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ARBITRAGE-FREE OPTION PRICING MODELS

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Abstract

We describe a scheme for constructing explicitly solvable arbitrage-free models for stock price. This is used to study a model similar to one introduced by Cox and Ross, where the volatility of the stock is proportional to the square root of the stock price. We derive a formula for the value of a European call option based on this model and give a procedure for estimating parameters and for testing the validity of the model.

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1. Introduction

The famous Black–Scholes model [1] for option pricing is based on the assumption that the value S of a stock follows an Itô equation of the form

$$dS = \mu S\,dt + \sigma S\,dw. \tag{1}$$

Here w is a standard Wiener process and μ and σ are two parameters representing, respectively, the drift and the volatility of the stock. This leads to the well-known Black–Scholes formula for determining the value V of a European call option, that is, the right to purchase the stock at a price k at a future time T,

$$\begin{aligned} V = {} & S_0 \Phi\left(\frac{\log(S_0/k) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right) \\ & - ke^{-rT}\,\Phi\left(\frac{\log(S_0/k) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right). \end{aligned} \tag{2}$$

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In this formula, S_0 denotes the present value of the stock, Φ is the cumulative distribution function of the standard Gaussian distribution and r is the risk-free interest rate.

An essential feature of option pricing is the calculation of expectation with respect to an underlying probability measure with respect to which the model (1) is *arbitrage free*. This is the condition that the process $e^{-rt}S_t$ is a *martingale*. In particular, the expected value of S_t at any time t is precisely the future value at time t of a risk-free bond with present value S_0, that is,

$$E[S_t] = S_0 e^{rt}.$$

The arbitrage-free condition effectively forces the replacement of μ in equation (1) by r in the subsequent computations of expectations.

Alternatives to the Black–Scholes model (1) have been proposed. In particular Cox and Ross [2] introduced the model

$$dS = \mu S\,dt + \sigma\sqrt{S}\,dw. \tag{3}$$

They obtained this equation as the diffusion limit of a sequence of jump processes with intensity tending to infinity. In contrast to (1), equation (3) is not solvable in closed form by elementary means. Thus there is no direct analogue of the Black–Scholes formula (2) for the Cox–Ross model.

In Section 2 we describe a method for constructing a class of *solvable* arbitrage-free models for stock price. Our starting point is the following stochastic Bernoulli equation of *Stratonovich* type

$$d\tilde{S} = \mu\tilde{S}\,dt + \sigma\tilde{S}^p \circ dw, \tag{4}$$

where $1/2 \le p \le 1$. In view of the fact that Stratonovich differentials transform in the same way as classical differentials, equation (4) can be solved explicitly by elementary methods (see the Zvonkin, Doss–Sussmann method, Karatzas and Shreve [5, Proposition 2.21], Rogers and Williams [7, Theorem 28.2]). This is done in Theorem 1.

The process $\tilde{S}$ will not generally satisfy the arbitrage-free condition. In Theorem 2, we construct a function G such that the process $S_t \equiv G(\tilde{S}_t, t)$ *does* satisfy this condition. This yields a second-order partial differential equation for G that is similar to the classical Black–Scholes equation.

In Section 3 we consider the extreme values $p = 1$ and $p = 1/2$, which are particularly tractable to this analysis. In the linear case, where $p = 1$, our approach is shown to yield the Black–Scholes model. We then focus on the case where $p = 1/2$ studied by Cox and Ross. In this case the partial differential equation for G turns out to have an especially simple form and we can solve it explicitly. This results in a formula (16) for the value of a European call option analogous to the Black–Scholes formula (2).

Finally, in Section 4, we give a method for estimating the volatility parameter σ from a set of data and for testing the validity of the model.

It should be pointed out that the idea of using Stratonovich calculus in mathematical finance is not new. A main objective of this paper is to use this methodology to provide a unified treatment of the Black–Scholes and Cox–Ross models that is self-contained and, at the same time, elementary enough to be comprehensible to a wide variety of readers.

