m(x) &= \Lambda_m(\Lambda_m^k \varepsilon_m(x)) \\ &= 2\Lambda_m^k\varepsilon_m((m+1)x) + \Lambda_m^k\varepsilon_m(mx) + \Lambda_m^k\varepsilon_m(-mx) + \Lambda_m^k\varepsilon_m((2m+1)x) \\ &\le [s_1(m+1)s_2(m)][2s_1(m+1)s_2(m+1) + s_1(m)s_2(m) \\ &\qquad + s_1(-m)s_2(-m) + s_1(2m+1)s_2(2m+1)]^k \\ &\qquad \times [2u((m+1)x)v((m+1)x) + u(mx)v(mx) + u(-mx)v(-mx) \\ &\qquad + u((2m+1)x)v((2m+1)x)] \\ &\le [s_1(m+1)s_2(m)][2s_1(m+1)s_2(m+1) + s_1(m)s_2(m) \\ &\qquad + s_1(-m)s_2(-m) + s_1(2m+1)s_2(2m+1)]^{k+1}u(x)v(x).\end{aligned}$$

This yields (2.11) for $n = k+1$, so (2.11) holds for all $n \in \mathbb{N}_0$. From (2.11),

$$\begin{aligned}\varepsilon^*(x) &= \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \\ &\le \sum_{n=0}^{\infty}[s_1(m+1)s_2(m)][2s_{12}(m+1) + s_{12}(m) + s_{12}(-m) + s_{12}(2m+1)]^n u(x)v(x) \\ &= \frac{[s_1(m+1)s_2(m)]u(x)v(x)}{1 - 2s_{12}(m+1) - s_{12}(m) - s_{12}(-m) - s_{12}(2m+1)}\end{aligned}$$

for all $x \in X$ and $m \in M_0$, where $s_{12}(n) := s_1(n)s_2(n)$. Thus, according to Theorem 1.1, for each $m \in M_0$ there exists a unique solution $F_m : X \to Y$ of the equation

$$F_m(x) = 2F_m((m+1)x) + F_m(mx) + F_m(-mx) + F_m((2m+1)x)$$

such that

$$\|f(x) - F_m(x)\| \le \frac{[s_1(m+1)s_2(m)]u(x)v(x)}{1 - 2s_{12}(m+1) - s_{12}(m) - s_{12}(-m) - s_{12}(2m+1)}, \quad x \in X.$$

Next, we will show that

$$\begin{aligned}\|\mathcal{T}_m^n f(x+y) + \mathcal{T}_m^n f(x-y) - 2\mathcal{T}_m^n f(x) - \mathcal{T}_m^n f(y) - \mathcal{T}_m^n f(-y)\| \\ \le [2s_{12}(m+1) + s_{12}(m) + s_{12}(-m) + s_{12}(2m+1)]^n u(x)v(y)\end{aligned} \tag{2.12}$$

for every $x, y \in X$ with $x+y, x-y \in X$ and $n \in \mathbb{N}_0$. If $n = 0$, (2.12) is simply (2.7). Take $r \in \mathbb{N}_0$ and assume that (2.12) holds for $n = r$ and $x, y \in X$ with $x+y, x-y \in X$.

