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A NOTE ON AN ILL-POSED PROBLEM FOR THE HEAT EQUATION

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Abstract

In this paper an ill-posed problem for the heat equation is investigated. Solutions u to the equation $u_t - u_{xx} = 0$, which are approximately known on the positive half-axis $t = 0$ and on some vertical lines $x = x_1, \dots, x = x_n$, are considered and stability estimates of these solutions are presented. We assume an *a priori* bound, governing the heat flow across the boundary $x = 0$.

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1. Introduction

Suppose that the temperature $u(x, t)$ of an oil well is initially known for any depth x and then the temperature is monitored at some depths $x_1, \dots, x_n$, for all times. However, the monitoring device measures the temperature only approximately. A natural question arises to what inference can be made from such a set of data, can the temperature $u(x, t)$ at any time t for any depth x be determined at least approximately?

A simple (though coarse) approach to this problem might be to determine a solution $u(x, t)$ to the heat equation

$$u_t = u_{xx} \qquad (0 < x < \infty, 0 < t < \infty) \tag{1.1}$$

such that: (i) the initial value $u(x, 0)$ is given: (ii) the values $u(x_1, t), \dots, u(x_n, t)$ of u on some vertical half-axes are known; (iii) u does not increase too fast as

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$x \to +\infty$. Without loss of generality, the initial condition can be assumed to have the form

$$u(x,0) = 0 \qquad (0 < x < \infty). \tag{1.2}$$

An appropriate growth condition is

$$\lim_{x\to\infty} x^{-2} \ln\left(\int_0^T u^2(x,t)\,dt\right) = 0 \quad \text{for any } T > 0. \tag{1.3}$$

If $f_1,\ldots,f_n$ denote the monitored measurements at given knots $x_1,\ldots,x_n$, ε and p_i, $i = 1,\ldots,n$, are positive numbers with $p_1 + \cdots + p_n = 1$, then a convenient way of formulating condition (ii) is

$$\sum_{i=1}^{n} p_i \int_0^\infty |u(x_i,t) - f_i(t)|^2\,dt \leqslant \varepsilon^2. \tag{1.4}$$

Thus the ratio $\varepsilon/\sqrt{p_i}$ represents an upper bound to the error of measurement of u at x_i.

In this paper we deal with the problem specified by (1.1)–(1.4). We indicate that this problem is an ill-posed problem in the sense of Hadamard, and that it has some instability. Our results are theoretical and we give additional conditions which restore stability of solutions. Moreover we give explicit stability estimates and by using the least squares method we establish a formula for obtaining stable solutions.

A key remark is in order at this stage. The set (1.1), (1.2), (1.3), (1.4) is a statement of an ill-posed problem in the sense of Hadamard. More precisely, the evaluation of any solution u to (1.1)–(1.4) on the left of the vertical line $x = x_1$ (the first vertical axis carrying data) may suffer from an arbitrarily large error, whatever the error in the data is. This assertion is easily proved with the help of the following argument. Let u satisfy equation (1.1) and boundary conditions (1.2), (1.3), and let $0 < a < b$ be any two fixed points. A well-known uniqueness theorem (see for example [10]) tells us that the values $u(a,t)$ of u on the line $x = a$ determine the values of u on the line $x = b$ according to the rule

$$u(b,t) = \int_0^t u(a,s)K(t-s)\,ds$$

where

$$K(t) = \begin{cases} h(4\pi)^{-1}t^{-3/2}\exp(-h^2/4t) & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0 \end{cases}$$

and $h = b - a$. In other words, the forward operator mapping the restriction of u to the line $x = a$ into the restriction of u to the line $x = b$ is the convolution with a kernel K whose Fourier transform is given by

$$\hat{K}(\xi) = \exp\left(-h\sqrt{i\xi}\right) \tag{1.5}$$

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where $\sqrt{i\xi} = |\xi|^{1/2}\exp[(i\pi/4)\operatorname{sign}\xi]$. Formula (1.5) shows that the operator in question is bounded in $L^2(0, +\infty)$, actually it is a contraction and it is one-to-one. Furthermore, it is a strongly smoothing operator which maps $L^2(0, +\infty)$ into a subclass of $C^\infty(0, +\infty)$ and has *not* a bounded inverse. These facts are a consequence of the *fast decay* of the Fourier transform of the kernel K. Hence we arrive at the following conclusion: *the backward operator, mapping the restriction of u to the line $x = b$ into the restriction of u to a line located at the left of $x = b$, is not bounded in* $L^2(0, +\infty)$. The last property amounts exactly to the previously mentioned instability. A completely explicit example may be the following:

