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ON SUBGROUPS OF CLIFFORD GROUPS DEFINED BY JORDAN PAIRS OF RECTANGULAR MATRICES

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Abstract

Some embeddings of general linear groups into hyperbolic Clifford groups are constructed generically by using Jordan pairs of rectangular and alternating matrices over a ring. In low rank cases through exceptional isomorphisms, their direct description and relationships to some automorphisms of Clifford groups are given. Generic norms are calculated in detail, and equivariant embeddings of representation spaces are constructed.

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1. Introduction

This paper studies some embeddings of general linear groups into Clifford groups. In general, these are explicit only on certain open dense parts of the group, and the formulae are quite complicated, even in examples of low rank, though in fact they are quite natural when interpreted Jordan theoretically. We use *Jordan pairs of rectangular matrices* in the process, embedded naturally into *alternating* matrices in a global setting. Examples in low rank suggest that our embeddings are involved with certain *automorphisms* of Clifford groups.

We work over an arbitrary commutative base ring k of scalars and take as our data two finitely generated projective modules, N and P. We consider the two direct sums, $M := N \oplus P$ and $Q := N \oplus P^*$, where P^* is the dual $\mathrm{Hom}(P, k)$ of P, and we construct the *special Clifford group* $\mathbf{S\Gamma}(\mathbf{H}(M))$ from M via the *hyperbolic quadratic module* $\mathbf{H}(M)$. In Section 3.4, we prove that the general linear group $\mathbf{GL}(Q)$ is embedded in $\mathbf{S\Gamma}(\mathbf{H}(M))$ by a unique homomorphism λ, characterized on an open dense subscheme. This is a nontrivial explicit example for Loos's theory [12] of Tits–Kantor–Koecher constructions at the *group level*, and the proof of existence involves some simple ideas from the theory of Jordan pairs, in particular, the space $(\mathrm{Hom}(P^*, N), \mathrm{Hom}(P, N^*))$ of 'rectangular matrices'. These are actually viewed by an embedding in the space $(\Lambda^2 M, \Lambda^2 M^*)$ of 'alternating matrices' (see Section 2.4). In general, the description of λ is rather vague; however, in Section 4, we consider some explicit examples of low rank. One of our examples is quite complex, as it involves a number of exceptional isomorphisms (see Section 4.3). These examples suggest that λ is related to an involutive automorphism of $\mathbf{S\Gamma}(\mathbf{H}(M))$ (Sections 4.1 and 4.7).

Besides the general construction and examples of λ, this paper contains other topics of interest, such as the calculation of the 'generic norm' δ_M of $(\Lambda^2 M, \Lambda^2 M^*)$ in terms of components of the decomposition $M = N \oplus P$ (Section 2.6), and the construction of an equivariant embedding ζ of representation spaces relative to λ (Section 3.8). The former provides tools for further studies of spin representations [8, 9]. The latter is reminiscent of the classical observation of Igusa [4] that some fundamental representations of symplectic groups occur in spin representations. Note that the symplectic group of $N \oplus N^*$ may be embedded in $\mathbf{S\Gamma}(\mathbf{H}(M))$ not only through $\mathbf{GL}(M)$ by taking $P = N^*$ as Igusa did, but also through $\mathbf{GL}(Q)$ by taking $P = N$.

2. Jordan pairs of rectangular matrices

2.1. Exterior powers of modules. We begin by fixing notation. Suppose that M is an arbitrary finitely generated projective k-module. We denote by M^* the dual k-module $\mathrm{Hom}(M, k)$, and by $\langle x, f\rangle$ the evaluation of the natural pairing $M \times M^* \to k$. For each $\nu \geq 0$, the pairing extends to the νth exterior power $(\Lambda^\nu M) \times (\Lambda^\nu M^*)$, using the sign convention

$$\langle x_1 \wedge \cdots \wedge x_\nu, f_1 \wedge \cdots \wedge f_\nu\rangle := \varepsilon_\nu \det(\langle x_i, f_j\rangle), \tag{2.1.1}$$

where $\varepsilon_\nu := (-1)^{\nu(\nu-1)/2}$. Further, setting $\langle z, z^*\rangle := 0$ if $z \in \Lambda^\nu M$ and $z^* \in \Lambda^\mu M^*$ where $\nu \neq \mu$, we may extend the pairing to all of $\Lambda M \times \Lambda M^*$. Besides the ordinary wedge product $z \wedge z' =: l_z \cdot z'$, we use the (left) *interior product* $z^* \lrcorner z' = d_{z^*} \cdot z'$ systematically, with which ΛM becomes a *left* ΛM^**-module* such that any element $f \in M^*$ of degree one operates as the unique anti-derivation d_f that extends the map $f : \Lambda^1 M \to \Lambda^0 M$. The left ΛM-module ΛM^* is considered similarly, and there arise natural identifications

$$\begin{gathered} \Lambda^2 M \subset \mathrm{Hom}(M^*, M), \quad \Lambda^2 M^* \subset \mathrm{Hom}(M, M^*), \\ A(f) := -f \lrcorner A, \quad B(x) := -x \lrcorner B, \end{gathered} \tag{2.1.2}$$

which equip the pair $(\Lambda^2 M, \Lambda^2 M^*)$ with the quadratic product

$$(A, B) \mapsto (-ABA, -BAB);$$

this is a *Jordan pair*, and (A, B) is quasi-invertible if and only if the endomorphism $1 + AB$ of M, dual to the endomorphism $1 + BA$ of M^*, is invertible, in which case the quasi-inverse A^B is given by $A^B = (1 + AB)^{-1} A = A(1 + BA)^{-1}$, and the quasi-inverse B^A is similar [5, Section 1.5]. We will be particularly interested in the polynomial function

$$\delta_M(A, B) := \langle \exp(A), \exp(B)\rangle$$

(where exp denotes *Chevalley's exponential map* [2, Section IV-2]), which we called the 'generic norm' in the introduction, and now label with M, as it will be varied later; actually it is the square root of the polynomial $\det(1 + AB)$ [5, Theorem 2.6]. This δ_M satisfies the 'cocycle relations'

$$\delta_M(A, B)\delta_M(A^B, B') = \delta_M(A, B + B'), \tag{2.1.3}$$

$$\delta_M(A', B^A)\delta_M(A, B) = \delta_M(A' + A, B) \tag{2.1.4}$$

for all quasi-invertible (A, B) and all (A', B') [5, Section 3.11]. Exponential maps relate the Jordan pair $(\Lambda^2 M, \Lambda^2 M^*)$ to the *hyperbolic quadratic module* $\mathbf{H}(M)$; recall that this module is the direct sum $M \oplus M^*$, equipped with the quadratic form defined by the pairing $\langle\cdot, \cdot\rangle$, and that the exterior power $\Lambda M = \Lambda^+ M \oplus \Lambda^- M$, decomposed by the parities of the degrees, plays the role of 'the space of spinors' once we identify $\mathrm{End}(\Lambda M)$ with the Clifford algebra; it has $M \oplus M^*$ embedded by $(x, f) \mapsto l_x + d_f$, and is graded by the 'checker-board grading' (see [10, IV, Section (2.1)] for details).

2.2. Exterior powers of linear maps. We shall consider the case where M is given as the direct sum of two arbitrary finitely generated and projective modules:

$$M = N \oplus P.$$

Applying the convention (2.1.1) to P, we identify $\Lambda^\nu P$ with the dual of $\Lambda^\nu P^*$, and identify $\mathrm{Hom}(\Lambda^\nu P^*, \Lambda^\nu N)$ with $(\Lambda^\nu N) \otimes (\Lambda^\nu P)$; moreover, embedding the latter into $\Lambda^{2\nu} M$ by the map $z \otimes w \mapsto z' \wedge w'$ induced by the νth exterior powers $z \mapsto z'$, $w \mapsto w'$ of the natural inclusions of N and P into M, together with the wedge product in ΛM, we construct a linear map

$$\lambda_+^\nu : \mathrm{Hom}(\Lambda^\nu P^*, \Lambda^\nu N) \longrightarrow \Lambda^{2\nu} M. \tag{2.2.1}$$

We may decompose $\Lambda^{2\nu} M$ as the sum of the spaces $(\Lambda^{2\nu - i} N) \otimes (\Lambda^i P)$, where $0 \le i \le 2\nu$. Then λ_+^ν is an isomorphism onto one of the direct summands; in particular, there is a decomposition

$$\Lambda^2 M \overset{\sim}{\to} (\Lambda^2 N) \oplus \mathrm{Hom}(P^*, N) \oplus (\Lambda^2 P). \tag{2.2.2}$$

