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SOME REMARKS ON TRANSLATIVE COVERINGS OF CONVEX DOMAINS BY STRIPS

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ABSTRACT. In the euclidean plane let K be a compact convex set and $S_1, S_2, \ldots$ strips of respective widths $w_1, w_2, \ldots$. Some conditions on $\sum w_i$ are given that imply that K can be covered by translates of the strips S_i. These conditions involve the perimeter, the diameter, or the minimal width of K and yield improvements of previously known results.

Let K be a compact convex subset of the euclidean plane E^2. The perimeter of K will be denoted by p, the diameter by D, and the thickness (minimal width) by Δ. A strip of width w is defined as a closed subset of E^2 consisting of all points between two parallel lines of mutual distance w. If (S_i) is a (finite or infinite) sequence of strips we say that (S_i) permits a translative covering of K if there are translations τ_i so that $K \subset \bigcup \tau_i S_i$. Let w_i denote the width of S_i. We concern ourselves with the problem of finding sufficient conditions on $\sum w_i$ in order that (S_i) permits a translative covering of K. It has been shown in [4] that $\sum w_i \geq 6D$ is a condition of this kind, and Makai and Pach [5] have proved that already the weaker inequality $\sum w_i \geq p$, and consequently also $\sum w_i \geq \pi D$, serves the same purpose. We establish here an improvement of the theorem of Makai–Pach and show some related results. Further properties of translative coverings by strips in E^2 or slabs in E^n satisfying more restrictive assumptions are discussed in [2] and [3].

THEOREM 1. *Let K denote a compact convex domain in E^2 of perimeter p, diameter D, and thickness Δ. Furthermore, let (S_i) be a sequence of strips in E^2 of respective widths w_i. Then, each of the conditions*

$$\sum w_i \geq \frac{3}{\pi} p, \tag{1}$$

$$\sum w_i \geq 2\sqrt{2}\, D, \tag{2}$$

$$\sum w_i \geq D + 2\Delta \tag{3}$$

implies that (S_i) permits a translative covering of K.

We add two remarks.

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REMARK 1. The constants $3/\pi$, $2\sqrt{2}$, 2, appearing in (1), (2), (3) are almost certainly not the best possible ones. From the example of a unit square and two mutually orthogonal strips that are parallel to the sides of the square it follows that the greatest lower bounds of the three sets of possible constants must lie in the intervals $[1/2, 3/\pi]$, $[\sqrt{2}, 2\sqrt{2}]$, $[2-\sqrt{2}, 2]$, respectively. The right hand side of (3) cannot be replaced by any expression of the form $\alpha D+\beta\Delta$ with $\alpha<1$. This can be seen by taking for K a line segment and for S_i strips orthogonal to K.

REMARK 2. The conditions (1), (2), (3) are independent in the sense that for any one of them it is possible to find a K and a sequence (S_i) so that it is satisfied, but the other two are not. This follows from the fact that for a circular disc $2\sqrt{2}\,D<\min\{3/\pi)p, D+2\Delta\}$, for a circular disc with four added "caps" of angle 120° $(3/\pi)p<\min\{2\sqrt{2}\,D, D+2\Delta\}$, and for a line segment $D+2\Delta<\min\{(3/\pi)p, 2\sqrt{2}D\}$.

Instead of looking for conditions of the type (1) one may also try, provided that (S_i) is finite, to find better conditions of the kind $\sum w_i \geq c(n)p$ where n denotes the number of strips and $c(n)$ is allowed to depend on n. The following theorem exemplifies this possibility.

THEOREM 2. *Let K be a compact convex domain in E^2 of perimeter p, and let (S_i) be a sequence of n strips of respective widths w_i. Then, the condition*

$$\sum w_i \geq \frac{n}{n+1}p \tag{4}$$

implies that (S_i) permits a translative covering of K.

Condition (4) is best possible in the trivial case when $n=1$, K is a line segment and S_1 a strip orthogonal to K. If $n\leq 21$ condition (4) is weaker than (1), but for $n>21$ Theorem 1 implies Theorem 2.

The following lemma forms the basis for the proofs of our theorems.

LEMMA. *Let M be a compact convex subset of E^2 and L a supporting line of M. The perimeter of M will be denoted by p_M and the length of $M\cap L$ by r_M. Furthermore, let (S_i) be a finite or infinite sequence of strips of respective widths w_i. Then, (S_i) permits a covering of M if*

$$\sum w_i > p_M - r_M. \tag{5}$$

Proof. If (5) is satisfied with an infinite series on the left hand side it is also satisfied for a sufficiently large partial sum. Thus, it suffices to consider only the case when (S_i) is finite. It may also be assumed that a cartesian (x, y)-coordinate system has been selected so that L is the y-axis and M is in the half plane defined by $x\leq 0$. Let α_i be the angle between a boundary line of S_i and the x-axis, measured so that $-\pi/2\leq\alpha_i<\pi/2$ where α_i is positive or negative depending on whether the slope is positive or negative. Without any loss of generality we may suppose that the given sequence $S_1, S_2, \ldots, S_n$ is ordered so that

$$(6) \qquad \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n.$$

Moreover, we may suppose that $\alpha_1 < \alpha_2$ and $\alpha_1 = -\pi/2$. If necessary, this can be achieved, without disturbing condition (5), by combining all strips with angle α_1 into one strip, and by adding a strip with angle $-\pi/2$ and of width 0. (If after these changes only one strip remains, the situation is completely trivial and can be excluded from further consideration.)

