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# LOCALIZATIONS OF $H_i$ AND $D_i$ RINGS

EVAN G. HOUSTON

**Introduction.** In this paper we study conditions under which certain localizations of semilocal $H_i$ and $D_i$ rings satisfy the second chain condition. Two of these results are as follows.

1) *If $P$ is a prime in a semilocal $H_i$ ring with* depth $P > i \geqq$ height $P - 1$, *then $R_P$ satisfies the second chain condition.*
2) *If $P$ is a prime in a semilocal $D_i$ ring with* depth $P \geqq i$, *then $R_P$ satisfies the second chain condition.*

In addition several conditions under which $H_i$ implies $D_i$ are obtained. Finally, we show that if $(R, M)$ is a local ring and $i$ is a positive integer, then $R$ is $D_i$ if and only if $R[x]_{(M,x)R[x]}$ is $D_{i+1}$.

**Preliminaries.** All rings mentioned in this paper are commutative with 1. We shall confine ourselves to semilocal (which we take as implying Noetherian) rings and to integral extensions thereof. Any undefined terminology may be found in [**2**] or [**8**]. We include the following definitions.

1) If $R$ is a ring, we denote by $S(R)$ the set of lengths of maximal chains of prime ideals (primes) in $R$.

2) A ring $R$ satisfies the *first chain condition* (f.c.c.) in case $S(R) = \{$altitude $R\}$.

3) A ring $R$ satisfies the *second chain condition* (s.c.c.) in case, for each minimal prime $Q$ of $R$, and for every integral extension domain $T$ of $R/Q$, we have $S(T) =$ altitude $R$.

4) A ring $R$ is *catenary* if, for every pair $P \subset Q$ of primes of $R$, $R_Q/P_Q$ satisfies the f.c.c.

5) A ring $R$ of finite altitude $a$ is called an *$H_i$ ring* if every height $i$ prime $P$ of $R$ satisfies height $P +$ depth $P = a$. In speaking of $H_i$ rings, we shall for convenience assume that $0 \leqq i \leqq a$.

6) A ring $R$ of finite altitude $a$ is called a *$D_i$ ring* if every depth $i$ prime $P$ of $R$ satisfies height $P +$ depth $P = a$. As with the $H_i$ notation, we take $i$ to satisfy $0 \leqq i \leqq a$.

We refer the reader to [**6**] for relevant facts about $H_i$ and $D_i$ rings.

LEMMA 1. *Let $P$ be a prime of depth $d > 0$ in a semilocal ring $R$ of altitude $a$, let $Q$ be a minimal prime of $R$ with $Q \subset P$, and let $T$ be an integral extension*

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*domain of* $R_P/Q_P$. *If* $m \in S(T)$ *then there is a prime* $q$ *in* $R$ *with* height $q = m$ *and* depth $q = d$.

*Proof*. Since $R_P/Q_P = (R/Q)_{P/Q}$, $T = D_N$, where $D$ is an integral extension domain of $R/Q$ and $N$ is the multiplicatively closed subset $R/Q - P/Q$ of $R/Q$. Thus since $m \in S(T)$, there is a saturated chain of primes $0 \subset P_1' \subset \ldots \subset P_m'$ in $D$ with $P_m' \cap R/Q = P/Q$. Of course depth $P_m' =$ depth $P/Q =$ depth $P = d > 0$. Hence by [**1**, Theorem 1.11] there is a prime $p'/Q$ in $R/Q$ such that height $p'/Q = m$ and depth $p'/Q = d$. If $m = 0$ we simply choose $q = p' = Q$, and the conclusion of the lemma is clearly satisfied. Otherwise, let $Q = p_0 \subset p_1 \subset \ldots \subset p_m = p'$ be a saturated chain of primes. By [**3**, Theorem 5] we may assume that height $p_{m-1} = m - 1$. By [**3**, Proposition 2] there are infinitely many primes $q \supset p_{m-1}$ such that depth $q = d$ and height $q/p_{m-1} = 1$. Since for all but finitely many such $q$ we have height $q = m$ [**3**, Theorem 1], this clearly yields a prime $q$ with height $q = m$ and depth $q = d$.