On the external two factors of the decomposition (2.2.2), the exponential map $\exp : \Lambda^2 M \to \Lambda^+ M$ considered for M induces those for N and P by naturalness; as for the central factor, given any element $u \in \mathrm{Hom}(P^*, N)$, we may decompose its exponential into homogeneous components, $\exp(\lambda_+^1(u)) =: \sum_{\nu \ge 0} \exp(\lambda_+^1(u))_{2\nu}$, and we claim that

$$\exp(\lambda_+^1(u))_{2\nu} = \lambda_+^\nu(\Lambda^\nu u) \in \Lambda^{2\nu} M. \tag{2.2.3}$$

To see this, take finite n-tuples $(x_1, \ldots, x_n) \in N^n$ and $(y_1, \ldots, y_n) \in P^n$ such that $u = \sum_i x_i \otimes y_i$, and abbreviate $x_\alpha \wedge x_\beta \wedge \cdots$ to x_I, whenever $I =: \{\alpha < \beta < \cdots\}$ is a finite set; it is then immediate that

$$\Lambda^\nu u = \varepsilon_\nu \sum_I x_I \otimes y_I,$$

where I ranges over the set of all ν-indices. Furthermore, $\lambda_+^1(u)$ is equal to $\sum_i (x_i, 0) \wedge (0, y_i)$, so its exponential is the product of the mutually commuting n-elements $1 + (x_i, 0) \wedge (0, y_i)$; thus its 2ν-component $\exp(\lambda_+^1(u))_{2\nu}$ is the sum of all partial products $p_I := \prod_{i\in I}(x_i, 0) \wedge (0, y_i)$, where I ranges over the set of all ν-indices. Since the signature ε_ν occurs in the relation $p_I = \varepsilon_\nu \lambda_+^\nu(x_I \otimes y_I)$, the proof is complete.

2.3. The Jordan pair $(\mathfrak{V}^+, \mathfrak{V}^-)$. The map λ_+^ν defined above may be coupled with the analogous map

$$\lambda_-^\nu : \mathrm{Hom}(\Lambda^\nu P, \Lambda^\nu N^*) \longrightarrow \Lambda^{2\nu} M^*$$

that is constructed similarly from (N^*, P^*), by making the obvious identifications $N^* \oplus P^* \cong M^*$ and $P^{**} \cong P$. To give equal weight to M and M^*, we will sometimes write $M^+ := M$ and $M^- := M^*$, with a mod 2 suffix $\sigma \in \{+, -\}$, as in [5, Section 1.6]. We now consider the pair

$$\mathfrak{V}^+ := \mathrm{Hom}(P^*, N), \quad \mathfrak{V}^- := \mathrm{Hom}(P, N^*)$$

of k-modules equipped with the quadratic product $(u, v) \mapsto (uv^*u, vu^*v)$. Needless to say, this is a Jordan pair, locally isomorphic to the Jordan pair of *rectangular matrices*, but it is possible to give global proofs of its fundamental properties involving quasi-inverses, without resorting to calculations in local coordinates. Note that since the Bergmann operators $B_+(u, v) \in \mathrm{End}(\mathfrak{V}^+)$ and $B_-(v, u) \in \mathrm{End}(\mathfrak{V}^-)$ are now of the form

$$B_+(u, v) : u_0 \mapsto (1 - uv^*)u_0(1 - v^*u)$$

and

$$B_-(v, u) : v_0 \mapsto (1 - vu^*)v_0(1 - u^*v),$$

a pair (u, v) is quasi-invertible if all the endomorphisms

$$\begin{matrix} 1 - uv^* \in \mathrm{End}(N^*), & 1 - vu^* \in \mathrm{End}(N), \\ 1 - v^*u \in \mathrm{End}(P^*), & 1 - u^*v \in \mathrm{End}(P) \end{matrix} \tag{2.3.1}$$

are invertible, and furthermore if this is so, then the quasi-inverses are of the form $u^v = (1 - uv^*)^{-1}u = u(1 - v^*u)^{-1}$, and so on.

2.4. A criterion for quasi-invertibility. The converse of the statement above appears likely but is difficult to state precisely in terms of matrices, so we shall subsume it into the following result.

PROPOSITION. *Let λ_+^1 and λ_-^1 be the maps defined above.*

(a) *The pair $(\lambda_+^1, \lambda_-^1)$ of linear maps is a homomorphism of Jordan pairs; in particular, if (u, v) is quasi-invertible, then so is $(\lambda_+^1(u), \lambda_-^1(v))$.*

(b) *For all (u, v) in $\mathfrak{V}^+ \times \mathfrak{V}^-$, the determinants of the four endomorphisms in (2.3.1) have the same value, which is equal to $\delta_M(\lambda_+^1(u), \lambda_-^1(v))$; this value is invertible if and only if (u, v) is quasi-invertible.*

PROOF. To prove (a), let $w = \sum_i x_i \otimes y_i$ be any element of $\mathfrak{V}^\sigma \cong N^\sigma \otimes P^\sigma$ and regard $\lambda^1_\sigma(w) =: W$ as a linear map $M^{-\sigma} \to M^\sigma$ by the rule (2.1.2); then for any $\xi = (f, g) \in M^{-\sigma} \cong N^{-\sigma} \oplus P^{-\sigma}$,

$$W(\xi) = \sum_i -(f, g) \lrcorner ((x_i, 0) \wedge (0, y_i)) = \sum_i (\langle y_i, g\rangle x_i, -\langle x_i, f\rangle y_i);$$

therefore $W(f, g) = (w(g), -w^*(f))$. Interpreted as matrices of left actions and duplicated for the two cases $\sigma = \pm$, this shows that

$$\lambda^1_+(u) = \begin{pmatrix} 0 & u \\ -u^* & 0 \end{pmatrix}, \quad \lambda^1_-(v) = \begin{pmatrix} 0 & v \\ -v^* & 0 \end{pmatrix}. \tag{2.4.1}$$

The identities $-\lambda^1_+(u)\lambda^1_-(v)\lambda^1_+(u) = \lambda^1_+(uv^*u)$ and $-\lambda^1_-(v)\lambda^1_+(u)\lambda^1_-(v) = \lambda^1_-(vu^*v)$ follow immediately, and show that $(\lambda^1_+, \lambda^1_-)$ is a homomorphism. The last statement is then a consequence of the naturalness of quasi-inverses; see [11, Proposition 3.2].

We now prove (b). Since $1 - uv^*$ and $1 - vu^*$ are dual to each other, as are $1 - u^*v$ and $1 - v^*u$, in order to prove the equality of the determinants, it suffices to show that

$$\det(1 - uv^*) = \delta_M(\lambda^1_+(u), \lambda^1_-(v)) = \det(1 - u^*v); \tag{2.4.2}$$

furthermore, in light of the last statement of part (a), these identities will also establish the 'if' part of the invertibility. We prove (2.4.2). Since the formula (2.2.3) and its analogue for $\lambda^\nu_-(v)$ convert $\delta_M(\lambda^1_+(u), \lambda^1_-(v))$ to the sum of the $\langle\lambda^\nu_+(\Lambda^\nu u), \lambda^\nu_-(\Lambda^\nu v)\rangle$ where $\nu \geq 0$, by a well-known formula for determinants of type $\det(1 + h)$ (see for example formula (19) of [1, Section III-8.5]), it suffices to prove that

$$(-1)^\nu \operatorname{trace}(XY^*) = \langle\lambda^\nu_+(X), \lambda^\nu_-(Y)\rangle = (-1)^\nu \operatorname{trace}(X^*Y), \tag{2.4.3}$$

for all X in $\operatorname{Hom}(\Lambda^\nu P^*, \Lambda^\nu N)$ and all Y in $\operatorname{Hom}(\Lambda^\nu P, \Lambda^\nu N^*)$. The problem is now linear in both arguments X and Y, so we may suppose that $X = z \otimes w$ and $Y = z^* \otimes w^*$, where z, w, z^*, w^* are arbitrary elements of the νth exterior powers of N, P, N^*, and P^*. In this case, $XY^* = \langle w, w^*\rangle z \otimes z^*$ and $X^*Y = \langle z, z^*\rangle w \otimes w^*$ have the same trace $\langle z, z^*\rangle\langle w, w^*\rangle$. Since $\lambda^\nu_+(X) = z \wedge w$ and $\lambda^\nu_-(Y) = z^* \wedge w^*$ under the identifications that we have made when defining λ^ν_+ in (2.2.1), their pairing $\langle\lambda^\nu_+(X), \lambda^\nu_-(Y)\rangle$ is now equal to $\langle w^* \lrcorner (z \wedge w), z^*\rangle$ by the duality between left interior and right wedge products. As $w^* \lrcorner (z \wedge w) = (-1)^\nu \langle w, w^*\rangle z$, this completes the proof. □