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Then

$$\begin{aligned}
&\|\mathcal{T}_m^{r+1}f(x+y)+\mathcal{T}_m^{r+1}f(x-y)-2\mathcal{T}_m^{r+1}f(x)-\mathcal{T}_m^{r+1}f(y)-\mathcal{T}_m^{r+1}f(-y)\| \\
&\quad =\|2\mathcal{T}_m^{r}f((m+1)(x+y))+\mathcal{T}_m^{r}f(m(x+y))+\mathcal{T}_m^{r}f(-m(x+y)) \\
&\qquad -\mathcal{T}_m^{r}f((2m+1)(x+y))+2\mathcal{T}_m^{r}f((m+1)(x-y))+\mathcal{T}_m^{r}f(m(x-y)) \\
&\qquad +\mathcal{T}_m^{r}f(-m(x-y))-\mathcal{T}_m^{r}f((2m+1)(x-y)) \\
&\qquad -2(2\mathcal{T}_m^{r}f((m+1)x)+\mathcal{T}_m^{r}f(mx)+\mathcal{T}_m^{r}f(-mx)-\mathcal{T}_m^{r}f((2m+1)x)) \\
&\qquad -2\mathcal{T}_m^{r}f((m+1)y)-\mathcal{T}_m^{r}f(my)-\mathcal{T}_m^{r}f(-my)+\mathcal{T}_m^{r}f((2m+1)y) \\
&\qquad -2\mathcal{T}_m^{r}f((m+1)(-y))-\mathcal{T}_m^{r}f(m(-y))-\mathcal{T}_m^{r}f(-m(-y))+\mathcal{T}_m^{r}f((2m+1)(-y))\| \\
&\quad \le [2s_{12}(m+1)+s_{12}(m)+s_{12}(-m)+s_{12}(2m+1)]^r[2u((m+1)x)v((m+1)y) \\
&\qquad +u(mx)v(my)+u(-mx)v(-my)+u((2m+1)x)v((2m+1)y)] \\
&\quad =[2s_{12}(m+1)+s_{12}(m)+s_{12}(-m)+s_{12}(2m+1)]^{r+1}u(x)v(y).
\end{aligned}$$

This means that (2.12) holds for all $n\in\mathbb{N}_0$. Letting $n\to\infty$ in (2.12) yields

$$F_m(x+y)+F_m(x-y)=2F_m(x)+F_m(y)+F_m(-y)$$

for all $x,y\in X$ with $x+y,x-y\in X$. Thus, we have a sequence $\{F_m\}_{m\in M_0}$ of functions satisfying (2.8) for which

$$\|f(x)-F_m(x)\|\le\frac{[s_1(m+1)s_2(m)]u(x)v(x)}{1-2s_{12}(m+1)-s_{12}(m)-s_{12}(-m)-s_{12}(2m+1)},\qquad x\in X.$$

It follows, by letting $m\to\infty$, that f also satisfies (2.8) for $x,y\in X$. □

By using Theorems 2.1 and 2.2 and the same technique as in the proof of Brzdęk [4, Corollary 4.8], we have the following hyperstability results for the inhomogeneous Drygas functional equation. To avoid repetition, the details are omitted.

Corollary 2.3. *Let X be a nonempty subset of a normed space such that $0\notin X$ and X is symmetric with respect to 0 and let Y be a Banach space and $C: X\times X\to Y$ be a given mapping. Suppose that there exist $n_0\in\mathbb{N}$ with $nx\in X$ for all $x\in X$, $n\in\mathbb{N}_{n_0}$, and a function $h: X\to\mathbb{R}_+$ such that*

$$M_0:=\{n\in\mathbb{N}_{n_0}: 2s(n+1)+s(n)+s(-n)+s(2n+1)<1\}$$

is an infinite set, where

$$s(n):=\inf\{t\in\mathbb{R}_+: h(nx)\le th(x)\text{ for all }x\in X\}$$

and $s(n)$ satisfies the following conditions for all $n\in\mathbb{N}$:

$$\lim_{n\to\infty}s(n)=0\quad\text{and}\quad\lim_{n\to\infty}s(-n)=0.$$

If $f: X\to Y$ satisfies the inequality

$$\|f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)-C(x,y)\|\le h(x)+h(y)$$

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for all $x, y \in X$ *with* $x + y, x - y \in X$ *and the functional equation*

$$g(x + y) + g(x - y) = 2g(x) + g(y) + g(-y) + C(x, y)$$

has a solution $g_0 : X \to Y$*, then* f *satisfies the equation*

$$f(x + y) + f(x - y) = 2f(x) + f(y) + f(-y) + C(x, y)$$

for all $x, y \in X$.