THEOREM 2. *Let* $P$ *be a prime of depth* $d$ *in a semilocal* $H_i$ *ring of altitude* $a$, *let* $Q$ *be a minimal prime of* $R$ *with* $Q \subset P$, *and let* $T$ *be an integral extension domain of* $R_p/Q_p$. *If* $m \in S(T)$ *and* $m + d < a$, *then either* $m > i$ *or* $m + d \leqq i$.

*Proof*. We first note that the theorem is trivial if $d = 0$. Thus we assume $d > 0$.

By the lemma there is a prime $q$ in $R$ with height $q = m$ and depth $q = d$. We shall assume that $m \leqq i$ and $m + d > i$; we are required to show that $m + d = a$. Since $d > i - m$ we have a saturated chain of primes

$$q = p_0 \subset \ldots \subset p_{i-m} \subset \ldots \subset p_d.$$

By [**4**, Lemma 1] we may assume height $p_{i-m} = i$, and it is clear that depth $p_{i-m} = d - (i - m)$. Since $R$ is $H_i$, however, we must have $d - (i - m) =$ depth $p_{i-m} = a - i$, from which it follows that $m + d = a$.

As a first application of Theorem 2, we give a generalization of [**5**, Corollary 4.2].

PROPOSITION 3. *Let* $R$ *be a semilocal* $H_i$ *ring of altitude* $a$, *and let* $P$ *be a prime of* $R$ *with* height $P = h$ *and* depth $P = d$. *If* $d \geqq i \geqq h - 1$, *then* $R_P$ *satisfies the* s.c.c.

*Proof*. Let $Q$ be a minimal prime of $R$ with $Q \subset P$, and let $T$ be an integral extension domain of $R_P/Q_P$. We wish to show that $S(T) = \{h\}$. Let $m \in S(T)$. We have $m \leqq h \leqq i + 1$ so that if $m = i + 1$ we are done. Thus we assume $m \leqq i$. We may also assume $m > 0$, for if $m = 0$ then $T$ is a field, whence $R_P/Q_P$ is a field [**2**, Theorem 16]. Hence $m + d > i$ since $d \geqq i$. We thus have $m + d = a \geqq h + d$ by Theorem 2. Therefore, $m = h$ as desired.

PROPOSITION 4. *Let* $R$ *be a semilocal* $H_i$ *ring of altitude* $a$ *with* $i <$ min $S(R)$. *If* $P$ *is a prime of* $R$ *with* height $P \leqq i$, *then* $R_P$ *satisfies the* s.c.c.

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*Proof.* As in the proof of Proposition 3, let $Q \subset P$ be a minimal prime of $R$, let $T$ be an integral extension domain of $R_P/Q_P$, and let $m \in S(T)$. By Lemma 1 one can produce a prime $q$ in $R$ with height $q = m$ and depth $q = d =$ depth $P$. In particular, $m + d \in S(R)$. Thus $m + d \geqq \min S(R) > i$ by hypothesis. Since also $m \leqq$ height $P \leqq i$, we must have $m + d = a$ by Theorem 2. Thus we again have $m = h$.

PROPOSITION 5. *Let $R$ be a semilocal ring of altitude $a$ which is $H_i$ for $i = i_1 < i_2 < \ldots < i_n = a$. Set $i_0 = 0$. If $P$ is a prime with $i_{r-1} <$* height $P \leqq i_r$ *and* depth $P \geqq i_j - i_{j-1}$ *for $j = 1, \ldots, r$, then $R_P$ satisfies the* s.c.c.

*Proof.* Let $m > 0$ be as in the proof above. Then $i_{k-1} < m \leqq i_k$ for some $k$ with $1 \leqq k \leqq r$. Since depth $P \geqq i_k - i_{k-1} > i_k - m$, we have $m +$ depth $P > i_k$. But then $m +$ depth $P = a$ by Theorem 2, and we again have $m =$ height $P$.

To discover further conditions on $H_i$ rings under which certain localizations satisfy the s.c.c., we study $D_i$ rings. The following lemma was proved in the local case in [**6**, Remark 2.7a(ii)].