2.5. More on δ_M. Let us examine the polynomial $\delta_M(A, B)$ in more detail than the previous case, where $(A, B) = (\lambda^1_+(u), \lambda^1_-(v))$. Coupling the decomposition (2.2.2) with the analogous one for $\Lambda^2 M^*$, we shall note here the following addendum to (2.4.1): as intertwining maps between $M = N \oplus P$ and $M^* = N^* \oplus P^*$, according to the rule (2.1.2), the elements $A = a \oplus u \oplus c \in \Lambda^2 M$ and $B = b \oplus v \oplus d \in \Lambda^2 M^*$ with components

$$\begin{aligned} &a \in \Lambda^2 N, \quad u \in \operatorname{Hom}(P^*, N), \quad c \in \Lambda^2 P, \\ &b \in \Lambda^2 N^*, \quad v \in \operatorname{Hom}(P, N^*), \quad d \in \Lambda^2 P^*, \end{aligned}$$

are represented as alternating matrices of the type

$$a \oplus u \oplus c = \begin{pmatrix} a & u \\ -u^* & c \end{pmatrix}, \quad b \oplus v \oplus d = \begin{pmatrix} b & v \\ -v^* & d \end{pmatrix}. \tag{2.5.1}$$

In Section 2.6 we will give a method of calculating $\delta_M(A, B)$ in terms of these components 'generically'. We begin by noting that, from the naturalness of exponential maps, $\delta_M(A, B)$ reduces to $\delta_N(a, b)$ or $\delta_P(c, d)$ when (A, B) is either $(a \oplus 0 \oplus 0, b \oplus 0 \oplus 0)$ or $(0 \oplus 0 \oplus c, 0 \oplus 0 \oplus d)$; both $\delta_M(a \oplus 0 \oplus 0, 0 \oplus 0 \oplus d)$ and $\delta_M(0 \oplus 0 \oplus c, b \oplus 0 \oplus 0)$ are 1, since all pairings coming from $(\Lambda N^{\pm}) \times (\Lambda P^{\mp})$ vanish, except for constant terms; henceforth, we shall tacitly forget the order of arguments in the pairing $\langle \cdot, \cdot \rangle$, which will thus appear 'symmetric'; thus we shall write $\langle z, z^* \rangle =: \langle z^*, z \rangle$. Along the same lines, combined also with (2.4.2), we get the following generalization:

$$\begin{aligned} &\delta_M(0 \oplus u \oplus c, b \oplus 0 \oplus 0) = \delta_M(a \oplus u \oplus 0, 0 \oplus 0 \oplus d) \\ &\quad = \delta_M(a \oplus 0 \oplus 0, 0 \oplus v \oplus d) = \delta_M(0 \oplus 0 \oplus c, b \oplus v \oplus 0) = 1, \end{aligned} \tag{2.5.2}$$

$$\begin{aligned} &\delta_M(a \oplus u \oplus c, 0 \oplus 0 \oplus d) = \delta_M(0 \oplus 0 \oplus c, b \oplus v \oplus d) = \delta_P(c, d), \\ &\delta_M(a \oplus 0 \oplus 0, b \oplus v \oplus d) = \delta_M(a \oplus u \oplus c, b \oplus 0 \oplus 0) = \delta_N(a, b), \end{aligned} \tag{2.5.3}$$

$$\begin{aligned} \delta_M(a \oplus u \oplus c, 0 \oplus v \oplus 0) &= \delta_M(0 \oplus u \oplus 0, b \oplus v \oplus d) \\ &= \det(1 - uv^*) = \det(1 - u^*v). \end{aligned} \tag{2.5.4}$$

Note, however, that values such as $\delta_M(a \oplus 0 \oplus c, 0 \oplus v \oplus 0)$ are no longer trivial, since, for example, $\exp(a \oplus 0 \oplus c)$ has components coming from $\mathrm{Hom}(\Lambda^\nu(P^*), \Lambda^\nu(N))$. For these, we shall prove that

$$\begin{aligned} &\delta_M(a \oplus 0 \oplus c, 0 \oplus v \oplus 0) = \delta_N(a, (\Lambda^2 v)c) = \delta_P(c, (\Lambda^2 v^*)a), \\ &\delta_M(0 \oplus u \oplus 0, b \oplus 0 \oplus d) = \delta_N((\Lambda^2 u)d, b) = \delta_P((\Lambda^2 u^*)b, d). \end{aligned} \tag{2.5.5}$$

Indeed, let $\delta^{(1)}$ denote the left-hand side $\delta_M(a \oplus 0 \oplus c, 0 \oplus v \oplus 0)$ of the first line. We first regard $\delta^{(1)}$ as $\delta_M(A + A', B)$, where $A := a \oplus 0 \oplus 0$, $A' := 0 \oplus 0 \oplus c$ and $B := 0 \oplus v \oplus 0$. In this case, $\delta_M(A, B) = 1$ by (2.5.2) and the quasi-inverse $B^A = B(1 + AB)^{-1}$, calculated in terms of matrices of type (2.5.1), has components $0 \oplus v \oplus (\Lambda^2 v^*)a$; thus, the cocycle relation (2.1.4), together with (2.5.3), converts $\delta^{(1)}$ to $\delta_P(c, (\Lambda^2 v^*)a)$. Similarly, if we start from $A := 0 \oplus 0 \oplus c$ and $A' := a \oplus 0 \oplus 0$, in which case $B^A = (\Lambda^2 v)c \oplus v \oplus 0$, then $\delta^{(1)} = \delta_N(a, (\Lambda^2 v)c)$. The proof of the second line of (2.5.5) is similar; we merely apply the formula (2.1.4) to the case where $A := 0 \oplus u \oplus 0$ and $\{B, B'\} := \{b \oplus 0 \oplus 0, 0 \oplus 0 \oplus d\}$.

2.6. Generic calculation of δ_M. We shall proceed similarly to prove the following result.

PROPOSITION. *Let δ_M be as defined above.*

(a) *If $(c, d) \in (\Lambda^2 P) \times (\Lambda^2 P^*)$ is quasi-invertible, then*

$$\delta_M(a \oplus u \oplus c, b \oplus v \oplus d) = \delta_M(a_1 \oplus u_1 \oplus 0, b_1 \oplus v \oplus 0)\delta_P(c, d), \quad (2.6.1)$$

where $a_1 := a + (\Lambda^2 u)d^c$, $u_1 := u(1 + dc)^{-1}$, and $b_1 := b + (\Lambda^2 v)c^d$; in particular,

$$\delta_M(a \oplus u \oplus c, b \oplus v \oplus 0) = \delta_M(a \oplus u \oplus 0, (b + (\Lambda^2 v)c) \oplus v \oplus 0). \quad (2.6.2)$$

(b) *If $(u, v) \in \mathfrak{V}^+ \times \mathfrak{V}^-$ is quasi-invertible, then*

$$\delta_M(a \oplus u \oplus 0, b \oplus v \oplus 0) = (\det h)\delta_N(a', b),$$

where $h := 1 - uv^*$ and $a' := (\Lambda^2 h)^{-1} a$.

PROOF. To prove (a), we begin with the special case (2.6.2), which is proved similarly to (2.5.5), by taking $A' := a \oplus u \oplus 0$, $A := 0 \oplus 0 \oplus c$, and $B := b \oplus v \oplus 0$, and applying (2.1.4). In order to prove (2.6.1), we take $A := a \oplus u \oplus c$, $B := 0 \oplus 0 \oplus d$, and $B' := b \oplus v \oplus 0$. In this case, $\delta_M(A, B) = \delta_P(c, d)$, which is invertible by hypothesis, by (2.5.3). We use (2.1.4) to decompose the left-hand side of (2.6.1) to $\delta_M(A^B, B')\delta_P(c, d)$. Calculation shows that

$$1 + AB = \begin{pmatrix} 1 & ud \\ 0 & 1 + cd \end{pmatrix} = \begin{pmatrix} 1 & -ud(1 + cd)^{-1} \\ 0 & (1 + cd)^{-1} \end{pmatrix}^{-1}$$

and noting that, from the equality of the two expressions $d(1 + cd)^{-1}$ and $(1 + dc)^{-1} d$ for d^c, the expression $1 - d(1 + cd)^{-1} c$ is equal to $(1 + dc)^{-1}$, we see that the quasi-inverse $A^B = (1 + AB)^{-1} A$ has components $a_1 \oplus u_1 \oplus c^d$, with a_1 and u_1 as defined above; thus, in light of (2.6.2), we get the desired relation

$$\delta_M(A^B, B') = \delta_M(a_1 \oplus u_1 \oplus 0, b_1 \oplus v \oplus 0).$$

To prove (b), we put $A := a \oplus u \oplus 0$, $B := 0 \oplus v \oplus 0$, and $B' := b \oplus 0 \oplus 0$; in this case, $\delta_M(A, B) = \det h$ by (2.5.4), and this is invertible by hypothesis. Now we use (2.1.4) to reduce the problem to proving that $\delta_M(A^B, B') = \delta_N(a', b)$. Again we calculate $1 + AB$, which, since $v(1 - u^* v)^{-1} = (1 - vu^*)^{-1} v = v^u$, is now of the form

$$\begin{pmatrix} 1 - uv^* & av \\ 0 & 1 - u^* v \end{pmatrix} = \begin{pmatrix} (1 - uv^*)^{-1} & -(1 - uv^*)^{-1} a v^u \\ 0 & (1 - u^* v)^{-1} \end{pmatrix}^{-1}.$$

