Canad. J. Math. Vol. **65** (3), 2013 pp. 655–674
http://dx.doi.org/10.4153/CJM-2012-026-7

Proof of the Completeness of Darboux Wronskian Formulae for Order Two

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Abstract. Darboux Wronskian formulas allow us to construct Darboux transformations, but Laplace transformations, which are Darboux transformations of order one, cannot be represented this way. It has been a long-standing problem to discover what other exceptions exist. In our previous work we proved that among transformations of total order one there are no other exceptions. Here we prove that for transformations of total order two there are no exceptions at all. We also obtain a simple explicit invariant description of all possible Darboux transformations of total order two.

1 Introduction

Classical Darboux transformations and their generalizations are methods for obtaining analytic solutions of linear Partial Differential Equations (PDEs). They also serve as leverage for larger theories for solution of non-linear PDEs; see, for example, [7] and references therein.

In this paper we are concerned with the intertwining relations $\mathcal{N} \circ \mathcal{L} = \mathcal{L}_1 \circ \mathcal{M}$ for operators of the form

$$\mathcal{L} = D_x D_y + a D_x + b D_y + c, \tag{1.1}$$

where the coefficients may be non constant. Since they were introduced in the classical work of [2], we shall call them Darboux transformations as well. PDEs corresponding to such operators appear also as part of the problem of the search of flat metrics; see [5].

Given the Linear Partial Differential Operator (LPDO) $\mathcal{L}$ and some LPDO $\mathcal{M}$, the coefficients of the resulting operator $\mathcal{L}_1$ and of the auxiliary operator $\mathcal{N}$ can be found algebraically. There are two choices of $\mathcal{M}$ that always lead to a DT for a given operator (1.1): $\mathcal{M} = D_x + b$, and $\mathcal{M} = D_y + a$. These Darboux transformations have a special name: Laplace transformations. The latter are not be confused with Laplace integral transforms.

There is also a large class of Darboux transformations generated by operators $\mathcal{M}$ that are constructed using so-called Darboux Wronskian formulas. These are based on the assumption that we know some number of linearly independent particular solutions of the initial PDE, $\mathcal{L}\psi = 0$.

Received by the editors January 11, 2012; revised June 4, 2012.
Published electronically July 27, 2012.
AMS subject classification: **53Z05**, 35Q99.
Keywords: completeness of Darboux Wronskian formulas, completeness of Darboux determinants, Darboux transformations, invariants for solution of PDEs.

This class is very large, and Darboux transformations of arbitrary orders can be constructed provided we know enough particular solutions. Laplace transformations, which are Darboux transformations of order one do not belong to this class.

Laplace transformations (see [12]) are particularly good Darboux transformations, and they have been the only known examples of Darboux transformations that cannot be described by Darboux Wronskian formulas. In [10] we proved that a Darboux transformation of total order one is either described by Darboux Wronskians or is a Laplace transformation. The problem reduces to solution of a non-linear PDE. The PDE was not so large, and, noticing some interesting structure, we were able to tackle the problem.

After that work it was still unclear whether there were some exceptional transformations, that is, that cannot be described by Darboux Wronskian formulas among Darboux transformations of orders higher than one. This problem reduces to solution of a system of two large non-linear PDEs, for which methods of the previous work [10] were hard to apply. However, we succeeded in proving that Darboux Wronskian formulas complete for transformations of order two in a different and rather elegant fashion, and that we present proof in this paper.

Recently several new ideas have been used to tackle Darboux transformations and related problems. Thus, [11] and [1] have made very important progress in the description of factorizable operators corresponding to linear PDEs in terms of certain abelian categories and algebraic groups, respectively. In this paper, we adopt an approach that is based on the ideas of differential geometry and is constructive.

Our main result is an elegant proof that all Darboux transformations of total order two can be described by Wronskian formulae (Theorem 7.5). The second achievement is an easy to use invariant description of all these Darboux transformations (Theorem 7.4).

The paper is organized as follows. Darboux transformations of total order two are those that have $\mathcal{M}$ in one of the following forms:

$$\mathcal{M} = m_{20}D_{xx} + m_{10}D_x + m_{00},$$

$$\mathcal{M} = m_{02}D_{yy} + m_{01}D_y + m_{00},$$

$$\mathcal{M} = m_{10}D_x + m_{01}D_y + m_{00},$$

where the $m_{ij} \in K$ are not necessarily constant. In Section 3 we show that to cover all Darboux transformations of total order two it is enough to consider $\mathcal{M}$ of the form D_x+qD_y+r, where p and q are some functions. After some preparation in the next two sections, we introduce in Section 6 new transformations of the pair $\{\mathcal{L}, \mathcal{M}\}$, which we call *gauged evolution*. We determine the generating invariants uniquely defining the equivalence classes under these transformations and use them to invariantize the nonlinear system of PDEs defining all possible Darboux transformations of total order two. The invariantized system is easier and can be solved explicitly by classical methods; however, even though we have a technical solution, it is in quadratures and it is useful neither for invariant description of Darboux transformations, nor to judge whether or not Wronskian formulae give all such Darboux transformations.

Therefore, we need a further invention, Theorem 7.3, through which we are able to obtain an elegant general solution (Theorem 7.4) of the invariantized system of PDE.

We still have to remember that even if the invariantized system of PDEs has solutions, the existence of Darboux transformations depends also on the existence of a solution of a nonlinear PDE system (6.3), where we return from gauged evolution invariants to the coefficients of operators $\mathcal{L}$ and $\mathcal{M}$. In the proof of Theorem 7.5 we resolve this problem and conclude that for every Darboux transformation of total order 2 there exist two linearly independent partial solutions of $\mathcal{L}u = 0$ such that it can be constructed using Darboux Wronskian formulas.

2 Preliminaries

Let K be a differential field of characteristic zero with commuting derivations ∂_x, ∂_y. Let $K[D] = K[D_x, D_y]$ be the corresponding ring of linear partial differential operators over K, where D_x, D_y correspond to derivations ∂_x, ∂_y.

Operators $\mathcal{L} \in K[D]$ have the general form $\mathcal{L} = \sum_{i+j=0}^{d} a_{ij} D_x^i D_y^j$, where $a_{ij} \in K$. The formal polynomial $\mathrm{Sym}_{\mathcal{L}} = \sum_{i+j=d} a_{ij} X^i Y^j$ in some formal variables X, Y is called the *symbol* of $\mathcal{L}$.

One can assume field K to be differentially closed, in other words containing all the solutions of, in general nonlinear, PDEs with coefficients in K, or simply assume that K contains the solutions of those PDEs that we encounter on the way.

Definition 2.1 An operator $\mathcal{L}_1 \in K[D]$ is called a *Darboux transformation* of an operator $\mathcal{L} \in K[D]$ if $\mathrm{Sym}(\mathcal{L}) = \mathrm{Sym}(\mathcal{L}_1)$, and there exist operators $\mathcal{N} \in K[D]$ and $\mathcal{M} \in K[D]$ such that

$$\mathcal{N} \circ \mathcal{L} = \mathcal{L}_1 \circ \mathcal{M}. \tag{2.1}$$

In this case we say that this Darboux transformation corresponds to pair $\{\mathcal{L}, \mathcal{M}\}$, and that operator $\mathcal{L}_1$ is *associated with*, or *Darboux-conjugated to*, operator $\mathcal{L}$ and use the notation

$$\mathcal{L}_1 = \varphi(\mathcal{L}, \mathcal{M}, \mathcal{N}).$$

Note that coefficients of the operators are not required to be constants.