2. Arbitrage-free models

Throughout this section, let p denote a rational number m/n in the interval $[1/2, 1]$, with m odd and n even. We introduce the following stochastic differential equation as a tentative model for stock price:

$$d\tilde{S} = r\tilde{S}\,dt + \sigma\tilde{S}^p \circ dw, \tag{5}$$

where $\circ\, dw$ denotes the Stratonovich differential. The following relationship between Itô and Stratonovich differentials holds (see Klebaner [6, Theorem 5.20]): for a random process ξ with a stochastic differential $d\xi = a\,dw + b\,dt$, where a and b are continuous adapted processes,

$$\begin{aligned} \xi \circ dw &= \xi\,dw + \tfrac{1}{2}d[\xi, w] \\ &= \xi\,dw + \tfrac{1}{2}a\,dt \end{aligned} \tag{6}$$

where the bracket $[\xi, w]$ denotes the quadratic covariation of the semimartingales ξ and w. It follows from this and Itô's formula (see, for instance [6, Theorem 4.16]) that Stratonovich differentials transform under composition with smooth maps in the same way as classical differentials, that is, by the standard chain rule. Thus equation (5) may formally be regarded as an ordinary differential equation in $\tilde{S}$. As such, it is solvable by elementary differential equation methods, that is, variation of parameters and separation of variables. This gives the following result.

THEOREM 1. *Define*

$$\tilde{S}_t = e^{rt}\left\{(1-p)\sigma\int_0^t e^{r(p-1)u}\,dw_u + \tilde{S}_0^{1-p}\right\}^{1/(1-p)}, \tag{7}$$

for all $t \geq 0$. *Then* $\tilde{S}_t$ *is the solution to equation (5).*

PROOF. Note that, although the quantity inside the rational power $1/(1-p)$ in (7) may be negative, the assumption on p ensures that $\tilde{S}_t$ is *real* and *nonnegative*.

We verify that (7) is the solution to equation (5) as follows. Write

$$E_t \equiv \left\{(1-p)\sigma\int_0^t e^{r(p-1)u}\,dw_u + \tilde{S}_0^{1-p}\right\}^{1/(1-p)}$$

and note that the function $f(x) = x^{1/(1-p)}$ is C^2 on $\mathbb{R}$. Applying Itô's formula to compute the stochastic differential of the process $\tilde{S}_t = e^{rt}E_t$ and using (6) gives

$$
\begin{aligned}
d\tilde{S} &= re^{rt}E_t\,dt + \sigma e^{rpt}E_t^p\,dw + \frac{p\sigma^2}{2}e^{rt(2p-1)}E_t^{2p-1}\,dt \\
&= \left(r\tilde{S} + \frac{p\sigma^2}{2}\tilde{S}^{2p-1}\right)dt + \sigma\tilde{S}^p\,dw \\
&= r\tilde{S}\,dt + \sigma\tilde{S}^p\,dw + \frac{1}{2}d[\sigma\tilde{S}^p, w] \\
&= r\tilde{S}\,dt + \sigma\tilde{S}^p \circ dw
\end{aligned}
$$

as required. □

As remarked in Section 1, the process $\tilde{S}_t$ will not generally satisfy the arbitrage-free condition and hence is not a feasible model for stock price. We therefore seek a function G such that $S_t \equiv G(\tilde{S}_t, t)$ is arbitrage-free.

THEOREM 2. *Suppose that $G : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is C^2 in s, C^1 in t, and satisfies the partial differential equation*

$$G_t + \left(rs + \frac{p\sigma^2 s^{2p-1}}{2}\right)G_s + \frac{\sigma^2 s^{2p}}{2}G_{ss} = rG. \tag{8}$$

Suppose further that there exists n such that for every $T > 0$

$$\sup_{0\le t\le T} |G_s(s,t)| \le C|s|^n, \quad \forall s \in \mathbb{R} \tag{9}$$

where C is a constant depending only on T.

Let $S_t = G(\tilde{S}_t, t)$. Then the process $e^{-rt}S_t$ is a martingale.