Therefore, as $A^B = (1 + AB)^{-1} A$ and $1 + v^u u^* = (1 + vu^*)^{-1}$, it follows that $A^B = a' \oplus u^v \oplus 0$. In light of (2.5.3), this proves the desired result. □

3. Embeddings

3.1. The homomorphisms $X_\pm$, X_0. We turn our attention to group schemes and adapt the functorial point of view [3]. Put

$$Q := N \oplus P^* \quad (3.1.1)$$

and express elements of $\mathrm{End}(Q)$ as matrices with entries in the array

$$\begin{pmatrix} \mathrm{End}(N) & \mathrm{Hom}(P^*, N) \\ \mathrm{Hom}(N, P^*) & \mathrm{End}(P^*) \end{pmatrix}, \tag{3.1.2}$$

which act on the left. Similar matrix notation is used for all scalar extensions too, and so we have homomorphisms

$$\begin{gathered} \mathbf{W}(\mathfrak{V}^+) \xrightarrow{X_+} \mathbf{GL}(Q) \xleftarrow{X_-} \mathbf{W}(\mathfrak{V}^-) \\ X_+(u) := \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad X_-(v) := \begin{pmatrix} 1 & 0 \\ -v^* & 1 \end{pmatrix} \end{gathered} \tag{3.1.3}$$

of k-group schemes, where $\mathbf{W}$ denotes the covariant functor constructing vector bundles from finitely generated projective modules, that is, $\mathbf{W}(\mathfrak{V}^+)(k') := \mathfrak{V}^+ \otimes k'$ for each scalar extension k' of k. When describing scheme morphisms, letters like u, v in (3.1.3) are to be understood as ranging over all scalar extensions too. Analogously, we introduce a homomorphism

$$\begin{gathered} X_0 : \mathbf{GL}(N) \times \mathbf{GL}(P) \longrightarrow \mathbf{GL}(Q) \\ X_0(g, h) := \begin{pmatrix} g & 0 \\ 0 & h^{*-1} \end{pmatrix}. \end{gathered} \tag{3.1.4}$$

Actual computation of the product $X_-(v)X_0(g, h)X_+(u) =: \xi(v, (g, h), u)$ shows immediately that ξ is an *open embedding* and its image is the principal open subscheme $\Omega \subset \mathbf{GL}(Q)$ defined by the condition that the $(1, 1)$th entry in the sense of matrix (3.1.2) is invertible; since the k-group scheme $\mathbf{GL}(Q)$ is smooth with connected fibers, this shows that Ω is scheme theoretically dense in $\mathbf{GL}(Q)$.

3.2. Relation to the Jordan pair $(\mathfrak{V}^+, \mathfrak{V}^-)$. We need to know about the induced group germ structure on Ω, for which it is convenient to use the notation of [12]. The decomposition (3.1.1) is now used to endow $\mathbf{GL}(Q)$ with the action ψ of the multiplicative group $\mathbf{G}_{\mathbf{m}k}$ such that ψ_t, for any invertible scalar t, multiplies each (i, j)th entry by t^{j-i}; this amounts to setting $\psi_t := \mathrm{Int}(X_0(t, 1))$ and makes the spaces $\mathbf{W}(\mathfrak{V}^\pm)$ be of weights ± 1, while $\mathbf{GL}(N) \times \mathbf{GL}(P)$ remains invariant. There is an action $\rho := (\rho_+, \rho_-)$ of the k-group $\mathbf{GL}(N) \times \mathbf{GL}(P)$ on the pair $(\mathfrak{V}^+, \mathfrak{V}^-)$, given by

$$\rho_+(g, h) \cdot u := guh^*, \quad \rho_-(g, h) \cdot v := h^{*-1}vg^{-1}, \tag{3.2.1}$$

which might also be written using conjugations in $\mathbf{GL}(Q)$:

$$\begin{aligned} \mathrm{Int}(X_0(g, h)) \cdot X_+(u) &=: X_+(\rho_+(g, h) \cdot u), \\ \mathrm{Int}(X_0(g, h)) \cdot X_-(v) &=: X_-(\rho_-(g, h) \cdot v). \end{aligned}$$

This action is compatible with the Jordan product $(u, v) \mapsto (uv^*u, vu^*v)$. Moreover, from Section 2.4, on the open subscheme $\mathcal{W} \subset \mathbf{W}(\mathfrak{V}^+) \times \mathbf{W}(\mathfrak{V}^-) =: \mathbf{X}$ of quasi-invertible pairs, there exists a well-defined morphism

$$\begin{gathered} b : \mathcal{W} \longrightarrow \mathbf{GL}(N) \times \mathbf{GL}(P) \\ b(u, v) := (1 - uv^*, 1 - u^*v). \end{gathered} \tag{3.2.2}$$

For the morphism $p:\mathbf{X}\to\mathbf{GL}(Q)$, defined by $p(u,v):=X_+(u)X_-(v)$, easy computation shows that $\mathcal{W}$ is the inverse image $p^{-1}(\Omega)$, on which p induces the morphism $(u,v)\mapsto X_-(v^u)X_0(b(u,v))X_+(u^v)$. In the terminology of [12], the datum $(\mathbf{GL}(Q),\psi)$ is thus an 'elementary system' and $(\mathfrak{V}^+,\mathfrak{V}^-)$ is the associated Jordan pair, which becomes a 'Jordan system' once it is equipped with the action ρ and the morphism b (to be called the *Bergmann morphism*). A theorem on generators and relations may be found in [12, Theorem 4.14]; we shall apply this to construct an embedding λ of $\mathbf{GL}(Q)$.

3.3. Partial description of λ. Provisionally, we suppose that M is any finitely generated projective module and we consider the hyperbolic quadratic module $\mathbf{H}(M)$. First, we recall the description [5], similar to that above, of the *special Clifford group* $\mathbf{S\Gamma}(\mathbf{H}(M))$, the normalizer in $\mathbf{GL}(\Lambda^+M)\times\mathbf{GL}(\Lambda^-M)$ of the embedded subset $M\oplus M^*$ of $\mathrm{End}(\Lambda M)$: there are two homomorphisms,

$$\mathbf{W}(\Lambda^2M)\xrightarrow{Y_+}\mathbf{GL}(\Lambda M)\xleftarrow{Y_-}\mathbf{W}(\Lambda^2M^*)$$
$$Y_+(A):=l_{\exp(A)},\qquad Y_-(B):=d_{\exp(B)},$$

which in fact factors, through the *spin group*, the kernel of the *norm character* $\mu:\mathbf{S\Gamma}(\mathbf{H}(M))\to\mathbf{G}_{\mathbf{m}k}$, and

$$\begin{gathered}Y_0:\mathbf{G}_{\mathbf{m}k}\times\mathbf{GL}(M)\longrightarrow\mathbf{GL}(\Lambda M)\\ Y_0(t,T):=t(\det T)^{-1}\Lambda T,\end{gathered}\tag{3.3.1}$$

which, after composition with μ, yields the character $(t,T)\mapsto t^2(\det T)^{-1}$. Furthermore, Y_0 normalizes $Y_\pm$, the induced action on the Jordan pair $(\Lambda^2M,\Lambda^2M^*)$ is $(t,T)\mapsto(\Lambda^2T,\Lambda^2T^{*-1})$, and the Bergmann morphism is $(A,B)\mapsto(\delta_M(A,B),1+AB)$. Let us now suppose that $M=N\oplus P$, as in Section 2.2. Then there are homomorphisms $\lambda^1_\pm$, which may be understood as having sources $\mathbf{W}(\mathfrak{V}^\pm)$ and targets $\mathbf{W}(\Lambda^2(M^\pm))$; analogously, we introduce the homomorphism

$$\begin{gathered}\lambda_0:\mathbf{GL}(N)\times\mathbf{GL}(P)\longrightarrow\mathbf{G}_{\mathbf{m}k}\times\mathbf{GL}(M)\\ \lambda_0(g,h):=(\det g,g\oplus h).\end{gathered}\tag{3.3.2}$$

3.4. The main theorem. Our main theorem establishes the existence of an embedding which extends $\lambda_\pm$ and λ_0.

THEOREM. *Let N and P be finitely generated projective modules, and denote $M:=N\oplus P$ and $Q:=N\oplus P^*$. Then there exists one and only one homomorphism*

$$\lambda:\mathbf{GL}(Q)\longrightarrow\mathbf{S\Gamma}(\mathbf{H}(M))$$

of group schemes such that $\lambda\circ X_\pm=Y_\pm\circ\lambda^1_\pm$ and $\lambda\circ X_0=Y_0\circ\lambda_0$. This is a monomorphism, and converts the norm μ to the determinant, that is, $\det=\mu\circ\lambda$.