Darboux transformation implies the transformations of kernels

$$\operatorname{Ker} \mathcal{L} \to \operatorname{Ker} \mathcal{L}_1 \colon \psi \mapsto \mathcal{M}(\psi)$$

and requires $\mathrm{Sym}(\mathcal{M}) = \mathrm{Sym}(\mathcal{N})$.

Definition 2.2 The Darboux transformation of an operator (1.1), where a, b, c are not required to be constants, is called a *Laplace transformation* if the corresponding operator $\mathcal{M}$ is either $\mathcal{M} = D_x + b$, or $\mathcal{M} = D_y + a$.

Laplace transformations are the most well-studied case of Darboux transformation and have several important properties; see [2].

One of the most famous results in [2] concerns Darboux transformations for operators of the form (1.1) and can be formulated as follows.

https://doi.org/10.4153/CJM-2012-026-7 Published online by Cambridge University Press

Theorem 2.3 (Darboux) *Let $\mathcal{L}$ be an operator of the form (1.1) and $\psi_1, \dots, \psi_{m+n} \in \operatorname{Ker} \mathcal{L}$ be linearly independent. Then*

$$\mathcal{M}(\psi) = W_{m,n}(\psi, \psi_1, \dots, \psi_{m+n}) \tag{2.2}$$

defines some Darboux transformation for operator $\mathcal{L}$.

Here $W_{m,n}$ is a Wronskian-like function

$$W_{m,n}(\psi, \psi_1, \dots, \psi_{m+n}) = \begin{vmatrix} \psi & D_x\psi & \dots & D_x^m\psi & D_y\psi & \dots & D_y^n\psi \\ \psi_1 & D_x\psi_1 & \dots & D_x^m\psi_1 & D_y\psi_1 & \dots & D_y^n\psi_1 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \psi_{m+n} & D_x\psi_{m+n} & \dots & D_x^m\psi_{m+n} & D_y\psi_{m+n} & \dots & D_y^n\psi_{m+n} \end{vmatrix};$$

that is, a Darboux transformation of order $m + n$ can be built using $m + n$ particular solutions of the initial equation $\mathcal{L}(\psi) = 0$; see [2]. Darboux Wronskian-like formulas (2.2) provide a large class of possible Darboux transformations.

3 Normalization of Darboux Transformations

3.1 Normalization of Darboux Transformations Using Expansion

Lemma 3.1 *Let $\mathcal{L}$ be of the form (1.1) and let $\mathcal{M}$ be an operator of arbitrary order from $K[D]$. Let $\mathcal{M}$ define at least one Darboux transformation for operator $\mathcal{L}$. Then for any given operator $\mathcal{A} \in K[D]$ there exists also a Darboux transformation for the same $\mathcal{L}$ with $\mathcal{M} = \mathcal{M} + \mathcal{A} \circ \mathcal{L}$.*

Proof Equality (2.1) implies that

$$\mathcal{L}_1 \circ (\mathcal{M} + \mathcal{A} \circ \mathcal{L}) = \mathcal{M}_1 \circ \mathcal{L} + \mathcal{L}_1 \circ \mathcal{A} \circ \mathcal{L} = (\mathcal{M}_1 + \mathcal{L}_1 \circ \mathcal{A}) \circ \mathcal{L}$$

is true for an arbitrary operator $\mathcal{A} \in K[D]$. Therefore, there exists a Darboux transformation for $\mathcal{L}$ with $\mathcal{M} = \mathcal{M} + \mathcal{A} \circ \mathcal{L}$. ∎

Definition 3.2 Lemma 3.1 describes transformations of pairs of operators $\{\mathcal{L}, \mathcal{M}\}$. It shows that such transformations preserve the property of the existence of Darboux transformations for a given operator $\mathcal{L}$, and splits the operators $\mathcal{M}$ into equivalence classes. We call this transformation an *expansion*.

Remark 3.3 Notice that the resulting operators of the initial Darboux transformation and of the one generated for $\mathcal{L}$ by $\mathcal{M} + \mathcal{A} \circ \mathcal{L}$ are the same.

Given an operator $\mathcal{L}$ of the form (1.1), we shall be considering different pairs $\{\mathcal{L}, \mathcal{M}\}$, where $\mathcal{M}$ is an operator in $K[D]$. Using expansion we can eliminate all the mixed derivatives in $\mathcal{M}$ in the case $\mathcal{L}$ has the form (1.1).

Definition 3.4 Let $\mathcal{L} \in K[D]$ be of the form (1.1) and let $\mathcal{M} \in K[D]$ be an arbitrary operator. Then denote the result of elimination of the mixed derivatives in $\mathcal{M}$ using $\mathcal{L}$ as $\pi_{\mathcal{L}}(\mathcal{M})$.

Definition 3.5 Given $\mathcal{L} \in K[D]$ of the form (1.1), we define the *bi-degree* $\deg_{\mathcal{L}} \mathcal{M} = (m, n)$ of operator $\mathcal{M}$ with respect to $\mathcal{L}$ as follows: m is the highest derivative with respect to D_x in $\pi_{\mathcal{L}}(\mathcal{M})$, and n is that with respect to D_y. We shall say that $m + n$ is the *total degree* of $\mathcal{M}$.

Definition 3.6 By the *degree* or *total degree* of a Darboux transformation $\mathcal{L}_1 = \varphi(\mathcal{L}, \mathcal{M}, \mathcal{N})$ we shall understand the degree or the total degree of $\mathcal{M}$.

3.2 Normalization of Darboux Transformations Using Composition with Laplace Transformations

Definition 3.7 Let there be a Darboux transformation of arbitrary $\mathcal{L} \in K[D]$ defined by some $\mathcal{M} \in K[D]$; that is, (2.1) holds. Let the result, operator $\mathcal{L}_1 \in K[D]$, be transformed into some $\mathcal{L}_2 \in K[D]$ by a Darboux transformation defined by some $\mathcal{M}_1 \in K[D]$, that is, $\mathcal{N}_1 \circ \mathcal{L}_1 = \mathcal{L}_2 \circ \mathcal{M}_1$ for some $\mathcal{N}_1 \in K[D]$. Then *the composition of these two Darboux transformations* is a Darboux transformation transforming $\mathcal{L}$ into $\mathcal{L}_2$ defined by $\mathcal{N}_1 \circ \mathcal{N} \circ \mathcal{L} = \mathcal{L}_2 \circ \mathcal{M}_1 \circ \mathcal{M}$.

The following lemma allows us to use expansion and composition together.

Lemma 3.8 (Correctness of the composition of two Darboux transformation with expansion) *The result of composition of two Darboux transformations does not depend on the choice of the operator $\mathcal{M}$ within its class of equivalence under expansion.*

Proof Let there be a Darboux transformation of arbitrary $\mathcal{L} \in K[D]$ defined by some $\mathcal{M} \in K[D]$; *i.e.,* (2.1) holds. Let the result, operator $\mathcal{L}_1 \in K[D]$, be transformed into some $\mathcal{L}_2 \in K[D]$ by a Darboux transformation defined by some $\mathcal{M}_1 \in K[D]$; *i.e.,* $\mathcal{N}_1 \circ \mathcal{L}_1 = \mathcal{L}_2 \circ \mathcal{M}_1$ for some $\mathcal{N}_1 \in K[D]$. Consider $(\mathcal{M} + \mathcal{A} \circ \mathcal{L})$ and $\mathcal{M}_1 + \mathcal{B} \circ \mathcal{L}_1$ for some $\mathcal{A}, \mathcal{B} \in K[D]$, which belongs to the same classes of equivalence under the expansion as $\mathcal{M}$ and $\mathcal{M}_1$ correspondingly. That is, we have

$$
\begin{aligned}
(3.1) \qquad (\mathcal{N} + \mathcal{L}_1 \circ \mathcal{A}) \circ \mathcal{L} &= \mathcal{L}_1 \circ (\mathcal{M} + \mathcal{A} \circ \mathcal{L}),\\
(\mathcal{N}_1 + \mathcal{L}_2 \circ \mathcal{B}) \circ \mathcal{L}_1 &= \mathcal{L}_2 \circ (\mathcal{M}_1 + \mathcal{B} \circ \mathcal{L}_1).
\end{aligned}
$$