$$u_n = \left(x - a + \frac{1}{n}\right)t^{-3/2}\exp\left[-\frac{1}{4t}\left(x - a + \frac{1}{n}\right)^2\right]$$

are solutions of (1.1), (1.2), (1.3) such that

$$\int_0^\infty u_n^2(b, t)\,dt = 4\left(h + \frac{1}{n}\right)^{-2} \to 4/h^2,$$
$$\int_0^\infty u_n^2(a, t)\,dt = 4n^2 \to \infty.$$

As is well known, an ill-posed problem in the sense of Hadamard is a process in which effectual information on solutions is not available from the data. In other words instability, namely the impossibility of efficiently recovering a solution from conventional data, can be thought of as a consequence of a loss of information. Experience has shown that the stability of solutions can often be restored by complementing conventional data with *a priori* bounds on the solutions, the role of these *a priori* bounds being to replace the lost information. In the present paper we assume that the heat flow at the bottom of the oil well does not exceed a fixed quantity. More explicitly, we shall be concerned with solutions to problem (1.1), (1.2), (1.3), (1.4) which satisfy the *a priori* bound

$$\int_0^\infty u_x^2(0, t)\,dt \leqslant E^2 \tag{1.6}$$

where E is a given constant.

Our results are presented in the next sections. We must mention that the problem we are dealing with was discussed by Tihonov and Glasko [9], who especially stressed its numerical aspects. Stability estimates have been obtained, by methods quite different from ours and in a different functional setting, by Cannon [2], [3], [4], [5]. Similar problems have been discussed by Anderssen and Saull [1], Glasko, Zaharov and Kolp [6] and P. Manselli and K. Miller [7].

2. Stability estimates

The aim of stability estimates is to describe how much the development of solution from data magnifies errors, when noise affects the data. In other words, a stability estimate should tell how much any two solutions, which fit the data up to some error (and possibly satisfy reasonable *a priori* bounds), differ from each other. When the problem is linear in nature, stability estimates can be derived by estimating the size of solutions to the corresponding homogeneous problem. Thus we shall be concerned in this section with solutions u to the heat equation (1.1), which satisfy the homogeneous boundary conditions (1.2), (1.3) and the following inequality

$$\sum_{k=1}^{n} p_k \int_0^\infty u^2(x_k, t)\, dt \leqslant \varepsilon^2. \tag{2.1}$$

We retain condition (1.3), which plays the role of an *a priori* bound.

THEOREM. *Let u satisfy* (1.1), (1.2), (1.3), (1.6) *and* (2.1). *Then the following estimates hold*:

(i) $$\left[\int_0^\infty u^2(0, t)\, dt\right]^{1/2} \leqslant E\left(\frac{\sqrt{2}}{m} \ln \frac{E/\varepsilon}{(2/m)\ln(E/\varepsilon)}\right)^{-1}\left[1 + \mathrm{o}\left(\left(\ln \frac{E/\varepsilon}{\ln(E/\varepsilon)}\right)^{-1}\right)\right];$$

(ii) $$\left[\int_0^\infty u^2(x, t)\, dt\right]^{1/2} \leqslant \varepsilon^{x/m} E^{1-x/m}\left[\frac{2}{m} \ln(E/\varepsilon)\right]^{-1+x/m}[1 + \mathrm{o}(1)];$$

(iii) $$\sup\{|u(x, t): t \geqslant 0\} \leqslant \frac{e}{\sqrt{\pi}\, m} \varepsilon^{x/m} E^{1-x/m}\left(\frac{2}{m} \ln(E/\varepsilon)\right)^{-1+x/m}\left(\ln \frac{1}{\frac{\sqrt{2}}{m} \frac{\varepsilon}{E} \ln \frac{E}{\varepsilon}}\right)[1 + \mathrm{o}(1)].$$

Here m is the mean

$$m = p_1 x_1 + \cdots + p_n x_n \tag{2.2}$$

and the abscissa x, that is involved in (ii), (iii), *is to be kept fixed between* 0 *and* m. *The symbols* $\mathrm{o}(1)$ *stand for quantities which are infinitesimal as* $\varepsilon/E \to 0$ (*we do not claim that these infinitesimals are uniform with respect to* x).