PROOF. The uniqueness and the properties that λ is monomorphic and $\det=\mu\circ\lambda$ are all evident if we restrict to the dense subscheme $\Omega\subset\mathbf{GL}(Q)$ (this is harmless);

note that, by [12, Proposition 3.8], the homomorphism f defined in [12, Theorem 4.14] is monomorphic whenever f_0 and $f_\pm$ are monomorphic. Furthermore, since everything has been interpreted in the category of Jordan systems, in order to prove the existence of λ, it suffices to verify that *the pair* $(\lambda_0, (\lambda^1_+, \lambda_-))$ *is a homomorphism of Jordan systems*; in other words [12, Equations 5.1(6), (7)], that

$$\lambda^1_+(\rho_+(g,h)\cdot u) = (\Lambda^2 T)\cdot A, \quad \lambda^1_-(\rho_-(g,h)\cdot v) = (\Lambda^2 T^{*-1})\cdot B, \tag{3.4.1}$$

$$\lambda_0(b(u,v)) = (\delta_M(A,B), 1+AB), \tag{3.4.2}$$

for all u, v, and (g,h), where $A := \lambda^1_+(u)$, $B := \lambda^1_-(v)$, and $T := g \oplus h$. Now, (3.4.1) follows from (3.2.1), and a matrix computation like (2.4.1) and (3.1.2). Similarly, we take (g,h) to be $b(u,v)$, and then $1 + AB = g \oplus h$ (see (3.2.2)); it follows that $\det(g) = \delta_M(A,B)$ by Section 2.4, whence (3.4.2) holds, in light of (3.3.2).

REMARK. The mapping λ resembles Λ. More precisely, switching P and P^* induces a natural isometry $\mathbf{H}(M) \xrightarrow{\sim} \mathbf{H}(Q)$, which in turn induces an isomorphism $I : \mathbf{S\Gamma}(\mathbf{H}(M)) \xrightarrow{\sim} \mathbf{S\Gamma}(\mathbf{H}(Q))$. We may very well ask what is the composite $i \circ \lambda$. We conjecture that this is $i \circ \kappa \circ Y_0(\det(\cdot), \cdot)$, where $\kappa : \mathbf{S\Gamma}(\mathbf{H}(Q)) \xrightarrow{\sim} \mathbf{S\Gamma}(\mathbf{H}(M))$ is an *isomorphism*, and more precisely that there exists an isomorphism κ such that $\lambda(\gamma) = \kappa(\Lambda\gamma)$ for all points γ of $\mathbf{GL}(Q)$. We shall give an explicit description of κ in Section 4, in the case where M has rank two or three.

3.5. Close-up of End(ΛQ). We now give a method to calculate the exterior powers of $X_\pm$ in terms of $\lambda^1_\pm$, which we will use in proving the next theorem (in Section 3.8). We begin by rewriting (3.1.3) as

$$X_+(u) = 1 + \tilde{u}, \quad X_-(v) = 1 + \tilde{v}, \tag{3.5.1}$$

where $w \mapsto \tilde{w}$ denotes the map from $\mathfrak{V}^\sigma \cong N^\sigma \oplus P^\sigma$ to $\mathrm{End}(Q) \cong Q \otimes Q^*$ such that $(n \otimes p)^\sim = (n \oplus 0) \otimes (0 \oplus p)$ when $\sigma = +$ and $(n^* \otimes p^*)^\sim = -(0 \oplus p^*) \otimes (n^* \oplus 0)$ when $\sigma = -$; here and henceforth, elements of Q and Q^* are written as $n \oplus p^*$ and $n^* \oplus p$ and those of M and M^* are written as (n, p) and (n^*, p^*). Introducing the direct sum k-module

$$E := N \oplus P^* \oplus N^* \oplus P,$$

there is an injection $j : \mathrm{End}(Q) \to \Lambda^2 E$ just as in Section 2.2; furthermore, since $M^\pm$ are direct summands in E, there are also induced injections

$$\Lambda M \xrightarrow{\iota_+} \Lambda E \xleftarrow{\iota_-} \Lambda M^*;$$

it is then easy to see that, if $w \in \mathfrak{V}^\sigma$, where $\sigma = \pm$, then

$$j(\tilde{w}) = \iota_\sigma(\lambda^1_\sigma(w)). \tag{3.5.2}$$

Furthermore, since E is the module underlying $\mathbf{H}(Q)$ and the embedded subspaces $M^\pm \subset E$ are totally singular, both of them generate subalgebras isomorphic to exterior algebras; explicitly, there are injective homomorphisms

$$\Lambda M \xrightarrow{\varphi_+} \mathrm{End}(\Lambda Q) \xleftarrow{\varphi_-} \Lambda M^*$$

of k-algebras, which, on the degree-one parts, are given by

$$\varphi_+(\xi) = l_{n\oplus 0} + d_{0\oplus p}, \quad \varphi_-(\xi^*) = l_{0\oplus p^*} + d_{n^*\oplus 0} \tag{3.5.3}$$

when $\xi = (n, p)$ in M and $\xi^* = (n^*, p^*)$ in M^*. As yet another description, we recall [6, Section 1.6] that there exists a module isomorphism

$$\varphi : \Lambda E \xrightarrow{\sim} \mathrm{End}(\Lambda Q)$$

characterized by the unital property $\varphi(1) = 1$ and the identity

$$\varphi((n, p^*, n^*, p) \wedge Z) = (l_{n\oplus p^*} + d_{n^*\oplus p}) \circ \varphi(Z) - \varphi(F \lrcorner Z) \tag{3.5.4}$$

when (n, p^*, n^*, p) in E and Z in ΛE, where F denotes the linear form on E taking the value $\langle n_0, n^*\rangle + \langle p, p_0^*\rangle$ in k at (n_0, p_0^*, n_0^*, p_0). This F annihilates the embedded subspaces M and M^* if $n^* = 0$ or $p = 0$, respectively. It follows, in particular, that φ is an algebra homomorphism when it is restricted to either ΛM or ΛM^*; in fact, comparing (3.5.3) and (3.5.4), we find that $\varphi_\pm = \varphi \circ \iota_\pm$.

3.6. Calculating exterior powers of $X_\pm$ in terms of $\lambda^1_\pm$. We are now in a position to prove the following result.

PROPOSITION. *For all $u \in \mathfrak{V}^+$ and $v \in \mathfrak{V}^-$,*

$$\begin{aligned} \Lambda X_+(u) &= \varphi_+(\exp(\lambda^1_+(u))),\\ \Lambda X_-(v) &= \varphi_-(\exp(\lambda^1_-(v))). \end{aligned} \tag{3.6.1}$$

PROOF. Since $\varphi_\pm = \varphi \circ \iota_\pm$, it follows that $\varphi_\pm \circ \exp = \mathrm{Exp} \circ \iota_\pm$, where Exp denotes the map $\Lambda^2 E \to \mathrm{End}(\Lambda Q)$ introduced in [6, Section 2]. This, combined with (3.5.2), proves that $\varphi_\sigma(\exp(\lambda^1_\sigma(w))) = \mathrm{Exp}(j(\tilde{w}))$ when $w \in \mathfrak{V}^\sigma$ and $\sigma = \pm$, and $\mathrm{Exp}(j(\tilde{w}))$ is $\Lambda(1 + \tilde{w})$ by [6, Section 2.2.1]; in view of (3.5.1), we are done. □

3.7. The embedding ζ. Let L denote the invertible module $\Lambda^d P$, where d is the rank of P. Given the natural decomposition of $\Lambda^d Q$ as the direct sum of spaces $(\Lambda^\nu N) \otimes (\Lambda^{d-\nu} P^*)$ where $0 \le \nu \le d$, there exists an isomorphism

$$\begin{aligned} (\Lambda^{d-\nu} P^*) \otimes L &\xrightarrow{\sim} \Lambda^\nu P\\ z^* \otimes \omega &\mapsto z^* \lrcorner \omega \end{aligned} \tag{3.7.1}$$

for each ν. Thus, tensoring $\Lambda^d Q$ by L on the right, and (3.7.1) by $\Lambda^\nu N$ on the left, after which the target $(\Lambda^\nu N) \otimes (\Lambda^\nu P)$ becomes embedded in $\Lambda^{2\nu} M \subset \Lambda^+ M$ as in Section 2.2, we obtain a linear map

$$\zeta : (\Lambda^d Q) \otimes L \longrightarrow \Lambda^+ M,$$

which is by construction an isomorphism onto a direct factor. Furthermore, the k-group $\mathbf{GL}(Q)$ acts on the source $(\Lambda^d Q) \otimes L$ by the dth exterior power of the standard representation tensored with Id_L, while $\mathbf{S\Gamma}(\mathbf{H}(M))$ acts on the target by the even spin representation.