Then the composition is

$$(\mathcal{N}_1 + \mathcal{L}_2 \circ \mathcal{B}) \circ (\mathcal{N} + \mathcal{L}_1 \circ \mathcal{A}) \circ \mathcal{L} = (\mathcal{N}_1 + \mathcal{L}_2 \circ \mathcal{B}) \circ \mathcal{L}_1 \circ (\mathcal{M} + \mathcal{A} \circ \mathcal{L}),$$

which, using equality (3.1), can be re-written as

$$(\mathcal{N}_1 + \mathcal{L}_2 \circ \mathcal{B}) \circ (\mathcal{N} + \mathcal{L}_1 \circ \mathcal{A}) \circ \mathcal{L} = \mathcal{L}_2 \circ (\mathcal{M}_1 + \mathcal{B} \circ \mathcal{L}_1) \circ (\mathcal{M} + \mathcal{A} \circ \mathcal{L}).$$

After expanding some multiples and re-grouping we have

$$(\mathcal{N}_1 \circ \mathcal{N} + \mathcal{L}_2 \circ \mathcal{C} + \mathcal{N}_1 \circ \mathcal{L}_1 \circ \mathcal{A}) \circ \mathcal{L} = \mathcal{L}_2 \circ (\mathcal{M}_1 \circ \mathcal{M} + \mathcal{E} \circ \mathcal{L} + \mathcal{B} \circ \mathcal{L}_1 \circ \mathcal{M}),$$

where $\mathcal{C} = \mathcal{B} \circ \mathcal{N} + \mathcal{B} \circ \mathcal{L}_1 \circ \mathcal{A}$, and $\mathcal{E} = \mathcal{M}_1 \circ \mathcal{A} + \mathcal{B} \circ \mathcal{L}_1 \circ \mathcal{A}$. Finally, substituting $\mathcal{N} \circ \mathcal{L}$ for $\mathcal{L}_1 \circ \mathcal{M}$ we obtain that the "$\mathcal{M}$" operator of this Darboux transformation belongs to the same equivalence class that $\mathcal{M}_1 \circ \mathcal{M}$ does under the expansion transformation. ■

Then one of the results of [2] can be interpreted as follows.

Theorem 3.9 *Let $\mathcal{L} \in K[D]$ be of the form (1.1) and let $\mathcal{M} \in K[D]$ define a Darboux transformation for $\mathcal{L}$, and $\deg_{\mathcal{L}} \mathcal{M} = (m, n)$. Let $\mathcal{M}_x = D_y + a$ and $\mathcal{M}_y = D_x + b$ define LTs for operator (1.1). Then*

(i) $\deg \pi(\mathcal{M} \circ \mathcal{M}_x) = (m-1, n+1)$,
(ii) $\deg \pi(\mathcal{M} \circ \mathcal{M}_x) = (m+1, n-1)$,
(iii) $\pi(\mathcal{M}_x \circ \mathcal{M}_y) = b_y - c + ab$, *which is an operator of order zero,*
(iv) $\pi(\mathcal{M}_y \circ \mathcal{M}_x) = a_x - c + ab$, *which is an operator of order zero.*

Summarizing all the results we can formulate the following theorem.

Theorem 3.10 *Let $\mathcal{L} \in K[D]$ be of the form (1.1) and let $\mathcal{M} \in K[D]$ define a Darboux transformation for $\mathcal{L}$. Let $\deg_{\mathcal{L}} \mathcal{M} = (m, n)$. Then for every $i = 1, \dots, \min(m, n)$, there exists an operator $\mathcal{M}_i$ without mixed derivatives having the property that $\deg_{\mathcal{L}} \mathcal{M}_i = (m-i, n+i)$ and $\mathcal{M}_i$ defines a Darboux transformation of $\mathcal{L}$.*

Lemma 3.11 ($\mathcal{M}$ can be multiplied by a function on the left) *Let there exist a Darboux transformation of operator $\mathcal{L} \in K[D]$ with some operator $\mathcal{M} \in K[D]$. Then for every invertible element $p \in K$ there exists a Darboux transformation of operator $\mathcal{L}$ with operator $p\mathcal{M}$.*

Proof The conditions of the lemma imply that, for some $\mathcal{N}, \mathcal{L}_1 \in K[D]$, equality (2.1) holds. Therefore, $p \circ \mathcal{N} \circ \mathcal{L} = p \circ \mathcal{L}_1 \circ p^{-1} \circ p \circ \mathcal{M}$ is true also. Since the symbol of $\mathcal{L}_1$ is not altered under gauge transformations, operator $p \circ \mathcal{L}_1 \circ p^{-1}$ is an operator of the form (1.1), and we have proved the statement of the lemma. ■

Let $\mathcal{L} \in K[D]$ be of the form (1.1) and $\mathcal{M} \in K[D]$ of arbitrary form and order defining a Darboux transformation for $\mathcal{L}$. Theorem 3.10 and Lemma 3.11 imply that using operations of expansion, composition with LTs, and division by a function on the left, we can bring such Darboux transformation into a normalized form with $\mathcal{M}$ having no mixed derivatives and having one of the following symbols:

$$
\begin{aligned}
&\operatorname{Sym}(\mathcal{M}) = X^k, && k > 0,\\
&\operatorname{Sym}(\mathcal{M}) = Y^k, && k > 0,\\
&\operatorname{Sym}(\mathcal{M}) = X^k + qY^k, && k > 0,\ q \neq 0.
\end{aligned}
$$

Before we decide which of these to use in further considerations, let us consider the uniqueness problem for Darboux transformations.

3.3 Uniqueness of Darboux Transformations for Given $\mathcal{L}$ and $\mathcal{M}$

Theorem 3.12 *Let $\mathcal{L} \in K[D]$ be of the form (1.1) and let $\mathcal{M} \in K[D]$ define some Darboux transformation. Then, unless for its normalized form we have $\mathrm{Sym}(\mathcal{M}) = X^k$ or $\mathrm{Sym}(\mathcal{M}) = Y^k$, such a Darboux transformation is unique.*

If for its normalized form $\mathrm{Sym}(\mathcal{M}) = X^k$ (resp. $\mathrm{Sym}(\mathcal{M}) = Y^k$) and there exist two Darboux transformations: $\mathcal{L}_1 = \varphi(\mathcal{L}, \mathcal{M}, \mathcal{N})$ and $\mathcal{L}_1 + \mathcal{L}_1' = \varphi(\mathcal{L}, \mathcal{M}+\mathcal{M}', \mathcal{N}+\mathcal{N}')$, then

$$\mathrm{Sym}(M_1') = X^{k-1}, L_1' = D_y + \gamma, \quad \big(\textit{resp.} \quad \mathrm{Sym}(M_1') = Y^{k-1}, L_1' = D_x + \gamma\big).$$

for some $\gamma \in K$.