REMARK 1. We have not specified the function class where our solutions u are sought. For our purposes, any function class (that is any meaning of the boundary conditions) is allowed which leads to the usual representations of solutions to the heat equation in terms of Poisson type integrals.

REMARK 2. As it will be clear from our proofs, the previous estimates could be given in a more precise (though less explicit) form. For instance, the following inequality is proved below:

$$\phi\left(\frac{1}{E^2}\int_0^\infty u^2(0,t)\,dt\right) \leqslant \left(\frac{\varepsilon}{E}\right)^2, \tag{2.3}$$

where

$$\phi(\lambda) = \lambda \exp\left(-m\sqrt{2/\lambda}\right), \tag{2.4a}$$

an increasing convex function on $0 \leqslant \lambda < \infty$ having a zero of infinite order at $\lambda = 0$. An elementary argument shows the following behaviour of the inverse function

$$\phi^{-1}(\mu) = \left(\frac{1}{2m}\ln\left(\frac{1}{-\mu\frac{1}{2m}\ln 1/\mu}\right)\right)^{-2} + o\left(\left(\ln\frac{1}{-\mu\ln\mu}\right)^{-3}\right), \tag{2.4b}$$

hence (2.3) is a more comprehensive version of (i).

REMARK 3. The above theorem shows that the difference between any two solution u_1 and u_2 to problem (1.1), (1.2), (1.3), (1.4), (1.6) satisfies

$$\phi\left(\frac{1}{4E^2}\int_0^{+\infty}|u_1(0,t) - u_2(0,t)|^2\,dt\right) \leqslant \left(\frac{\varepsilon}{E}\right)^2 \tag{2.5a}$$

where ϕ is given by (2.3). We want to stress here that the same proof allows us to replace such a ϕ with the following one:

$$\phi(\lambda) = \lambda\sum_{k=1}^{n} p_k \exp\left(-x_k\sqrt{2/\lambda}\right). \tag{2.5b}$$

The latter form of ϕ, though more involved, brings into evidence how error bounds may depend upon the (position and number of the) knots $x_1, \ldots, x_n$ as well as the precision p_k of the measurement at x_k. In particular, formulas (2.5a), (2.5b) may help in investigating those arrangements of knots which lead to the minimum error estimate.

PROOF. A well-known uniqueness theorem (see [11] for example) guarantees that any solution u to (1.1), (1.2), (1.3) can be represented by the formula

$$u(x,t) = \int_0^t u(0,s)K(x,t-s)\,ds \tag{2.6a}$$

where

$$K(x,t) = \begin{cases} -\dfrac{\partial}{\partial x}\left[(\pi t)^{-1/2}\exp(-x^2/4t)\right] & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0 \end{cases} \tag{2.6b}$$

is a derivative of the fundamental solution of the heat equation. A more comprehensive form of (2.6) is

$$\int_{-\infty}^{+\infty} \hat{\Phi}(\xi) \exp\left(it\xi - x\sqrt{i\xi}\right) \frac{d\xi}{2\pi} = \begin{cases} u(x, t) & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0 \end{cases} \tag{2.7}$$

where $\hat{}$ denotes the Fourier transform and

$$\Phi(t) = \begin{cases} u(0, t) & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0. \end{cases} \tag{2.8}$$

In the derivation of (2.7) the formula

$$K(x, \cdot)\hat{}(\xi) = \exp\left(-x\sqrt{i\xi}\right)$$

is involved, as well as Paley-Wiener theorem.