3.8. Second main theorem. We now come to our second main theorem: we establish an important property of the mapping ζ.

THEOREM. *The mapping ζ is equivariant relative to the group homomorphism λ.*

PROOF. This amounts to saying that

$$\zeta(((\Lambda^d \gamma) \cdot Z) \otimes \omega) = \lambda(\gamma) \cdot \zeta(Z \otimes \omega) \tag{3.8.1}$$

for all points (γ, Z, ω) of the k-scheme $\mathbf{GL}(Q) \times \mathbf{W}(\Lambda^d Q) \times \mathbf{W}(L)$; without loss of generality, we may suppose that these points are k-points. Furthermore, since the problem is homomorphic in γ and linear in Z, we may suppose that γ is of one of the forms

$$X_+(n \otimes p), \quad X_-(n^* \otimes p^*), \quad X_0(g, h), \tag{3.8.2}$$

as in the explanation of (3.5.1) or in (3.1.4), and that Z is given by

$$Z = (z_\nu \oplus 0) \wedge (0 \oplus z^*_{d-\nu}), \tag{3.8.3}$$

whence, by (3.7.1),

$$\zeta(Z \otimes \omega) = (z_\nu, 0) \wedge (0, z^*_{d-\nu} \lrcorner\, \omega), \tag{3.8.4}$$

where z_ν is in $\Lambda^\nu N$ and $z^*_{d-\nu}$ in $\Lambda^{d-\nu} P$, for some ν; here and henceforth, in an abuse of notation, previously used symbols such as $z \oplus 0$, $(z, 0)$, and so on denote *exterior tensors* over Q, M, and so on.

In order to prove (3.8.1) for the first two cases in (3.8.2), we rewrite γ as $X_\sigma(w)$ for short, as in (3.5.2), and note that $\exp(\lambda^1_\sigma(w)) = 1 + \lambda^1_\sigma(w)$. Then $\Lambda\gamma$ is equal to $\varphi_\sigma(1 + \lambda^1_\sigma(w))$ by (3.6.1), while $\lambda^1_\sigma(w)$ is now either $(n, 0) \wedge (0, p)$ or $(n^*, 0) \wedge (0, p^*)$; combining (3.5.3) with the formula $\lambda \circ X_\sigma = Y_\sigma \circ \lambda^1_\sigma$, we see that the pair $(\Lambda\gamma, \lambda(\gamma))$ is either $(1 + l_{n\oplus 0} \circ d_{0\oplus p}, 1 + l_{(n,0)\wedge(0,p)})$ or $(1 + d_{n^*\oplus 0} \circ l_{0\oplus p^*}, 1 + d_{(n^*,0)\wedge(0,p^*)})$. Thus (3.8.1) now takes the form

$$\zeta(Z' \otimes \omega) = (n, 0) \wedge (0, p) \wedge \zeta(Z \otimes \omega), \tag{3.8.5}$$

$$\zeta(Z'' \otimes \omega) = (n^*, 0) \lrcorner\, ((0, p^*) \lrcorner\, \zeta(Z \otimes \omega)), \tag{3.8.6}$$

where $Z' := (n \oplus 0) \wedge ((0 \oplus p) \lrcorner\, Z)$ and $Z'' := (n^* \oplus 0) \lrcorner\, ((0 \oplus p^*) \wedge Z)$. To prove (3.8.5), we deduce from (3.8.3) that $Z' = (-1)^\nu((n \wedge z_\nu) \oplus 0) \wedge (0 \oplus (p \lrcorner\, z^*_{d-\nu}))$, so that

$$\zeta(Z' \otimes \omega) = (-1)^\nu(n \wedge z_\nu, 0) \wedge (0, (p \lrcorner\, z^*_{d-\nu}) \lrcorner\, \omega) \tag{3.8.7}$$

by (3.7.1). For any $(\nu + 1)$-tensor $z^*_{\nu+1}$ over M^*, it is clear that $p \lrcorner\, (z^*_{\nu+1} \wedge z^*_{d-\nu}) = 0$ and so

$$(p \lrcorner\, z^*_{\nu+1}) \wedge z^*_{d-\nu} = (-1)^\nu z^*_{\nu+1} \wedge (p \lrcorner\, z^*_{d-\nu});$$

acting by the pairing with ω proves that $(p \lrcorner z^*_{d-\nu}) \lrcorner \omega$ and $p \wedge (z^*_{d-\nu} \lrcorner \omega)$, which is equal to $(-1)^\nu (z^*_{d-\nu} \lrcorner \omega) \wedge p$, take the *same* value at $z^*_{\nu+1}$. Thus

$$(p \lrcorner z^*_{d-\nu}) \lrcorner \omega = p \wedge (z^*_{d-\nu} \lrcorner \omega).$$

Therefore, from (3.8.7) and (3.8.4), the following calculation proves (3.8.5):

$$\begin{aligned}\zeta(Z' \otimes \omega) &= (-1)^\nu (n \wedge z_\nu, 0) \wedge (0,\, p \wedge (z^*_{d-\nu} \lrcorner \omega)) \\ &= (-1)^\nu (n, 0) \wedge (z_\nu, 0) \wedge (0,\, p) \wedge (0,\, z^*_{d-\nu} \lrcorner \omega) \\ &= (n, 0) \wedge (0,\, p) \wedge (z_\nu, 0) \wedge (0,\, z^*_{d-\nu} \lrcorner \omega).\end{aligned}$$

As to (3.8.6), on the one hand, $Z'' = (-1)^\nu ((n^* \lrcorner z_\nu) \oplus 0) \wedge (0 \oplus (p^* \wedge z^*_{d-\nu}))$ from (3.8.3), while on the other, $(0,\, p^*) \lrcorner \zeta(Z \otimes \omega) = (-1)^\nu (z_\nu, 0) \wedge (0,\, p^* \lrcorner (z^*_{d-\nu} \lrcorner \omega))$ from (3.8.4). It is easy to show that both these expressions coincide with the expression $(-1)^\nu (n^* \lrcorner z_\nu, 0) \wedge (0,\, p^* \lrcorner (z^*_{d-\nu} \lrcorner \omega))$.

For the last case in (3.8.2), write Π for the product $((\Lambda^\nu g) \cdot z_\nu, 0) \wedge (0,\, w_\nu)$, where

$$w_\nu := (\Lambda^{d-\nu}(h^{*-1}) \cdot z^*_{d-\nu}) \lrcorner \omega = (\det h)^{-1} (\Lambda^\nu h) \cdot (z^*_{d-\nu} \lrcorner \omega) \in \Lambda^\nu P,$$

and Γ for the mapping $(\det h)^{-1} \Lambda(g \oplus h)$. Using (3.1.4), we calculate that the left-hand side of (3.8.1) is equal to Π, thus, in light of (3.8.4), when $\Gamma \in \mathrm{GL}(\Lambda M)$, it follows that $\Pi = \Gamma \cdot \zeta(Z \otimes \omega)$; however, (3.3.2) combined with the formula $\lambda \circ X_0 = Y_0 \circ \lambda_0$ shows that Γ is $Y_0(\lambda_0(g, h)) = \lambda(\gamma)$. This completes our proof. □

4. Examples

4.1. The case of two invertible modules. Consider the case where both N and P are of *rank one*. We have $\Lambda^- M = M$, and since λ^1_+ identifies $\mathfrak{V}^+ = \mathrm{Hom}(P^*, N)$ with $\Lambda^2 M$, there arises an identification

$$\Lambda^+ M \cong k \oplus \mathfrak{V}^+. \tag{4.1.1}$$

We take $\mathfrak{V}^- = \mathrm{Hom}(P, N^*)$, the dual to $\mathfrak{V}^+$ under the pairing $\{u, v\}$, which is defined either by $uv^* \in \mathrm{End}(N) \cong k$ or by $u^* v \in \mathrm{End}(P) \cong k$. We write elements of $\mathrm{End}(\Lambda^+ M)$ using matrices with entries in the array

$$\begin{bmatrix} k & \mathfrak{V}^- \\ \mathfrak{V}^+ & k \end{bmatrix},$$

acting from the left; we use similar matrix notation implicitly for all scalar extensions too. Then $\langle \lambda^1_+(u), \lambda^1_-(v) \rangle = -\{u, v\}$ by (2.4.3), so that the action of $Y_-(\lambda^1_-(v))$ on $\Lambda^+ M$ is given by $[\alpha_0, u_0] \mapsto [\alpha_0 - \{u_0, v\}, u_0]$. Evidently, $Y_+(\lambda^1_+(u))$ is given by $[\alpha_0, u_0] \mapsto [\alpha_0, u_0 + \alpha_0 u]$; and since a point (g, h) of $\mathbf{GL}(N) \times \mathbf{GL}(P)$ is now a pair of invertible scalars, $\lambda_0(g, h) = (t, T)$ where $t := g$ and $T := \mathrm{diag}(g, h)$, whence $Y_0(\lambda_0(g, h))$ is $[\alpha_0, u_0] \mapsto [h^{-1}\alpha_0, g u_0]$. Compared with (3.1.3) and (3.1.4), these prove, by evaluation on the dense subscheme Ω, that the composite $\rho_+ \circ \lambda$ with the even spin representation $\rho_+ : \mathbf{S\Gamma}(\mathbf{H}(M)) \to \mathbf{GL}(\Lambda^+ M)$ is the isomorphism

$$\gamma = \begin{pmatrix} \alpha & u \\ v^* & \beta \end{pmatrix} \mapsto \hat{\gamma} := \begin{bmatrix} \beta & v \\ u & \alpha \end{bmatrix} : \mathbf{GL}(Q) \xrightarrow{\sim} \mathbf{GL}(\Lambda^+ M). \tag{4.1.2}$$