Proof Since $\mathcal{L}$ is of the form (1.1), there are only four possibilities for $\mathcal{L}_1'$:

$$\begin{aligned}
\mathcal{L}_1' &= D_x + \beta D_y + \gamma, \quad \beta \neq 0,\\
\mathcal{L}_1' &= D_x + \gamma,\\
\mathcal{L}_1' &= D_y + \gamma,\\
\mathcal{L}_1' &= 1.
\end{aligned}$$

Then $\mathcal{L}_1 = \varphi(\mathcal{L}, \mathcal{M}, \mathcal{N})$, and $\mathcal{L}_1 + \mathcal{L}_1' = \varphi(\mathcal{L}, \mathcal{M}+\mathcal{M}', \mathcal{N}+\mathcal{N}')$ implies

$$\mathcal{M}_1' \circ \mathcal{L} = \mathcal{L}_1' \circ \mathcal{M}.$$

Case $\mathcal{L}_1' = 1$ cannot take place, because if it does, then $\mathcal{M}_1' \circ \mathcal{L} = \mathcal{M}$, which is impossible, as $\mathrm{Sym}(M)$ cannot be divisible by XY.

Let $\mathrm{Sym}(\mathcal{M}) = X^k$, $k > 0$, then

$$\mathrm{Sym}(\mathcal{M}_1') \cdot X \cdot Y = \mathrm{Sym}(\mathcal{L}_1') \cdot X^k,$$

which implies that $\mathrm{Sym}(\mathcal{L}_1')$ must be divisible by Y, which is only possible if $\mathcal{L}_1' = D_y + \gamma$. Then $\mathrm{Sym}(\mathcal{M}_1')$ cannot contain extra Y-s, and therefore, $\mathrm{Sym}(\mathcal{M}_1') = X^{k-1}$.

Analogously, if $\mathrm{Sym}(\mathcal{M}) = Y^k$, $k > 0$, we have $\mathcal{L}_1' = D_x + \gamma$ and $\mathrm{Sym}(\mathcal{M}_1') = Y^{k-1}$.

Let $\mathrm{Sym}(M) = X^k + qY^k$, then

$$\mathrm{Sym}(\mathcal{M}_1') \cdot X \cdot Y = \mathrm{Sym}(\mathcal{L}_1') \cdot (X^k + qY^k),$$

which means that $\mathrm{Sym}(\mathcal{L}_1')$ must be divisible by XY, which is impossible. ■

Theorem 3.12 guarantees the uniqueness of a Darboux transformation for given $\mathcal{M}$ and $\mathcal{L}$ if $\mathrm{Sym}(\mathcal{M}) = X^k + qY^k$. Further below we shall be interested in Darboux transformation of the total degree two, and we choose the normal form for such transformations with $\mathrm{Sym}(\mathcal{M}) = X + qY$.

4 Existence of a Darboux Transformation Defined by $\mathcal{M}$ of Bi-degree $(1,1)$

Even for the simplest case of $\mathcal{M}$ of total degree 1, the problem of describing all Darboux transformations is not easy [10]. For the case of $\mathcal{M}$ of total degree 2, which is considered here, the problem becomes very difficult.

Theorem 4.1 *Let $\mathcal{L}$ be of the form (1.1) and $\mathcal{M} \in K[D]$ in the form*

$$\mathcal{M} = D_x + qD_y + r.$$

If there exists a corresponding Darboux transformation, then the corresponding operator $\mathcal{N}$ is given by $\mathcal{N} = \mathcal{M} - (\ln q)_x + q_y$. The necessary and sufficient conditions for the existence of a Darboux transformation for such pair $(\mathcal{L}, \mathcal{M})$ are

$$\begin{aligned} &- qr_x + q^2 r_y + q_x r - bq_x + b_x q + q^2(b_y - aq_y - a_x) - q^3 a_y \\ &\qquad + q_y q_x - q_{xy} q = 0, \\ &- cq_x + (c - ar)q_y q + (ar + r_y)q_x + (c_y - ra_y)q^2 \\ &\qquad + (rr_y - ar_x - r_y b - r_{xy} - ra_x + c_x)q = 0. \end{aligned} \tag{4.1}$$

Proof Compare the corresponding coefficients on the both sides of equality (2.1). ■

Darboux Theorem 2.3 provides us with a particular solution of system (4.1). The following statement is Theorem 2.3 written out more explicitly for the case of $\mathcal{M}$ of bi-degree $(1,1)$.

Theorem 4.2 (Darboux main theorem for bi-degree $(1,1)$) *Let $\mathcal{L} \in K[D]$ be an arbitrary operator of the form (1.1) and let ψ_1, ψ_2 be two linearly independent solutions of $\mathcal{L}\psi = 0$. Then there exists a Darboux transformation with*

$$\mathcal{M} = D_x + \frac{\alpha}{d} D_y + \frac{\beta}{d},$$

where

$$d = -\psi_1\psi_{2y} + \psi_2\psi_{1y}, \quad \alpha = \psi_1\psi_{2x} - \psi_2\psi_{1x}, \quad \beta = -\psi_{2x}\psi_{1y} + \psi_{2y}\psi_{1x}.$$

Remark 4.3 If we denote by ψ the ratio of these particular solutions, $\psi = \frac{\psi_2}{\psi_1}$, then $\mathcal{M}$ in the statement of Theorem 4.2 can be written in more simple form:

$$\mathcal{M} = D_x - \frac{\psi_x}{\psi_y} D_y + \frac{\psi_{1y}\psi_x}{\psi_1\psi_y} - \frac{\psi_{1x}}{\psi_1}.$$

Remark 4.4 In order to describe all Darboux transformations for $\mathcal{M}$ of bi-degree $(1,1)$, we need to solve system (4.1) for q, r, where a, b, c are known and are not constants in general. The usual differential elimination techniques do not lead to a general solution.

A different approach can be to notice that in the system (4.1) the second equation is non-linear in both q, r, while the first equation is nonlinear in q only. The first equation is a linear first-order non-homogeneous PDE on r, and since we know its particular solutions (Theorem 4.2), one may solve it in quadratures. These quadratures are expressed in terms of q, and therefore, after substituting the expression for r into the second equation, one gets even more nonlinear, rather large, PDE.

In the rest of the paper we will prove that the general solution of system (4.1) is given by the class of particular solutions from Theorem 4.2.

5 Gauge Transformations of Pairs and Corresponding Invariants

Our plan is to address our problem using invariants methods. In this section we study gauge transformations of pairs $(\mathcal{L}, \mathcal{M})$, which are almost classical, the only difference being that we apply them to the pairs of operators. These transformations are not strong enough to simplify our system significantly, and completely new transformations will be introduced in Section 6. However, we shall use gauge transformations of pairs too.

Definition 5.1 Given some operator $\mathcal{R} \in K[D]$ and invertible function $g \in K$, the corresponding *gauge transformation* is defined as

$$\mathcal{R} \to \mathcal{R}^g, \quad \mathcal{R}^g = g^{-1} \circ R \circ g,$$

where $\circ$ denotes the operation of the composition of operators in $K[D]$. It is convenient to take g in the form $g = \exp(\alpha)$. Then we shall avoid fractions while writing this transformation out on the coefficients of $\mathcal{R}$.

Our first step towards simplification of the problem is the following simple observation.