Formula (2.7) lumps together equation (1.1), initial condition (1.2) and growth condition (1.3). The limitation (1.6) goes into a constraint for Φ, since (2.4) and the Plancherel theorem yield

$$\int_{-\infty}^{+\infty} |\xi| \, |\hat{\Phi}(\xi)|^2 \frac{d\xi}{2\pi} = \int_0^\infty u_x^2(0, t)\, dt. \tag{2.9}$$

On the other hand, (2.1) implies

$$\int_{-\infty}^{+\infty} |\hat{\Phi}(\xi)|^2 \exp\left(-m\sqrt{2|\xi|}\right) \frac{d\xi}{2\pi} \leqslant \varepsilon^2 \tag{2.10}$$

where m is given by (2.2). Indeed (2.4) and the Plancherel theorem yield

$$\begin{aligned} \sum_{k=1}^{n} p_k \int_0^\infty u^2(x_k, t)\, dt &= \int_{-\infty}^{+\infty} |\hat{\Phi}(\xi)|^2 \sum_{k=1}^{n} p_k \exp\left(-x_k\sqrt{2|\xi|}\right) \frac{d\xi}{2\pi} \\ &\geqslant \int_{-\infty}^{+\infty} |\hat{\Phi}(\xi)|^2 \exp\left(-\sum_{k=1}^{n} p_k x_k \sqrt{2|\xi|}\right) \frac{d\xi}{2\pi} \end{aligned}$$

because of the arithmetic-geometric inequality.

Now we are in a position to prove assertions (i) and (ii). Fix x such that $0 \leqslant x < m$ and define a function ϕ with the following rule:

$$\phi(\lambda) = \frac{1}{\xi} \exp\left(-m\sqrt{2\xi}\right), \qquad \lambda = \frac{1}{\xi} \exp\left(-x\sqrt{2\xi}\right), \qquad \xi > 0. \tag{2.11}$$

The properties of ϕ that we need are listed in the lemma below. In particular we use in a crucial way the *convexity* of ϕ. Note that if $x = 0$ (2.11) simply becomes (2.4a).

Denote the $L^2(0,\infty)$ norm by $\| \ \|$. We have the following chain of inequalities:

$$\phi\left(\frac{\|u(x,\cdot)\|^2}{\|u_x(0,\cdot)\|^2}\right)=\phi\left(\frac{\int_{-\infty}^{+\infty}|\hat{\Phi}(\xi)|^2\exp\left(-x\sqrt{2|\xi|}\right)d\xi/2\pi}{\int_{-\infty}^{+\infty}|\xi||\hat{\Phi}(\xi)|^2\,d\xi/2\pi}\right)$$

(because of formulas (2.7) and (2.9))

$$\leqslant\frac{\int_{-\infty}^{+\infty}\phi\left(1/|\xi|\exp\left(-x\sqrt{2|\xi|}\right)\right)|\xi||\hat{\Phi}(\xi)|^2\,d\xi/2\pi}{\int_{-\infty}^{+\infty}|\xi||\hat{\Phi}(\xi)|^2\,d\xi/2\pi}$$

(2.12) (Jensen inequality for convex functions)

$$=\frac{\int_{-\infty}^{+\infty}|\hat{\Phi}(\xi)|^2\exp\left(-m\sqrt{2|\xi|}\right)d\xi/2\pi}{\int_{-\infty}^{+\infty}|\xi||\hat{\Phi}(\xi)|^2\,d\xi/2\pi}\quad\text{(see equation (2.11))}$$

$$\leqslant\frac{\varepsilon^2}{\|u_x(0,\cdot)\|^2}\quad\text{(see inequality (2.10) and formula (2.9)).}$$

As $\phi(\lambda)/\lambda$ increases with λ, (2.12) and (1.6) give

$$\phi\left(\frac{1}{E^2}\|u(x,\cdot)\|^2\right)\leqslant\frac{\varepsilon^2}{E^2}. \tag{2.13}$$

In view of the expansion of ϕ^{-1} (see lemma below) the estimates (i), (ii) have been established.

A proof of (iii) runs as follows. From (2.7) we get

$$|u(x,t)|\leqslant\int_{-\infty}^{+\infty}|\hat{\Phi}(\xi)|\exp\left(-x\sqrt{|\xi|/2}\right)\frac{d\xi}{2\pi},$$

hence the Cauchy-Schwarz inequality gives

$$|u(x,t)|\leqslant\left[\int_{-\infty}^{+\infty}\exp\left(-(1-\lambda)x\sqrt{2|\xi|}\right)\frac{d\xi}{2\pi}\right]^{1/2}$$
$$\times\left[\int_{-\infty}^{+\infty}|\hat{\Phi}(\xi)|^2\exp\left(-\lambda x\sqrt{2|\xi|}\right)\frac{d\xi}{2\pi}\right]^{1/2}$$
$$=\frac{1}{\sqrt{\pi}\,(1-\lambda)x}\left[\int_0^{\infty}u^2(\lambda x,t)\,dt\right]^{1/2}$$