LEMMA 6. *If $R$ is a semilocal $D_i$ ring of altitude $a$* ($i > 0$) *and $j$ is an integer $\geqq i$, then $R$ is $D_j$.*

*Proof.* It clearly suffices to prove that $R$ is $D_{i+1}$. Thus let $P$ be a prime of height $h$ and depth $i + 1$; we must show that $h + i + 1 = a$. Since depth $P = i + 1$ we may clearly find a prime $P' \supset P$ such that height $P'/P = 1$ and depth $P' = i > 0$. In fact there are infinitely many such $P'$ by [**3**, Proposition 2], and thus by [**3**, Theorem 1] we may find such a $P'$ with height $P' = h + 1$. Since $R$ is $D_i$, however, we must have height $P' = a - i$. Thus $h + 1 = a - i$ and the result follows.

PROPOSITION 7. *Let $R$ be a semilocal $D_i$ ring of altitude $a$ with $i > 0$. If $P$ is a prime with* depth $P \geqq i$, *then $R_P$ satisfies the* s.c.c.

*Proof.* By the lemma we need only show that $R_P$ satisfies the s.c.c. when $P$ is a prime with depth $P = i$. Thus let $Q \subset P$ be a minimal prime of $R$, and let $T$ be an integral extension domain of $R_P/Q_P$ with $m \in S(T)$. By Lemma 1 this yields in $R$ a prime $q$ with height $q = m$ and depth $q = i$. Since $R$ is $D_i$, we have $m = a - i =$ height $P =$ altitude $R_P$. Thus $R_P$ satisfies the s.c.c.

The next three results give conditions under which an $H_i$ ring is also a $D_j$ ring. Of course, by Proposition 7 this automatically yields conditions under which certain localizations of $H_i$ rings will satisfy the s.c.c.

PROPOSITION 8. *Let $R$ be a semilocal $H_i$ ring of altitude $a$. If $2i + 1 \geqq a$ then $R$ is $D_j$ for each $j \geqq i + 1$. On the other hand, if $2i + 1 < a$ then $R$ is $D_j$ for each $j \geqq a - i - 1$.*

*Proof.* First, we assume that $2i + 1 \geqq a$. Let $P$ be a prime with depth $P = j \geqq i + 1$ and height $P = h$. If $h > i$ then $h + j > i + i + 1 \geqq a$, a

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contradiction. Thus $h \leqq i$. Since also $h + j > i$, we have $h + j = a$ by Theorem 2.

We now assume $2i + 1 < a$ and take $j \geqq a - i - 1$. Let $P$ be a prime of height $h$ and depth $j$. If $h > i$ then $h + j > i + a - i - 1 = a - 1$, whence $h + j = a$. Thus we may as well assume that $h \leqq i$. By Theorem 2 we need only show that $h + j > i$. But this is clear since $h + j \geqq h + a - i - 1 > h + 2i + 1 - i - 1 = h + i \geqq i$.

*Remark.* In the above proposition, we can improve the first statement if there are no depth $i$ minimal primes. With this added assumption, one easily shows that $R$ is $D_i$, rather than merely $D_{i+1}$.

PROPOSITION 9. *Let $R$ be a semilocal $H_i$ ring of altitude $a$ with $i < \min S(R)$. Then $R$ is $D_j$ for each $j \geqq a - i - 1$.*

*Remark.* In view of Proposition 8, this result is of interest only in the case $2i + 1 \geqq a$.

*Proof.* Let $P$ be a prime with height $P = h$ and depth $P = j \geqq a - i - 1$. Suppose that $h + j < a$. Then $h + a - i - 1 < a$ whence $h \leqq i$. By Theorem 2 we must have $h + j \leqq i < \min S(R)$, a contradiction. Thus $h + j = a$ and $R$ is $D_j$.

PROPOSITION 10. *Let $R$ be a semilocal ring of altitude $a$ which is $H_i$ for $i = i_1 < i_2 \ldots < i_n = a$. Set $i_0 = -1$. If $j \geqq i_k - i_{k-1}$ for $k = 1, \ldots, n$, then $R$ is $D_j$.*

*Proof.* Let $P$ be a prime of height $h$ and depth $j$. For some $k > 0$ we must have $i_{k-1} < h \leqq i_k$. Thus $j \geqq i_k - i_{k-1} > i_k - h$ and $h + j > i_k$. Since also $h \leqq i_k$, we have $h + j = a$ by Theorem 2.

*Example.* By setting $i_0 = -1$ in Proposition 10, we are requiring that $j > i_1$. The following ring (with $i_1 = 1$) shows why this is necessary. Let $K$ be a field and put $R = K[x, y, z]_{(x,y,z)}/(xz, yz)_{(x,y,z)}$. Then $R$ is a 2 dimensional local ring and is therefore obviously $H_1$ and $H_2$. However, $R$ is not $D_1$ since $(x, y)_{(x,y,z)}/(xz, yz)_{(x,y,z)}$ is a minimal prime of depth 1.