Lemma 5.2 *Let $\mathcal{L} = D_xD_y + aD_x + bD_y + c \in K[D]$ and $\mathcal{M} = D_x + qD_y + r \in K[D]$. If a Darboux transformation exists for the pair $(\mathcal{L}, \mathcal{M})$, then one also exists for $(\mathcal{M}^g, \mathcal{L}^g)$, where g is an arbitrary invertible element of K.*

Proof Indeed, from the Darboux equality (2.1) for the pair $(\mathcal{M}, \mathcal{L})$, we have

$$g^{-1} \circ \mathcal{N} \circ g \circ g^{-1} \circ \mathcal{L} \circ g = g^{-1} \circ \mathcal{L}_1 \circ g \circ g^{-1} \circ \mathcal{M} \circ g,$$

and, therefore, $\mathcal{N}^g \circ \mathcal{L}^g = \mathcal{L}_1^g \circ \mathcal{M}^g$. Recalling that gauge transformations do not change the symbol of an operator, we conclude the proof of the lemma. ∎

Therefore, it is natural to consider our problem for the equivalence classes of the pairs $(\mathcal{M}, \mathcal{L})$. In order to define every class uniquely we determine a generating set of all the invariants of these pairs under the gauge transformations.

Definition 5.3 Let $\mathcal{R} \in K[D]$ be an operator and let T be some transformation acting on $K[D]$. Then a function of the coefficients of $\mathcal{R}$ and of the derivatives of these coefficients is called a *differential invariant* if it is unaltered under the action of T on $\mathcal{R}$. The sum and the product of two differential invariants is an invariant; a derivative of an invariant is also an invariant. In the infinite set of all possible differential invariants there is some subset (not necessarily proper) of differential invariants which generate all others using algebraic operations and derivatives. We shall call such a subset a *generating set of invariants.*

Theorem 5.4 *Let* $\mathcal{L} = D_xD_y+aD_x+bD_y+c \in K[D]$ *and* $\mathcal{M} = D_x+qD_y+r \in K[D]$. *On the set of all pairs* $(\mathcal{L}, \mathcal{M})$ *of such operators, consider the gauge transformation of those with function* $\exp(\alpha)$ *:*

$$\varphi(\alpha) \colon (\mathcal{M}, \mathcal{L}) \to (\mathcal{M}^{\exp(\alpha)}, \mathcal{L}^{\exp(\alpha)}).$$

The following functions are invariants and in addition form a generating set of all differential invariants for such transformations:

$$(5.1)\qquad \begin{aligned} &q, \\ &m = a_x - b_y, \\ &h = ab - c + a_x, \\ &R = r - b - qa. \end{aligned}$$

Remark 5.5 Functions $h = ab-c+a_x$, $k = ab-c+b_y$ are known as h- and k-Laplace invariants, as they are invariants of operator $\mathcal{L}$ considered individually (without $\mathcal{M}$) under the gauge transformations. Both of them are present here: h is present in its original form, and k is hidden in m as $m = h - k$.

Proof To find a generating set of differential invariants we use the method of regularized moving frames introduced by Fels and Olver in [3]. A good overview of recent developments in the area can be found in [6]. Note that our case is infinite dimensional, so the connected difficulties have been treated in [9].

The transformations in question can be defined coordinate-wise as follows.

$$\begin{gathered} a_1 = a + \alpha_y, \quad b_1 = b + \alpha_x, \\ c_1 = c + a\alpha_x + b\alpha_y + \alpha_{xy} + \alpha_x\alpha_y, \\ q_1 = q, \quad r_1 = r + \alpha_x + q\alpha_y, \end{gathered}$$

where $\mathcal{L}^{\exp(\alpha)} = D_xD_y + a_1D_x + b_1D_y + c_1$ and $\mathcal{M}^{\exp(\alpha)} = D_x + q_1D_y + r_1$. We are choosing a cross-section as follows

$$(5.2)\qquad (a_1)_J = 0, \quad (b_1)_X = 0,$$

where J is a string of the form

$$\underbrace{x \ldots x}_{n}\underbrace{y \ldots y}_{m},$$

where $n = 0, 1, 2, \dots$, and $m = 0, 1, 2, \dots$ and X is a string of the form $\underbrace{x \dots x}_{l}$, where $l = 0, 1, 2, \dots$.

For every $f \in K[D]$, the notation f_J stands for the mixed derivative of f: of order n order with respect to x and m with respect to y, while f_X stands for the lth derivative of f with respect to x.

This gives us consistently all the values for the parameters of the pseudo-group action, $\alpha_x, \alpha_y, \alpha_{xx}, \dots$:

$$\alpha_x = -b, \quad \alpha_y = -a, \quad \alpha_{xy} = -a_x, \quad \dots$$

while we choose the value for α e arbitrarily, as it does not appear explicitly in the definition of the pseudo-group action. Then we evaluate the edge invariants on the frame:

$$\begin{aligned}
(b_1)_y &= b_y + \alpha_{xy} = b_y - a_x, \\
c_1 &= c - ab - ab - a_x + ab = c - a_x - ab, \\
r_1 &= r - b - qa,
\end{aligned}$$

which constitute the generating set of differential invariants of the pair under the gauge-transformations of the pair. Invariants m and h differ by a sign from the first two we have just obtained; the third invariant we have obtained is exactly R from the statement of the theorem. ∎

Definition 5.6 We shall call invariants (5.1) the *gauge invariants of the pair*.

Express the coefficients of the pair $(\mathcal{L}, \mathcal{M})$ in terms of invariants

$$r = b + qa + R, \quad c = ab - h + a_x.$$

After these substitutions system (4.1) does not depend on b itself, but only on its derivatives b_y and b_{xy}. Therefore, we can effectively use gauge-invariant m by enforcing the substitution $b_y = a_x - m$. Then system (4.1) simplifies to the following one:

$$\Omega = 0,$$

$$\Omega a + q_x h - q h_x - q^2 h_y - q_x m + q_x R_y + +q m_x - q R_{xy} - q_y q h - q R m + q R R_y = 0,$$

where $\Omega = -2q^2 m + q^2 R_y + q_x R + q_y q_x - q R_x - q_{xy} q$. Therefore, this system can be simplified further:

$$\begin{aligned}
& -2q^2 m + q^2 R_y + q_x R + q_y q_x - q R_x - q_{xy} q = 0, \\
& q_x h - q h_x - q^2 h_y - q_x m + q_x R_y + \\
& + q m_x - q R_{xy} - q_y q h - q R m + q R R_y = 0,
\end{aligned} \tag{5.3}$$

In our Darboux transformation problem, operator $\mathcal{L}$ is considered to be given; therefore, the gauge-invariants h and m are given, and the problem is reduced to the search for the general solution to system (5.3) with respect to unknowns q and R.

Although system (5.3) is visually shorter than system (4.1), it is still hard to solve using the usual methods such as the differentiation-cancellation technique.

Note that invariantiation and in particular moving frames method have been useful for investigation of Darboux-like methods earlier; see for example [4, 8, 13].