COROLLARY 2.4. *Let* X *be a nonempty subset of a normed space such that* $0 \notin X$ *and* X *is symmetric with respect to* 0 *and let* Y *be a Banach space and* $C : X \times X \to Y$ *be a given mapping. Assume that there exist* $n_0 \in \mathbb{N}$ *with* $nx \in X$ *for all* $x \in X$*,* $n \in \mathbb{N}_{n_0}$*, and functions* $u, v : X \to \mathbb{R}_+$ *such that*

$$M_0 := \{n \in \mathbb{N}_{n_0} : 2s_{12}(n + 1) + s_{12}(n) + s_{12}(-n) + s_{12}(2n + 1)) < 1\}$$

is an infinite set, where

$$s_1(n) := \inf\{t \in \mathbb{R}_+ : u(nx) \leq tu(x) \text{ for all } x \in X\},$$
$$s_2(n) := \inf\{t \in \mathbb{R}_+ : v(nx) \leq tv(x) \text{ for all } x \in X\}$$

and $s_{12}(n) = s_1(n)s_2(n)$ *and* s_1, s_2 *satisfy the following conditions for all* $n \in \mathbb{N}$:

(W_1) $\lim_{n\to\infty} s_1(\pm n)s_2(\pm n) = 0$;

(W_2) $\lim_{n\to\infty} s_1(n) = 0$ *or* $\lim_{n\to\infty} s_2(n) = 0$.

If $f : X \to Y$ *satisfies the inequality*

$$\|f(x + y) + f(x - y) - 2f(x) - f(y) - f(-y) - C(x, y)\| \leq u(x)v(y)$$

for all $x, y \in X$ *with* $x + y, x - y \in X$ *and the functional equation*

$$g(x + y) + g(x - y) = 2g(x) + g(y) + g(-y) + C(x, y)$$

has a solution $g_0 : X \to Y$*, then* f *satisfies the equation*

$$f(x + y) + f(x - y) = 2f(x) + f(y) + f(-y) + C(x, y)$$

for all $x, y \in X$.

3. Some particular cases

In this section, we derive some hyperstability results for the Drygas functional equation and the inhomogeneous Drygas functional equation from our main results.

COROLLARY 3.1 [14]. *Let* X *be a nonempty subset of a normed space such that* $0 \notin X$ *and* X *is symmetric with respect to* 0 *and let* Y *be a Banach space,* $c \geq 0$ *and* $p < 0$. *Suppose that there exist* $n_0 \in \mathbb{N}$ *with* $nx \in X$ *for all* $x \in X$*,* $n \in \mathbb{N}_{n_0}$*, and* $f : X \to Y$ *satisfies the inequality*

$$\|f(x + y) + f(x - y) - 2f(x) - f(y) - f(-y)\| \leq c(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$ *with* $x + y, x - y \in X$. *Then* f *satisfies the equation*

$$f(x + y) + f(x - y) = 2f(x) + f(y) + f(-y)$$

for all $x, y \in X$.

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Proof. Define $h : X \to \mathbb{R}_+$ by $h(x) := c\|x\|^p$, $x \in X$. For each $n \in \mathbb{N}$,

$$\begin{aligned} s(n) &= \inf\{t \in \mathbb{R}_+ : h(nx) \le th(x) \text{ for all } x \in X\} \\ &= \inf\{t \in \mathbb{R}_+ : c\|nx\|^p \le tc\|x\|^p \text{ for all } x \in X\} \\ &= \inf\{t \in \mathbb{R}_+ : |n|^p \le t\} = |n|^p. \end{aligned}$$

In the same way, $s(-n) = |n|^p$ for all $n \in \mathbb{N}$, so

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty} |n|^p = 0 \quad \text{and} \quad \lim_{n\to\infty} s(-n) = \lim_{n\to\infty} |n|^p = 0$$

for all $n \in \mathbb{N}$. Moreover, we can see that M_0 is an infinite set. Thus, all the conditions in Theorem 2.1 hold and we have this result. □

The next result follows from Corollary 2.3 with Corollary 3.1.