The above propositions give conditions under which $H_i$ implies $D_j$. One is naturally curious to know when $D_j$ implies $H_i$. A really good theorem in this direction would seem to require the truth of the depth conjecture, which is studied in [**1**] and [**6**]. However, we record the following result.

PROPOSITION 11. *Let $R$ be a semilocal $D_j (j > 0)$ ring of altitude $a$. If $i, i + 1, \ldots, i + j - 1 \notin S(R)$, then $R$ is $H_i$.*

*Proof.* Let $P$ be a prime with height $P = i$ and depth $P = d$. Then $i + d \in S(R)$. Since $i, \ldots, i + j - 1 \notin S(R)$, we have $i + d > i + j - 1$, so that $d \geqq j$. However, $R$ is $D_j$ so $R$ is $D_d$ by Lemma 6, whence $i + d = a$.

In [**7**] it is shown that a local domain $(R, M)$ satisfies the s.c.c. if and only if $R[x]$ is catenary. In fact $R$ satisfies the s.c.c. if and only if $R[x]_{(M,x)R[x]}$ satisfies

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the f.c.c. [**1**, Theorems 2.2 and 2.4]. This result is easily generalized to local rings. We now prove an analogous theorem about $D_i$ rings.

THEOREM 12. *Let* $(R, M)$ *be a local ring of altitude* $a$ *and let* $i$ *be a positive integer. Then* $R$ is $D_i$ *if and only if* $R[x]_{(M,x)R[x]}$ *is* $D_{i+1}$.

*Proof.* Assume that $R$ is $D_i$. To prove that $R[x]_{(M,x)R[x]}$ is $D_{i+1}$, we take a prime $Q$ in $R[x]$ with height $(M, x)R[x]/Q = i + 1$; we must show that height $Q = (a + 1) - (i + 1) = a - i$. Let $P = Q \cap R$. If $Q = PR[x]$ then depth $P = i$ (by [**2**, Theorem 149] applied to $R/P$) and thus height $Q$ = height $P = a - i$, since $R$ is $D_i$. If $Q \neq PR[x]$ then by [**3**, Theorem 3] applied to $R/P$ there are infinitely many primes $P'$ in $R$ with height $P'/P = 1$ and depth $P' = i$. By [**3**, Theorem 1] there is such a $P'$ with height $P'$ = height $P + 1$. Since $R$ is $D_i$ we must have height $P' = a - i$, so that height $Q$ = height $P + 1$ = height $P' = a - i$, as desired. The converse is elementary and is done in [**6**, Theorem 7.4].

COROLLARY 13. *Let* $(R, M)$ *be a local ring, and let* $i$ *be a positive integer. Then* $R$ *satisfies the* f.c.c. *if and only if* $R[x]_{(M,x)R[x]}$ *is* $D_2$.

*Proof.* It is known (and essentially proved in [**4**, Proposition 7]) that $R$ satisfies the f.c.c. if and only if $R$ is $D_1$. Thus the result is immediate from Theorem 12.

*Remark.* Let $R$ be a local ring which satisfies the f.c.c. Then $T = R[x]_{(M,x)R[x]}$ is a local $D_2$ ring. Hence by Proposition 7, $T_Q$ satisfies the s.c.c. for each prime $Q$ of $T$ with depth $Q \geqq 2$. However, one can do better. Let $Q$ be a depth 1 prime of $T$, and let $Q \cap R = P$. If $Q = PR[x]$ then clearly $P = M$ and $T_Q$ satisfies the f.c.c. [**8**, Theorem 4.11]. If $Q \neq PR[x]$ then $P$ is a nonmaximal prime of $R$ whence $R_P$ satisfies the s.c.c. by Proposition 7. It now follows from [**7**, Corollary 3.7] that $T_Q$ satisfies the s.c.c. Thus $T_Q$ satisfies the f.c.c. for each nonmaximal prime $Q$ of $T$. One wonders whether this is true for arbitrary local $D_2$ rings. We leave this as an open question.

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*University of North Carolina at Charlotte,*
*Charlotte, North Carolina*