6 Gauged Evolution of Pairs and Corresponding Invariants

Lemma 6.1 *Let $\mathcal{L}, \mathcal{M} \in K[D]$ be two arbitrary operators in $K[D]$ and let $\mathcal{L}$ be of order larger than one, while $\mathcal{M}$ is a first-order operator. Then if a Darboux transformation exists for the pair $(\mathcal{L}, \mathcal{M})$, it also exists for the pair $(\mathcal{L} + \beta\mathcal{M}, \mathcal{M})$, where $\beta \in K$ is arbitrary.*

Proof The existence of a Darboux transformation for the pair $(\mathcal{L}, \mathcal{M})$ means that for some $\mathcal{N}, \mathcal{L}_1 \in K[D]$, where $\mathcal{L}_1$ has the same symbol as $\mathcal{L}$. Therefore,

$$\mathcal{N} \circ \mathcal{L} = \mathcal{L}_1 \circ \mathcal{M}.$$

Then

$$\mathcal{N} \circ (\mathcal{L} + \beta \circ \mathcal{M}) = \mathcal{L}_1 \circ \mathcal{M} + \mathcal{N} \circ \beta \circ \mathcal{M} = (\mathcal{L}_1 + \mathcal{N} \circ \beta) \circ \mathcal{M}. \tag{6.1}$$

Since $\mathcal{N}$ must be of the same order as $\mathcal{M}$, $\mathcal{N}$ is a first-order operator. In addition, β is zero-order operator. Therefore, the symbol of the operator $\mathcal{L}_1 + \mathcal{N} \circ \beta$ is the same as the symbol of $\mathcal{L}_1$, which is the same as the symbol of $\mathcal{L}$. Therefore, equality (6.1) defines a Darboux transformation for pair $(\mathcal{L} + \beta\mathcal{M}, \mathcal{M})$. ■

Definition 6.2 This transformation on the pairs, that is $(\mathcal{L}, \mathcal{M}) \to (\widetilde{\mathcal{L}}, \widetilde{\mathcal{M}})$,

$$\widetilde{\mathcal{L}} \to \mathcal{L} + \beta \circ \mathcal{M}, \quad \widetilde{\mathcal{M}} \to \mathcal{M}.$$

we shall call *evolution of the pair* (or *β-evolution of the pair*).

Definition 6.3 Let $\mathcal{L} = D_x D_y + aD_x + bD_y + c \in K[D]$ and let $\mathcal{M} \in K[D]$ be arbitrary. On the set of all the pairs $(\mathcal{L}, \mathcal{M})$ of such operators, consider the consequential application of the gauge transformations and of the evolution. For given $\alpha, \beta \in K$:

$$(\mathcal{L}, \mathcal{M}) \mapsto (\mathcal{L}^{\exp(\alpha)} + \beta\mathcal{M}^{\exp(\alpha)}, \mathcal{M}^{\exp(\alpha)}). \tag{6.2}$$

We shall call these transformations *gauged evolution of the pairs*.

Theorem 6.4 *Let $\mathcal{L} = D_x D_y + aD_x + bD_y + c \in K[D]$ and $\mathcal{M} = D_x + qD_y + r \in K[D]$. The gauged evolutions of the pairs $(\mathcal{L}, \mathcal{M})$ of such operators have the following generating set of differential invariants:*

$$I_1 = q, \quad I_2 = 2m - R_y + \left(\frac{R}{q}\right)_x, \quad I_3 = 2h + \left(\frac{R}{q}\right)_x - \frac{R^2}{2q}, \tag{6.3}$$

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Remark 6.5 Notice that gauged evolution generating invariants (6.3) are expressed in terms of generating gauge invariants q, h, m, R only. This means that the gauged evolutions split the set of pairs $(\mathcal{L}, \mathcal{M})$ into larger equivalence classes than the gauge transformations of pairs do. Also we see that those "small" gauge classes can belong to the "larger" gauged evolution classes only entirely.

Proof Evolution (6.2) can be defined coordinate-wise as follows:

$$\begin{aligned} a_1 &= a + \alpha_y + \beta, \quad b_1 = b + \alpha_x + \beta q, \\ c_1 &= c + a\alpha_x + b\alpha_y + \alpha_{xy} + \alpha_y\alpha_x + \beta r + \beta\alpha_x + \beta q\alpha_y, \\ q_1 &= q, \quad r_1 = r + \alpha_x + q\alpha_y, \end{aligned}$$

where $\mathcal{L}^{\exp(\alpha)} + \beta\mathcal{M}^{\exp(\alpha)} = D_xD_y + a_1D_x + b_1D_y + c_1$ and $\mathcal{M}^{exp(\alpha)} = D_x + q_1D_y + r_1$. We are setting a cross-section by setting most of the coordinate functions to zero:

$$(a_1)_J = 0, \quad (b_1)_J = 0, \quad (r_1)_X = 0,$$

where J and X are the same notations as in (5.2). Then at the beginning we have three equations,

$$a_1 = 0, \quad b_1 = 0, \quad r_1 = 0$$

and three variables, parameters to determine: β, α_x, and α_y. The determinant is not 0, so there is a unique solution for such a system. At the next step we consider first prolongations only, which gives us 5 equations for 5 variables, and this linear system has non-zero determinant. In general, considering the i-th prolongation we have $2i + 3$ variables and the same number of equations as well as a non-zero determinant of the corresponding linear system. Therefore, we have defined a frame, and the generating set of invariants in this case consists of the corner invariants:

$$\begin{aligned} I^q &= q, \\ I^r_y &= r_y - a_x - \frac{r_x + b_x}{q} + \frac{q_xr - q_xb}{q^2} - q_ya - qa_y + b_y, \\ I^c &= c - \frac{a_x}{2} - \frac{br}{2q} - \frac{q_xb}{q^2} + \frac{q_xr}{2q^2} - \frac{ab + ar}{2} + \frac{qa^2}{4} + \frac{b^2}{4q} - \frac{r_x}{2q} + \frac{b_x}{2q} + \frac{r^2}{4q}. \end{aligned}$$

Substituting

$$r = b + qa + R, \quad c = ab - h + a_x, \quad a_x = b_y + m,$$

we obtain (up to a sign and a multiplication by 2) the invariants claimed in the statement of the theorem. ■

Definition 6.6 We shall refer to invariants (6.3) as *gauged evolution invariants*.

https://doi.org/10.4153/CJM-2012-026-7 Published online by Cambridge University Press

7 Solution of the PDE System. Description of All Darboux Transformations of Total Order Two.

In Lemma 6.1 we showed that the property of the existence of a Darboux transformation for a pair is invariant under the gauged evolutions. This does not necessarily mean that there is some explicit invariant form for system (4.1). Theorem 7.1 demonstrates, however, that in this particular case, we can have such an explicit invariant form. We also see that the invariantizing system can be written in a much simpler form than system (4.1).

Theorem 7.1 (Necessary and sufficient conditions for the existence of a Darboux transformation in terms of evolution) *Given pair $(\mathcal{L}, \mathcal{M})$, where $\mathcal{L} = D_xD_y + aD_x + bD_y + c \in K[D]$ and $\mathcal{M} = D_x + qD_y + r \in K[D]$, there exists a corresponding Darboux transformation if and only if its evolution invariants (q, I_2, I_3) satisfy the following two conditions simultaneously:*

$$I_2 + Q_{xy} = 0\,, \tag{7.1}$$

$$I_{3,x} + qI_{3,y} + (q_y - q_x/q)I_3 = Q_xQ_{xy} - Q_{xxy}, \tag{7.2}$$

where $Q = \ln q$.

Proof Expressing m and h using the second and the third equations of (6.3) and using $b_y = a_x - m$ system (5.3) can be written as in the statement. ∎

We also invariantize the class of particular solutions for system (4.1) that we derived from Darboux Wronskian formulas.