Corollary 3.2. *Let X be a nonempty subset of a normed space such that $0 \notin X$ and X is symmetric with respect to 0 and let Y be a Banach space, $c \ge 0$, $p < 0$ and $C : X \times X \to Y$ be a given mapping. Suppose that there exist $n_0 \in \mathbb{N}$ with $nx \in X$ for all $x \in X$, $n \in \mathbb{N}_{n_0}$, and $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y) - C(x,y)\| \le c(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$ with $x+y, x-y \in X$ and the functional equation

$$g(x+y) + g(x-y) = 2g(x) + g(y) + g(-y) + C(x,y)$$

has a solution $g_0 : X \to Y$. Then f satisfies the equation

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y) + C(x,y)$$

for all $x, y \in X$.

The next two corollaries can be derived from Theorem 2.2 and Corollary 2.4.

Corollary 3.3. *Let X be a nonempty subset of a normed space such that $0 \notin X$ and X is symmetric with respect to 0 and let Y be a Banach space, $c \ge 0$ and $p, q \in \mathbb{R}$ with $p + q < 0$. Suppose that there exist $n_0 \in \mathbb{N}$ with $nx \in X$ for all $x \in X$, $n \in \mathbb{N}_{n_0}$, and $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \le c(\|x\|^p\|y\|^q)$$

for all $x, y \in X$ with $x+y, x-y \in X$. Then f satisfies the equation

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y)$$

for all $x, y \in X$.

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Proof. Define $u, v : X \to \mathbb{R}_+$ by $u(x) := s\|x\|^p$ and $v(x) := r\|x\|^q$, where $s, r \in \mathbb{R}_+$ with $sr = c$. As in the proof of Corollary 3.1, for each $n \in \mathbb{N}$, $s_1(n) = |n|^p$ and $s_2(n) = |n|^q$. So,

$$\lim_{n\to\infty} s_1(\pm n)s_2(\pm n) = \lim_{n\to\infty} |n|^{p+q} = 0$$

for all $n \in \mathbb{N}$. Next, we claim that $\lim_{n\to\infty} s_1(n) = 0$ or $\lim_{n\to\infty} s_2(n) = 0$ for each $n \in \mathbb{N}$. Since $p, q \in \mathbb{R}$ with $p + q < 0$, either $p < 0$ or $q < 0$. If $p < 0$,

$$\lim_{n\to\infty} s_1(n) = \lim_{n\to\infty} |n|^p = 0.$$

On the other hand, if $q < 0$, then

$$\lim_{n\to\infty} s_2(n) = \lim_{n\to\infty} |n|^q = 0.$$

It is easy to see that M_0 is an infinite set. Thus, all the conditions in Theorem 2.2 now hold. Therefore, we obtain the result. □

Corollary 3.4. *Let X be a nonempty subset of a normed space such that $0 \notin X$ and X is symmetric with respect to 0 and let Y be a Banach space, $c \geq 0$, $p, q \in \mathbb{R}$ with $p + q < 0$ and $C : X \times X \to Y$ be a given mapping. Suppose that there exist $n_0 \in \mathbb{N}$ with $nx \in X$ for all $x \in X$, $n \in \mathbb{N}_{n_0}$, and $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y) - C(x,y)\| \leq c(\|x\|^p\|y\|^q)$$

for all $x, y \in X$ with $x + y, x - y \in X$ and the functional equation

$$g(x+y) + g(x-y) = 2g(x) + g(y) + g(-y) + C(x,y)$$

has a solution $g_0 : X \to Y$. Then f satisfies the equation

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y) + C(x,y)$$

for all $x, y \in X$.

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