Let us now define translates S_i' of S_i according to the following rules:

First, we set $S_1' = \{(x, y) : -w_1 \le x \le 0\}$. Because of $\alpha_1 = -\pi/2$ the strip S_1' is indeed a translate of S_1. Then we continue inductively. If for some $k < n$ the strips $S_1', S_2', \ldots, S_k'$ have already been determined we let D_k denote the residual set $D_k = M \setminus (S_1' \cup S_2' \cup \cdots \cup S_k')$. In the case $D_k = \emptyset$ we simply define $S_{k+1}' = S_{k+1}$. If $D_k \ne \emptyset$ we let S_{k+1}' be that translate of S_{k+1} which has one boundary line, say G, as supporting line of the closure of D_k so that both S_{k+1}' and D_k are below G (with respect to the given coordinate system; note that $-\pi/2 < \alpha_{k+1} < \pi/2$ and that each D_k is convex).

We now show that $M \subset S_1' \cup S_2' \cup \cdots \cup S_n'$. If this were not the case then $D_n \ne \emptyset$. But under this assumption it would follow from (6) and our construction of S_k' that each S_k' contains a subarc, say C_k, of the convex curve $bdr\, M$ such that any two of these arcs are disjoint (with the possible exception of endpoints) and that each C_k has length at least w_k. Moreover, $bdr\, M$ would not be completely covered by $S_1' \cup S_2' \cup \cdots \cup S_n'$ since the convex set D_n has clearly the property that it contains with each point p the point on $bdr\, M$ vertically below it. Noting that C_k has length at least w_k if $1 < k \le n$, and at least $r_M + w_1$ if $k = 1$, we could infer that $r_M + w_1 + w_2 + \cdots + w_n < p_M$, which obviously contradicts (5). Thus, $D_n = \emptyset$ and the lemma is proved.

The principle idea of this proof, namely the ordering of the strips so that (6) holds, is due to Erdös and Straus (unpublished). See Groemer [4] and Makai–Pach [5] for earlier applications of this idea. Our proof follows quite closely that of Makai and Pach.

Proof of Theorem 1. To avoid trivialities we assume that K has interior points. p_M and r_M are defined as in the lemma.

First we concern ourselves with condition (1). It can be shown (see Firey–Groemer [1]) that there is a rectangle R that contains K and has side lengths, a, b such that $a \le b$ and $(a+b) = (2/\pi)p$. The four vertices of R cannot all be contained in K, since this would imply $a + b = \frac{1}{2}p < (2/\pi)p$. Thus, one can remove from R a small triangle by cutting off one vertex with a line through R so that the resulting pentagon does still contain K and has one side of length b. Let now M be this pentagon and L the line that contains the side of M of length b. Then, $p_M - r_M < 2a + b \le \frac{3}{2}(a+b) = (3/\pi)p$. Together with (1) it follows that $\sum w_i \ge (3/\pi)p > p_M - r_M$ and the lemma implies that (S_i) permits a covering of M and therefore also of K.

If (3) holds the corresponding proof is very similar. There exists obviously a rectangle R with $K \subset R$ and side lengths Δ and D. Again, the set K cannot contain all four vertices of R (otherwise the diameter of K would be greater than D. Using for M a pentagon of the same kind as before we see that $p_M - r_M < 2\Delta + D$. Thus we have $\sum w_i \ge 2\Delta + D > p_M - r_M$, and the desired result is again an immediate consequence of the above lemma.

The proof regarding (2) is slightly more complicated. We use an isosceles triangle T with the following properties: $K \subset T$, the two sides of T, say s_1, s_2, of equal length enclose an angle $2\sin^{-1}(1/3)$, and the inradius of T is $\frac{1}{2}D$. The existence of such a triangle, actually one of inradius $(1/2\pi)p$ (which is less than or equal to $\frac{1}{2}D$), follows again from a theorem in [1]. Let now c denote the side of T which is different from s_1 and s_2, and let Q be the quadrangle with one side equal to c, two sides in s_1, s_2 and a fourth side, say e, parallel to c and at distance D from c. The perimeter of Q is easily calculated to be $\lambda + 2\sqrt{2}\,D$, where λ denotes the length of c. It cannot happen that K contains both of those vertices of Q that are in e since at least one of the distances from these vertices to a point in $K \cap c$ is greater than D. Hence, one may again cut off from Q a little triangle so that one obtains a convex pentagon that contains K, has perimeter less than $\lambda + 2\sqrt{2}\,D$, and has c as one of its sides. If we define now M to be this pentagon and L the line containing c the proof can be completed by an application of the lemma; note that $p_M < \lambda + 2\sqrt{2}\,D$, $\lambda = r_M$ and therefore $\sum w_i \ge 2\sqrt{2}\,D > p_M - r_M$.

Proof of Theorem 2. We proceed similarly as in the proof of the lemma, but as S_1 we select a strip of maximum width and choose the coordinate system so that S_1 is orthogonal to the x-axis. Then

$$w_1 \ge w_i \qquad (i = 2, 3, \dots, n) \tag{7}$$

and one may also assume that $-\pi/2 = \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n < \pi/2$. If the translates S_i' of the strips S_i are defined as in the proof of the lemma, and if we note that S_1' covers an arc of *bdr* K of length at least $2w_1$, we can deduce that K will be covered by the strips S_i' if

$$2w_1 + w_2 + \cdots + w_n \ge p.$$

But this inequality is a consequence of (4) and (7) since $w_1 + (w_1 + \cdots + w_n) \ge 1/n(w_1 + \cdots + w_n) + (w_1 + \cdots + w_n) = (n+1)/n(w_1 + \cdots + w_n)$.

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