Theorem 7.2 (Darboux transformations constructed from Wronskians) *Let $\mathcal{L} = D_xD_y + aD_x + bD_y + c \in K[D]$ and let ψ_1, ψ_2 be two linearly independent elements of its kernel. Let $\psi = \frac{\psi_2}{\psi_1}$, $A = \frac{\psi_{xy}}{\psi_x}$, and $B = \frac{\psi_{xy}}{\psi_y}$. Then for $\mathcal{L}$ there exists a Darboux transformation such that the evolution invariants of the corresponding pair $(\mathcal{M}, \mathcal{L})$ are as follows:*

$$q = -\frac{B}{A}, \quad I_2 = B_y - A_x, \quad I_3 = -A_x + \frac{A \cdot B}{2}. \tag{7.3}$$

Proof Compute the values of the gauged evolution invariants (6.3) for $\mathcal{M}$ constructed using Darboux formulas given in Theorem 4.2. Then we have

$$q = \frac{-\psi_x}{\psi_y},$$

$$I_2 = \frac{-\psi_{xxy}}{\psi_x} + \frac{\psi_{xx}\psi_{xy}}{\psi_x^2} + \frac{\psi_{xyy}}{\psi_y} - \frac{\psi_{xy}\psi_{yy}}{\psi_y^2},$$

$$I_3 = \frac{-\psi_{xxy}}{\psi_x} + \frac{\psi_{xx}\psi_{xy}}{\psi_x^2} + \frac{\psi_{xy}^2}{2\psi_x\psi_y},$$

which can be rewritten in a very short form using notations A and B. ∎

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The value of I_3 from Theorem 7.2 is a particular solution of (7.2), a first-order linear non-homogeneous PDE on I_3.

Let us solve (7.2). One useful idea is to consider q in the form $q = -z_x/z_y$, where z is not required to be a ratio of two particular solutions of $\mathcal{L}(\psi) = 0$. For this $q \in K$, invariants I_1, I_2 can be computed straightforwardly: $I_1 = q, I_2 = -(\ln q)_{xy}$.

Equation (7.2) is a first order non-homogeneous, and the general solution can be obtained as the sum of its particular solution and of the general solution of the corresponding homogeneous PDE. As a particular solution we take the expressions from (7.3). Note that for this particular solution $z = \psi$, where ψ is some particular solution of $\mathcal{L}\psi = 0$.

Using expression for q in terms of z the homogeneous PDE corresponding to (7.2) can be written as

$$I_3\Big(\frac{z_{xx}}{z_x} - \frac{z_x z_{yy}}{z_y^2}\Big) - I_{3,x} + \frac{z_x}{z_y} I_{3,y} = 0.$$

Considering I_3 in the form $I_3 = e^J$ for suitable $J \in K[D]$ and assuming $z_y \neq 0$, the PDE is equivalent to

$$T - \frac{J_x}{z_x} + \frac{J_y}{z_y} = 0, \tag{7.4}$$

the non-homogeneous part of the equation being $T = z_{xx}/z_x^2 - z_{yy}/z_y^2$. After applying the method of characteristics, which we follow more carefully in the proof of Theorem 7.4, we choose change of variables $\xi = x, \eta = z(x, y)$, and, in the new variables, equation (7.4) has the form

$$\frac{J_\xi}{z_x(\xi, \eta)} = T(\xi, \eta),$$

and, therefore,

$$J = \int T\big(\xi, \widetilde{y}(\xi, \eta)\big)\, z_\xi\big(\xi, \widetilde{y}(\xi, \eta)\big)\, d\xi + F(\eta),$$

where $\widetilde{y} = \widetilde{y}(\xi, \eta)$ is the solution of $z(\xi, y) - \eta$ for y and $F(\eta)$ is an arbitrary function of η. Changing variables back we have

$$J = \int^x T\big(\xi, \widetilde{y}(\xi, z)\big)\, z_\xi\big(\xi, \widetilde{y}(\xi, z)\big)\, d\xi + F(z).$$

This means that the homogeneous part of PDE (7.2) has the general solution

$$I_3 = G(z) \cdot \exp\bigg(\int^x T(\xi, \widetilde{y}(\xi, z)) z_\xi(\xi, \widetilde{y}(\xi, z)) d\xi\bigg),$$

where $G(z)$ is an arbitrary function of z. Then since we know the particular solution, $I_{30}(\psi)$, where $\psi = \psi_2/\psi_1$ for two particular solutions of $\mathcal{L}u = 0$, the general

https://doi.org/10.4153/CJM-2012-026-7 Published online by Cambridge University Press

description of all Darboux transformations of total order two will be as follows:

$$q = -\frac{z_x}{z_y}, \quad I_2 = \left(\frac{z_{xy}}{z_y}\right)_y - \left(\frac{z_{xy}}{z_x}\right)_x, \tag{7.5}$$

$$I_3 = G(z) \cdot \exp\left(\int^x T(\xi, \widetilde{y}(\xi, z)) z_\xi(\xi, \widetilde{y}(\xi, z)) d\xi \right) + I_{30}(\psi_2/\psi_1).$$

However, formulae (7.5) do not serve either of our two purposes: short description of all possible Darboux transformations and proof of completeness. Indeed, we would have to decide whether the series of particular solutions (7.3) is the same as the obtained general solution.

Thus, we approach the solution of (7.2) in a different manner. The next theorem implies that we can construct a series of particular solutions of (7.2) using an arbitrary function z rather than ψ, which must be the ratio of two linearly independent particular solutions ψ_1 and ψ_2 of the initial PDE $\mathcal{L} = 0$.

Theorem 7.3 (Another large class of particular solutions for the PDE on I_3) *Let $z \in K$ be arbitrary and non-constant and let $F = F(z)$ be an arbitrary function of z, then*

$$q = -\frac{z_x}{z_y} = -\frac{(F(z))_x}{(F(z))_y}$$

and function

$$I_{30}(z) = -\frac{z_{xxy}}{z_x} + \frac{z_{xx} z_{xy}}{z_x^2} + \frac{z_{xy}^2}{2 z_x z_y} \tag{7.6}$$

gives particular solutions of (7.2). More strongly, if instead of z we use the argument $F(z)$, that is $I_{30}(z)$ is replaced by $I_{30}(F(z))$, we still have particular solutions of (7.2).

Proof The proof can be verified by direct substitution. ■

In other words, on the invariant level we can forget about the fact that ψ must be a ratio of two solutions. Of course such a trick would not work in general for the pre-invariantized system. If we took Darboux Wronskian Formulas and substitute arbitrary functions instead of solutions we would not necessarily get a Darboux transformation. So the invariantization using gauged evolutions factors out some meaningful conditions, which justifies giving them a separate name.

Using the new class of particular solutions discovered in Theorem 7.3, we can find the general solution for the invariantized system (see it in Theorem 7.1) in the following form.

Theorem 7.4 (Simple Description of all Darboux transformations of bi-degree $(1,1)$) *All Darboux transformations of some $\mathcal{L} = D_x D_y + a D_x + b D_y + c \in K[D]$ generated by some $\mathcal{M}$, $\mathcal{M} = D_x + q D_y + r \in K[D]$ are parametrized by $z \in K$ an arbitrary non constant and can be written as*

$$q = -\frac{B}{A}, \quad I_2 = B_y - A_x, \quad I_3 = -A_x + \frac{A \cdot B}{2},$$

where $A = z_{xy}/z_x$ and $B = z_{xy}/z_y$.

Proof Find $z \in K$ such that $q = -z_x/z_y$. Then according to (7.1) we must have $I_2 = -(\ln(q))_{xy}$, which in terms of z has the form given for I_2 in the statement of the theorem.

Now we solve (7.2) for I_3. The solution of this equation can be obtained as the sum of particular solution (7.6) and of the general solution of the corresponding homogeneous PDE,

$$I_3\Big(\frac{z_{xx}}{z_x} - \frac{z_x z_{yy}}{z_y^2}\Big) - I_{3,x} + \frac{z_x}{z_y} I_{3,y} = 0. \tag{7.7}$$

Considering I_3 in the form $I_3 = e^J$ for suitable $J \in K[D]$ this equation can be rewritten equivalently as

$$\frac{z_{xx}}{z_x^2} - \frac{z_{yy}}{z_y^2} - \frac{J_x}{z_x} + \frac{J_y}{z_y} = 0.$$

Again we have homogeneous and non-homogeneous parts, and therefore, the general solution can be represented as the sum of a particular solution of the non-homogeneous part and the general solution of the homogeneous one.