As for the composite $\rho_- \circ \lambda$, both $Y_+(\lambda_+^1(u))$ and $Y_-(\lambda_-^1(v))$ are trivial and $Y_0(\lambda_0(g,h))$ acts as $\mathrm{diag}(h^{-1}g, 1)$; since $h^{-1}g = \det(X_0(g,h))$, the result is the mapping $\gamma \mapsto \mathrm{diag}(\det(\gamma), 1)$. Therefore

$$\lambda(\gamma) = (\hat{\gamma}, \mathrm{diag}(\det \gamma, 1)),$$

which matches with the well-known fact that $\mathbf{S\Gamma}(\mathbf{H}(M))$ is the fiber product of $\mathbf{GL}(\Lambda^\pm M)$ relative to the determinants. In fact, the construction (4.1.2) may easily be adapted by changing the roles of Q and M, and this leads us to an isomorphism $\mathbf{S\Gamma}(\mathbf{H}(Q)) \xrightarrow{\sim} \mathbf{S\Gamma}(\mathbf{H}(M))$, induced from

$$\kappa : \mathbf{GL}(\Lambda^+ Q) \times \mathbf{GL}(\Lambda^- Q) \xrightarrow{\sim} \mathbf{GL}(\Lambda^+ M) \times \mathbf{GL}(\Lambda^- M)$$
$$\kappa(\hat{T}, \gamma) := (\hat{\gamma}, T).$$

The point is that $\lambda(\gamma) = \kappa(\Lambda(\gamma))$, since $\Lambda(\gamma) = (\mathrm{diag}(1, \det \gamma), \gamma)$. Incidentally, since $d = 1$, and $L = P$ in the notation of Section 3.7, the embedding ζ is now the natural isomorphism identifying $\Lambda^d Q \otimes L = (N \oplus P^*) \otimes P$ with $\Lambda^+ M$; see (4.1.1), and it is easy to verify directly that ζ is equivariant relative to the group isomorphism (4.1.2), proved in Section 3.8.

4.2. An embedding ι. Suppose that N is of rank two and P is invertible, so that M is of rank three. In this case, $\mathbf{S\Gamma}(\mathbf{H}(M))$ is faithfully represented in $\mathbf{GL}(\Lambda^- M) \times \mathbf{G}_{\mathbf{m}k}$ by the morphism with components (ρ_-, μ), as the closed subgroup scheme of those points (S, t) such that $\det S = t^2$ [7, Theorem 1.5]. We know that $\mu \circ \lambda = \det$, so it remains to make explicit $\rho_- \circ \lambda$; this should be something quadratic. We consider the closed embedding

$$\iota : \mathbf{GL}(\Lambda^2 Q) \longrightarrow \mathbf{GL}(\Lambda^- M) \tag{4.2.1}$$

constructed in the following way: we identify $\Lambda^2 Q$ with $(\Lambda^2 N) \oplus (N \otimes P^*)$, and tensoring P yields an isomorphism

$$(\Lambda^2 Q) \otimes P \xrightarrow{\sim} (L \otimes P) \oplus N \quad (\text{where } L := \Lambda^2 N); \tag{4.2.2}$$

its target, say R, occupies partial factors of $(\Lambda^3 M) \oplus M$, so that $\mathbf{GL}(R)$ is in natural way a closed subgroup of $\mathbf{GL}(\Lambda^- M)$; thus, $\mathbf{GL}$ acting on (4.2.2) takes the required form (4.2.1).

4.3. Description of $\rho_- \circ \lambda$. Note that only the invertibility of P has been used so far. Supposing now that N is of rank two, we shall prove the following result.

PROPOSITION. *If N is of rank two and P is invertible, then $\rho_- \circ \lambda$ factors through ι by the natural homomorphism $\Lambda^2 : \mathbf{GL}(Q) \to \mathbf{GL}(\Lambda^2 Q)$ of taking second exterior powers.*

This conclusion $\rho_- \circ \lambda = \iota \circ \Lambda^2$ is what we conjectured earlier. Again, in the proof, the harmless restriction to k-points of Ω applies; combined with Section 3.4, this reduces the problem to verifying that

$$\rho_-(Y_+(\lambda_+^1(u))) = \iota(\Lambda^2 X_+(u)), \tag{4.3.1}$$
$$\rho_-(Y_-(\lambda_-^1(v))) = \iota(\Lambda^2 X_-(v)), \tag{4.3.2}$$
$$\rho_-(Y_0(\det(g), g \bigoplus \eta)) = \iota(\Lambda^2 X_0(g, \eta)) \tag{4.3.3}$$

for all $u \in \mathfrak{V}^+$, all $v \in \mathfrak{V}^-$, all $g \in \mathrm{GL}(N)$, and all $\eta \in k^* \cong \mathrm{GL}(P)$. Since $\Lambda^2 Q$ decomposes to $L \oplus (N \otimes P^*)$ where L is invertible, elements of $\mathrm{End}(\Lambda^2 Q)$ are expressed as matrices with entries in

$$\begin{bmatrix} k & \mathrm{Hom}(N \otimes P^*, L) \\ \mathrm{Hom}(L, N \otimes P^*) & k \end{bmatrix} \tag{4.3.4}$$

acting from the left, while those of $\mathrm{End}(\Lambda^- M)$ are of type

$$\begin{pmatrix} \mathrm{End}(N) & \mathrm{Hom}(P, N) & \mathrm{Hom}(L \otimes P, N) \\ \mathrm{Hom}(N, P) & k & \mathrm{Hom}(L \otimes P, P) \\ \mathrm{Hom}(N, L \otimes P) & \mathrm{Hom}(P, L \otimes P) & k \end{pmatrix} \tag{4.3.5}$$

acting on $\Lambda^- M = N \oplus P \oplus (L \otimes P)$ from the left. We note that tensoring with Id_P gives rise to isomorphisms

$$\mathrm{Hom}(N \otimes P^*, L) \xrightarrow{\sim} \mathrm{Hom}(N, L \otimes P), \tag{4.3.6}$$
$$\mathrm{Hom}(L, N \otimes P^*) \xrightarrow{\sim} \mathrm{Hom}(L \otimes P, N), \tag{4.3.7}$$

as well as the obvious $\mathrm{End}(N \otimes P^*) \xrightarrow{\sim} \mathrm{End}(N)$; if these are treated as identifications, then ι becomes the simple insertion of the array (4.3.4) into the external four corners of (4.3.5). Hence, verification of the identity $\rho_-(\lambda(\gamma)) = \iota(\Lambda^2\gamma)$ consists at most of the following three steps: describe $\Lambda^2\gamma$ as a matrix of type (4.3.4); describe $\rho_-(\lambda(\gamma))$ as a matrix of type (4.3.5); and chase entries along the isomorphisms (4.3.6) and (4.3.7) as necessary. The first step is carried out using the identification $\Lambda^2 Q \subset \mathrm{Hom}(Q^*, Q)$, which describes the components $a \in L$ and $B \in N \otimes P^* \cong \mathrm{Hom}(P, N)$ of an arbitrary element $\mathfrak{p} \in \Lambda^2 Q$ as two by two alternating matrices of the form

$$\begin{bmatrix} a & B \\ -B^* & 0 \end{bmatrix},$$

similar to those in (2.5.1), but whose (2, 2)th entry is forced to be zero by the assumption that P is of rank one. The action of $\Lambda^2\gamma$ on these matrices is $\mathfrak{p} \mapsto \gamma\mathfrak{p}\gamma^*$ [5, Equation (1.4.6)]; on account of (3.1.3) and (3.1.4), we get

$$\Lambda^2 X_+(u) = \begin{bmatrix} 1 & (B \mapsto Bu^* - uB^*) \\ 0 & 1 \end{bmatrix}, \tag{4.3.8}$$

$$\Lambda^2 X_-(u) = \begin{bmatrix} 1 & 0 \\ (a \mapsto -av) & 1 \end{bmatrix},$$

$$\Lambda^2 X_0(g, \eta) = \begin{bmatrix} \det g & 0 \\ 0 & g \otimes \eta^{-1} \end{bmatrix}. \tag{4.3.9}$$

In fact, once the relations $t(\det T)^{-1} = \eta^{-1}$ and $\Lambda^- T = \operatorname{diag}(g, \eta, \eta \det g)$ have been noted when $(t, T) := (\det g, g \oplus \eta)$, it is also apparent from (3.3.1) that

$$\rho_-(Y_0(\det g, g \oplus \eta)) = \begin{pmatrix} \eta^{-1} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \det g \end{pmatrix}; \tag{4.3.10}$$

thus, dispensing with the third step, comparing (4.3.9) to (4.3.10) proves (4.3.3). Verification of (4.3.1), (4.3.2) is less apparent, and needs some auxiliary definitions and remarks.