PROOF. We have

$$
\begin{aligned}
d(e^{-rt}S_t) &= e^{-rt}((G_t(\tilde{S}_t,t) - rG(\tilde{S}_t,t))\,dt + G_s(\tilde{S}_t,t)\,d\tilde{S}_t) \\
&= e^{-rt}(G_t(\tilde{S}_t,t) - rG(\tilde{S}_t,t))\,dt \\
&\quad + e^{-rt}G_s(\tilde{S}_t,t)(r\tilde{S}\,dt + \sigma\tilde{S}^p \circ dw).
\end{aligned} \tag{10}
$$

Applying (6) to convert equation (10) to Itô form then using (8), we obtain

$$
\begin{aligned}
d(e^{-rt}S_t) &= e^{-rt}\Bigg[G_t(\tilde{S}_t,t) + \left(r\tilde{S}_t + \frac{p\sigma^2}{2}\tilde{S}_t^{2p-1}\right)G_s(\tilde{S}_t,t) \\
&\qquad + \frac{\sigma^2\tilde{S}_t^{2p}}{2}G_{ss}(\tilde{S}_t,t) - rG(\tilde{S}_t,t)\Bigg]dt + e^{-rt}G_s(\tilde{S}_t,t)\sigma\tilde{S}_t^p\,dw \\
&= e^{-rt}G_s(\tilde{S}_t,t)\sigma\tilde{S}_t^p\,dw.
\end{aligned}
$$

Thus

$$e^{-rt}S_t = S_0 + \sigma\int_0^t e^{-ru}G_s(\tilde{S}_u,u)\tilde{S}_u^p\,dw. \tag{11}$$

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Now the indefinite Itô integral $\int_0^s f\,dw$, where $s \le t$, is a martingale (see [6, p. 100]) provided that f satisfies

$$\int_0^t E[f(u)^2]\,du < \infty. \tag{12}$$

Using the estimate (see Gikhman and Skorohod [4])

$$E\left[\sup_{0\le s\le T}\left|\int_0^s f(u)\,dw\right|^{2m}\right] \le C_m \int_0^T E[|f(u)|^{2m}]\,du$$

for some constant C_m, and condition (9), it is easy to check that the integrand in (11) satisfies (12). Thus the result holds. □

3. Examples

There are two values of p, namely 1 and $1/2$, where equation (8) is solvable in closed form. Firstly, in the case where $p = 1$, (8) reduces to

$$G_t + \left(r + \frac{\sigma^2}{2}\right) s G_s + \frac{\sigma^2 s^2}{2} G_{ss} = rG.$$

It is easy to check that

$$G(s, t) = s e^{-\sigma^2 t/2}$$

is a solution to this equation, and this function G clearly satisfies condition (9). Solving equation (5) in the case where $p = 1$ yields

$$\tilde{S}_t = S_0 \exp(rt + \sigma w_t).$$

Thus

$$S_t = G(\tilde{S}_t, t) = S_0 \exp((r - \sigma^2/2)t + \sigma w_t).$$

We also note that when $p = 1$, the Itô equation for S is

$$dS = rS\,dt + \sigma S\,dw,$$

and the equation that gives rise to this, prior to the imposing of the arbitrage-free condition, is precisely (1). Thus both our model and its solution reduce to the classical Black–Scholes theory in the case where $p = 1$.

We now turn to the case where $p = 1/2$ studied by Cox and Ross (this will be referred to in the sequel as the *fractional* model). In this case, equation (8) becomes

$$G_t + \left(rs + \frac{\sigma^2}{4}\right) G_s + \frac{\sigma^2 s}{2} G_{ss} = rG$$

which has the simple solution

$$G(s, t) = s + \frac{\sigma^2}{4r}. \tag{13}$$

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Again, G satisfies (9).