for any λ between 0 and 1. Combining this estimate with (ii) gives

$$|u(x,t)|\leqslant\frac{mE}{\sqrt{2\pi}\,x\ln(E/\varepsilon)}\,\frac{1}{1-\lambda}\left[\frac{\sqrt{2}}{m}\,\frac{\varepsilon}{E}\ln\frac{E}{\varepsilon}\right]^{x\lambda/m}[1+o(1)],$$

hence (iii) follows when λ is so chosen that the dominating part of the last estimate is as small as possible.

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LEMMA. *Let $0 < a < b$ be two fixed numbers. Define a function ϕ by the formula*

$$\phi(\lambda) = r^{-2}e^{-br}, \qquad \lambda = r^{-2}e^{-ar}, r > 0. \tag{2.15}$$

The following properties hold:

(i) $\phi(\lambda)/\lambda$, *hence* $\phi(\lambda)$, *increases as* λ *increases*;

(ii) $\phi(\lambda)$ *is convex*;

(iii) $\phi(\lambda) = \lambda^{b/a}((-\ln\lambda)/a)^{2(-1+b/a)}(1 + o(1))$ *as* $\lambda \to 0$;

(iv) $\phi^{-1}(\mu) = \mu^{a/b}((-\ln\mu)/b)^{2(-1+a/b)}(1 + o(1))$ *as* $\mu \to 0$.

PROOF. (i) $(d/d\lambda)(\phi(\lambda)/\lambda) = (d/d\lambda)e^{(a-b)r} = (a-b)e^{(a-b)r} \cdot 1/(d\lambda/dr) = (b-a)r^3/(ar+2) > 0$.

(ii) $(d^2\phi/d\lambda^2) = (b-a)(2 + 2(b+a)r + abr^2)r^3e^{ar}/(2+ar)^3 > 0$.

(iii) We start from an expansion of r in terms of λ. Here r and λ are connected by (2.15b), namely

$$r^2e^{ar} = 1/\lambda. \tag{2.16}$$

Defining q by the formula

$$r = \frac{1}{a}\left(\ln\frac{1}{\lambda q}\right) \tag{2.17}$$

and inserting (2.17) into (2.16) gives the following equations for q:

$$a\sqrt{q} + \ln q = \ln(1/\lambda) \tag{2.18}$$

From (2.18) one easily sees (by taking derivatives, for example) that q is a decreasing function of λ. Thus q must have a limit as $\lambda \to 0$, and (2.18) tells us that this limit is $+\infty$. Consequently

$$q((\ln\lambda)/a)^{-2} \to 1 \quad \text{as } \lambda \to 0,$$

for (2.18) yields

$$q((\ln\lambda)/a)^{-2} = \left(1 + \frac{1}{a}\frac{\ln q}{\sqrt{q}}\right)^{-2}.$$

We have thus proved $\ln q = \ln(a^{-1}\ln(1/\lambda))^2 + o(1)$, hence

$$r = \frac{1}{a}\ln\frac{1}{\lambda(1/a\ln 1/\lambda)^2} + o(1) \tag{2.19}$$

because of (2.17). Combining (2.19) with (2.15a) gives easily the wanted property (iii).

Property (iv) follows from (iii), by interchanging a with b. Our lemma is fully proved.

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3. Least squares method

A constructive existence theorem for problem (1.1), (1.2), (1.3), (1.4), (1.6) should involve both (i) showing that (under suitable compatibility conditions on the data $f_1(t),\dots,f_n(t)$, ε, E) the set of all solutions u to the heat equation (1.1), which satisfy conditions (1.2), (1.3) together with (1.4) and (1.6), is not empty; and (ii) exhibiting a representative from such a set of solutions. The least squares method (see [7] for example) provides us with a strategy for discussing these questions. In our case, the method may consist of looking for the solution u to the heat equation (1.1), which satisfies conditions (1.1), (1.2) and minimizes the quadratic functional

$$\varepsilon^{-2}\sum_{k=1}^{n} p_k\int_0^\infty |u(x_k,t)-f_k(t)|^2\,dt + E^{-2}\int_0^\infty u_x^2(0,t)\,dt. \tag{3.1}$$