4.4. The 'hat' isomorphisms. For the moment, let N and P be arbitrary. As we saw in (4.3.8), there is a bilinear composition coupling $B \in N \bigotimes P^* \cong \operatorname{Hom}(P, N)$ and $u \in \mathfrak{V}^+ = \operatorname{Hom}(P^*, N)$ to the following tensor, which is evidently an alternating map:

$$[B, u] := Bu^* - uB^* \in \Lambda^2 N \subset \operatorname{Hom}(N^*, N). \tag{4.4.1}$$

This amounts to $[B, n \otimes p] = B(p) \wedge n$, because

$$(Bu^* - uB^*)(f) = \langle n, f \rangle B(p) - \langle B(p), f \rangle n = -f \lrcorner (B(p) \wedge n)$$

for all $u = n \otimes p$ and all $f \in N^*$. Furthermore, there are linear maps

$$\begin{gathered} u \mapsto \hat{u} : \mathfrak{V}^+ = \operatorname{Hom}(P^*, N) \longrightarrow \operatorname{Hom}(N, \Lambda^2(N) \otimes P) \\ \hat{u}(x) := (x \wedge n) \otimes p \quad \text{when } u = n \otimes p, \end{gathered} \tag{4.4.2}$$

$$\begin{gathered} v \mapsto \hat{v} : \mathfrak{V}^- = \operatorname{Hom}(P, N^*) \longrightarrow \operatorname{Hom}(\Lambda^2(N) \otimes P, N) \\ \hat{v}(a \otimes p) := v(p) \lrcorner a, \end{gathered} \tag{4.4.3}$$

both identified naturally with tensorings $r \otimes \mathrm{Id}_P$ and $d \otimes \mathrm{Id}_{P^*}$, where r denotes the right wedge product map $N \to \operatorname{Hom}(N, \Lambda^2(N))$, namely $r(n)(x) := x \wedge n$, and d denotes the map $N^* \to \operatorname{Hom}(\Lambda^2(N), N)$ constructing left interior products. In particular, both (4.4.2) and (4.4.3) are *isomorphisms* if N is of *rank two*. If P is *invertible*, on the other hand, we claim that (4.4.1) is connected to (4.4.2) by the following relation (where $\omega \in P$ is arbitrary):

$$[B, u] \otimes \omega = \hat{u}(B(\omega)). \tag{4.4.4}$$

Indeed, if $u = n \otimes p$, then $[B, u] \otimes \omega = (B(p) \wedge n) \otimes \omega$ by the remark after (4.4.1), while $\hat{u}(B(\omega)) = (B(\omega) \wedge n) \otimes p$ by (4.4.2); thus, the assertion is a consequence of the fact that every bilinear map defined on an invertible module is symmetric. The relation (4.4.4) may be read as an identity $(B \mapsto Bu^* - uB^*) = \hat{u}$, for which the isomorphism (4.3.6) makes sense; furthermore, so does the isomorphism (4.3.7) when $(a \mapsto -av) = \hat{v}$, since (4.4.3) reads $-a(v(\omega)) = \hat{v}(a \otimes \omega)$.

4.5. Verification of (4.3.1) and (4.3.2). Returning to the proposition in Section 4.3, we may now complete the proof. By the remark made above, it suffices to prove that

$$\rho_-(Y_+(\lambda_+^1(u))) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hat{u} & 0 & 1 \end{pmatrix}, \quad \rho_-(Y_-(\lambda_-^1(v))) = \begin{pmatrix} 1 & 0 & \hat{v} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By linearity, we may again suppose that $u = n \otimes p$, so that $\lambda_+^1(u) = (n, 0) \wedge (0, p)$, which acts on $M = N \oplus P$ by the wedge product, annihilating P and sending $(x, 0)$ (where $x \in N$) to the element of $\Lambda^3 M \cong L \otimes P$ identified with $(x \wedge n) \otimes p = \hat{u}(x)$; see (4.4.2). Since $Y_+(\lambda_+^1(u)) = \mathrm{Id} + l_{(n,0)\wedge(0,p)}$, this proves the first formula. As to the second, v may be supposed to be $n^* \otimes p^* \in N^* \otimes P^*$, and then taking the interior product with $\lambda_-^1(v) = (n^*, 0) \wedge (0, p^*)$ converts an element $a \otimes \omega \in \Lambda^3 M \cong L \otimes P$ to $\langle \omega, p^* \rangle n^* \lrcorner a = v(\omega) \lrcorner a = \hat{v}(a \otimes \omega)$; see (4.4.3). Since $Y_-(\lambda_-^1(v)) = \mathrm{Id} + d_{(n^*,0)\wedge(0,p^*)}$, this completes our proof. □

4.6. The isomorphism κ. Since N is of rank two with $L = \Lambda^2 N$, there is an isomorphism

$$\begin{aligned} \theta : N^* \otimes L &\xrightarrow{\sim} N \\ \theta(f \otimes a) &:= -f \lrcorner a \end{aligned} \tag{4.6.1}$$

given by the interior product, in which the minus sign is employed so that the transposed inverse $\theta^{*-1} : N \otimes L^* \to N^*$ is the map $x \otimes b \mapsto x \lrcorner b$; the identification $L^* \cong \Lambda^2 N^*$ is as in Section 2.1. The isomorphism θ, together with obvious isomorphisms $P^* \otimes L \xrightarrow{\sim} \Lambda^3 Q$ and $(L \otimes P)^* \otimes L \xrightarrow{\sim} P^*$, gives rise to an isomorphism

$$(\Lambda^- M)^* \otimes L \xrightarrow{\sim} \Lambda^- Q,$$

with which we view the operation $T \mapsto T^{*-1}$ as an isomorphism $\mathbf{GL}(\Lambda^- Q) \xrightarrow{\sim} \mathbf{GL}(\Lambda^- M)$ of group schemes. On account of our faithful representations of rank-three special Clifford groups, it is then natural to construct the isomorphism

$$\begin{aligned} \kappa : \mathbf{S\Gamma}(\mathbf{H}(Q)) &\xrightarrow{\sim} \mathbf{S\Gamma}(\mathbf{H}(M)) \\ \kappa(T, t) &:= (tT^{*-1}, t). \end{aligned}$$

4.7. The last main result. We conclude by stating formally the relationship between λ and Λ.

PROPOSITION. $\lambda(\gamma) = \kappa(\Lambda\gamma)$ *for any point* γ *of* $\mathbf{GL}(Q)$.

PROOF. Since we know that $\lambda(\gamma)$ and $\Lambda\gamma$ have the same norm $\det \gamma$, it remains to check that the identity $\rho_-(\lambda(\gamma)) = (\det \gamma)(\Lambda^- \gamma)^{*-1}$ in $\mathbf{GL}(\Lambda^- M)$. As in the proof of the proposition in Section 4.3, we may restrict ourselves to three types of k-points, namely, when γ is $X_0(g, \eta)$, $X_+(u)$, and $X_-(v)$, for each of which the left-hand side $\rho_-(\lambda(\gamma))$ has been explicitly determined. For the case where $\gamma = X_0(g, \eta)$, transporting structures through θ makes g act on $N^* \otimes L$ as $g^{*-1} \otimes (\det g)$, whence the transposed inverse $(\Lambda^- \gamma)^{*-1}$ multiplied by $\det \gamma$, which is equal to $(\det g)\eta^{-1}$, takes the required form (4.3.10). The remaining two cases, in both of which $\det \gamma$ is 1, involve the formula $\theta^{*-1}(x \otimes b) = x \lrcorner b$ that we mentioned after (4.6.1). Indeed, using θ^{*-1} and the obvious identification $(L \otimes P) \otimes L^* \cong P$, we verify easily from the definitions that $-u^* = \hat{u} \otimes 1$ and $v = \hat{v} \otimes 1$, which shows that the operation $T \mapsto T^{*-1}$ converts $\Lambda^- X_+(u)$ and $\Lambda^- X_-(v)$ to the matrices displayed in Section 4.5. This completes our proof. □

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