Assume that $S_0 > \sigma^2/4r$. Combining (13) and (7) with $p = 1/2$ gives

$$S_t = G(\tilde{S}_t, t) = e^{rt}\left\{\frac{\sigma}{2}\int_0^t e^{-ru/2}\,dw_u + \sqrt{S_0 - \frac{\sigma^2}{4r}}\right\}^2 + \frac{\sigma^2}{4r}. \tag{14}$$

The Itô equation for S is

$$dS = rS\,dt + \sigma\sqrt{S - \frac{\sigma^2}{4r}}\,dw.$$

A natural candidate for the original model of stock price that gives rise to this formula after the imposing of the arbitrage-free condition, is

$$dS = \mu S\,dt + \sigma\sqrt{S - \frac{\sigma^2}{4r}}\,dw. \tag{15}$$

Formula (14) yields a value V for a call option with exercise price k at future time T. The Itô integral in (14) is a Gaussian random variable with zero mean and variance $(1 - e^{-rt})/r$. An elementary calculation shows that

$$V = e^{-rT} E[\max(S_T - k, 0)]$$

is given by

$$\begin{aligned} V = {} & \frac{v}{2\pi}(e^{-d_1^2/2}(vd_1 + 2a) - e^{-d_2^2/2}(vd_2 + 2a)) \\ & + ((v^2 + a^2) + e^{-rT}(b - k))(1 - \Phi(d_1) + \Phi(d_2)) \end{aligned} \tag{16}$$

where Φ is the cumulative distribution function of the standard normal distribution,

$$a = \sqrt{S_0 - \frac{\sigma^2}{4r}}, \quad b = \frac{\sigma^2}{4r}, \quad v = \frac{\sigma}{2}\sqrt{\frac{1 - e^{-rT}}{r}},$$

$$d_1 = \frac{e^{-rT/2}\sqrt{k - b} - a}{v}, \quad d_2 = \frac{-e^{-rT/2}\sqrt{k - b} - a}{v}.$$

4. Parameter estimation and model testing

In order to use formula (16), a mechanism is required to estimate the volatility σ for a given stock from a set of data. We propose the following scheme, based on a list of opening prices S_t for the stock on successive days $t = 1, \ldots, N$. Since the data set is discrete, it is natural to approximate equation (15) by the *difference* equation

$$\Delta_t S = \mu S_t + \sigma\sqrt{S - \frac{\sigma^2}{4r}}\Delta_t w,$$

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where $\Delta_t S$ and $\Delta_t w$ denote $S_{t+1} - S_t$ and $w_{t+1} - w_t$, respectively. Solving for $\Delta_t w$, we see that, if the model is valid, then the quantities

$$Z_t \equiv \frac{\Delta_t S - \mu S_t}{\sigma\sqrt{S_t - \sigma^2/4r}} \tag{17}$$

form an (approximately) independent set of standard Gaussian random variables. Equating the mean to 0 and the variance to 1 of the sample values Z_t, $t = 1, \ldots, N$ gives the following equations in μ and σ

$$\sum_{t=1}^{N} Z_t = 0 \quad \text{and} \quad \sum_{t=1}^{N} Z_t^2 = N - 1.$$

These equations can be solved numerically to yield estimates $\hat{\mu}$ and $\hat{\sigma}$ of the parameters μ and σ and the estimated value $\hat{\sigma}$ can then be used in place of σ in (16) to compute the option value V. Furthermore, the validity of the fractional model can be tested by applying a standard normality test, such as the Shapiro–Wilk test, to the quantities Z_t in (17), again using the estimated values for μ and σ in place of the actual values.

In conclusion, we note that Delbaen and Shirakawa [3] have recently used a Bessel process to study the law of the process $\hat{S}_t$ defined by the *Itô* equation

$$d\hat{S} = rd\hat{S}\,dt + \sigma\hat{S}^p\,dw.$$

Here p is assumed to lie in the range $(0, 1)$. The analysis in [3] results in an expression for the law of the random variable $\hat{S}_t$ (where $t > 0$) as an infinite series. This result is then used to obtain a formula [3, Equation 3.21] for the price of a European call option based on the stock price $\hat{S}_t$, as the difference of two infinite series. The relationship between the option pricing formula in [3] and its counterpart (equation (16)) in the present work is unclear at this time. This issue will be studied in a later paper.

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