Presently, we shall prove the following two facts:

(i) the minimum (under the specified constraints) of the functional (3.1) is

$$\int_{-\infty}^{+\infty}\left\{\sum_{k=1}^{n} p_k|\hat f_k(\xi)|^2-\frac{|\Sigma_{k=1}^n p_k\exp\left(-x_k\sqrt{i\xi}\right)\overline{\hat f_k(\xi)}\,|^2}{\Sigma_{k=1}^n p_k\exp\left(-x_k\sqrt{2|\xi|}\right)+(\varepsilon/E)^2}\right\}\frac{d\xi}{2\pi\varepsilon^2}; \tag{3.2}$$

(ii) the minimizer u of (3.1) is given by

$$u(x,t)=\int_{-\infty}^{+\infty}\hat\Phi(\xi)\exp\left(it\xi-x\sqrt{i\xi}\right)\frac{d\xi}{2\pi}, \tag{3.3a}$$

$$\hat\Phi(\xi)=\frac{\Sigma_{k=1}^n p_k\exp\left(-x_k\sqrt{i\xi}\right)\hat f_k(\xi)}{\Sigma_{k=1}^n p_k\exp\left(-x_k\sqrt{2|\xi|}\right)-(\varepsilon/E)^2|\xi|}. \tag{3.3b}$$

Here $\hat f_k$ denotes the Fourier transform of f_k, it is understood that all f_k are continued by zero on the negative axis. We proceed as follows:

(i) The level set

$$\begin{aligned}\{&u \text{ solution to } (1.1)\colon u \text{ satisfies } (1.2) \text{ and } (1.3),\\ &\qquad\qquad \text{the functional } (3.1) \text{ at } u \text{ is less than } 1\}\end{aligned} \tag{3.4}$$

is not empty if and only if the following condition (where only data of our problem are involved) holds

$$\text{the integral (3.2) is less than 1.} \tag{3.5}$$

Obviously, (3.4) is a subset of the collection of all solutions to our problem (1.1), (1.2), (1.3), (1.4), (1.6). Hence (3.5) ensures that our problem has solutions.

(ii) Suppose that condition (3.5) holds. Then the function u defined by (3.3), is a solution to problem (1.1), (1.2), (1.3), (1.4), (1.6): actually (3.3) is a distinguished member of the level set (3.4), namely (3.3) is just the *Chebyshev centre* of (3.4) (see [9] for comments). Let us recall that the Chebyshev centre of a convex subset K of a Hilbert space is the point u from K, which minimizes the worst deviation between u and any other point from K; in other words, u is the solution of the minimax problem

$$\sup\{\|u - v\| : v \in K\} = \text{minimum}.$$

The above results are the main concern of this section. As far as proofs are concerned, it is enough to observe that the functional (3.1) takes the form

$$\int_{-\infty}^{+\infty}\left(\sum_{k=1}^{n} p_k \exp\left(-x_k\sqrt{2|\xi|}\right) + \lambda|\xi|\right)$$
$$\times\left|\hat{\Phi}(\xi) - \frac{\sum_{k=1}^{n} p_k \exp\left(-x_k\sqrt{i\xi}\right)\overline{\hat{f}_k(\xi)}}{\sum_{k=1}^{n} p_k \exp\left(-x_k\sqrt{2|\xi|}\right) + \lambda|\xi|}\right|^2 \frac{d\xi}{2\pi\varepsilon^2}$$
$$+\int_{-\infty}^{+\infty}\left\{\sum_{k=1}^{n} p_k|\hat{f}_k(\xi)|^2 - \frac{|\sum_{k=1}^{n} p_k \exp\left(-x_k\sqrt{i\xi}\right)\overline{\hat{f}_k(\xi)}|^2}{\sum_{k=1}^{n} p_k \exp\left(-x_k\sqrt{2|\xi|}\right) + (\varepsilon/E)^2}\right\}\frac{d\xi}{2\pi\varepsilon^2}$$

if one uses (3.3a) to represent u.

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