Using the methods of characteristics, one can find the general solution of the homogeneous part of (7.4). Consider the equality

$$\frac{dx}{-1/z_x} = \frac{dy}{1/z_y},$$

which can be rewritten in the form $z_x dx + z_y dy = d(z) = 0$, and, therefore, $z = z(x, y) = C$, where C is a constant. Therefore, we consider the following change of variables:

$$\left.\begin{aligned} \xi &= x, \\ \eta &= z(x, y), \end{aligned}\right\}$$

which is non-degenerate, since the Jacobian is nonzero:

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \eta_y = z_y \neq 0.$$

Expressed in the new variables equation (7.4) has the form $J_\xi / z_x(\xi, \eta) = 0$, and, therefore, the general solution is $J = H(z)$, where $H(z)$ is an arbitrary function of z.

Notice now that since both $I_{30}(z)$ and $I_{30}(F(z))$ are solutions of (7.2), therefore, their difference is a solution of (7.7). Subtracting (7.6) from $I_{30}(G(z))$ we have

$$I_{30d} = \frac{1}{2} z_x z_y \Big(\frac{2F'F''' - 3(F'')^2}{(F')^2}\Big) = z_x z_y G(z),$$

where $G(z)$ is an arbitrary function. Therefore,

$$J_{30d} = \ln(I_{30d}) = G_1(z) + \ln(z_x z_y),$$

https://doi.org/10.4153/CJM-2012-026-7 Published online by Cambridge University Press

where $G_1(z)$ is an arbitrary function, is a solution of (7.4). Therefore, the general solution of (7.4) is $J = G_1(z) + \ln(z_x z_y) + H(z)$. Correspondingly, the general solution of (7.7), which is the homogeneous part of the PDE we needed to solve (7.2), is $I = \exp(J)$, which is $I = G(z) z_x z_y$. Adding to this the particular solution of the non-homogeneous part, I_{30}, we conclude that

$$I_3 = z_x z_y G(z) + I_{30}(z) = z_x z_y \frac{2F'F''' - 3(F'')^2}{2(F')^2} + I_{30}(z).$$

Notice that this expression is exactly $I_{30}(F(z))$. Now we have proved that

$$q = -\frac{z_x}{z_y}, \quad I_2 = \left(\frac{z_{xy}}{z_y}\right)_y - \left(\frac{z_{xy}}{z_x}\right)_x, \quad I_3 = I_{30}(F(z)).$$

However, notice that the application of q to z and q to $F(z)$ gives the same result, that is $q(F(z)) = q(z)$. The same is true for I_2: $I_2(F(z)) = I_2(z)$. Therefore, we can substitute $\widetilde{z} = F(z)$ and have the statement of the theorem, which we write in terms of z again for convenience. ■

Comparing Theorem 7.4 with Theorem 7.2, we see that the general solution of the invariantized system (see it in Theorem 7.1) is much richer than the class of the invariantized particular solutions (7.3). Does this mean that our hypothesis was wrong and that there must be something else besides Darboux transformations generated by Darboux Wronskians formulas? In the following theorem, lifting our results back into pre-invariantized situation, we conclude the proof of our hypothesis.

Theorem 7.5 *Given operator $\mathcal{L} = D_x D_y + aD_x + bD_y + c \in K[D]$, every Darboux transformation generated by $\mathcal{M}$, $\mathcal{M} = D_x + qD_y + r \in K[D]$ is described by Darboux Wronskian formulas.*

Proof Let $\mathcal{N}\mathcal{L} = \mathcal{L}_1\mathcal{M}$ be some Darboux transformations with the gauged evolution invariants (q, I_2, I_3). According to Theorem 7.4 there exists $z \in K$ such that

$$\begin{aligned} q &= -\frac{z_x}{z_y} = -\frac{B}{A}, \\ I_2 &= \left(\frac{z_{xy}}{z_y}\right)_y - \left(\frac{z_{xy}}{z_x}\right)_x = B_y - A_x, \\ I_3 &= I_{30}(z) = -A_x + \frac{A \cdot B}{2}, \end{aligned}$$

where $A = z_{xy}/z_x$, $B = z_{xy}/z_y$. Comparing this with the invariants given in Theorem 7.2, we conclude that there are some Darboux transformations constructed by Darboux formulae that have the same gauged evolution invariants.

Now let us find among the pairs that are constructed using Darboux formulas and having these gauged evolution invariants (q, I_2, I_3) those that have the same gauge invariants as our initial pair $(\mathcal{L}, \mathcal{M})$.

Let us denote the gauge invariants of the pair $(\mathcal{L}, \mathcal{M})$ by (m_0, h_0, q_0, R_0). Then I_2 and I_3 can be expressed in terms of (m_0, h_0, q_0, R_0); see (6.3). Therefore,

$$2m_0 = I_2 - (R_0/q)_x + R_{0y}, \quad 2h_0 = I_3 + R_0^2/(2q) - (R_0/q)_x.$$

Now, since (I_2, I_3) are given in terms of z, then invariants (m_0, h_0) are given in terms of z and R_0. That is, it is enough to find among the pairs that are constructed using Darboux Wronskians formulas a pair with the same R_0.

Choose arbitrary functions z_1 and c_0 and construct an operator $\mathcal{L}'$ of the form (1.1) that has two solutions: z_1 and zz_1 and with coefficient $c = c_0$. Using the same pair of solutions construct $\mathcal{M}'$ using Darboux formulas; see Theorem 4.2.

Then R' corresponding to the pair $(\mathcal{L}', \mathcal{M}')$ can be expressed in terms of z, z_1, and c_0 in the form

$$R' = -\frac{2z_1 z_x}{-z_x z_{1,y} + z_y z_{1,x}} \cdot c_0 + T(z, z_1),$$

where $T(z, z_1)$ is a certain expression depending on z and z_1 only. This means that for every z_1 we can uniquely find such c that $R' = R_0$.

Therefore, for arbitrary z_1 there exist a Darboux transformation $(\mathcal{L}', \mathcal{M}')$ constructed using Darboux formulas for which all gauge invariants (h, k, q, R) of the pair are correspondingly the same as those of the initial pair $(\mathcal{L}, \mathcal{M})$.

Since those agree, then $(\mathcal{L}, \mathcal{M})$ is different from $(\mathcal{L}', \mathcal{M}')$ by a gauge transformation, and therefore, $(\mathcal{L}, \mathcal{M})$ can be also constructed using Darboux Wronskian formulas. ∎

8 Conclusions

This paper closes an essential question for the theory of Darboux transformations. Darboux Wronskians formulas are complete for Darboux transformation of total order two of operators $\mathcal{L} = D_x D_y + aD_x + bD_y + c$ with non-constant coefficients. Since for Darboux transformations of total order there are two famous exceptions, Laplace transformations, the case of the total order two has been crucial.

We saw that a newly introduced transformations of pairs, gauged evolutions, may have much deeper role than just a tool in the proof of our specific problem (Theorem 7.3 and the paragraph after it).

We found a very short invariant description of all possible Darboux transformation for $\mathcal{L} = D_x D_y + aD_x + bD_y + c$ generated by $\mathcal{M}$ in the form $\mathcal{M} = D_x + qD_y + r \in K[D]$ (Theorem 7.4).

Now it is natural to expect completeness of Darboux Wronskians formulas for transformations of orders higher than two. We expect this one to be rather difficult to prove. Simple repetition and adjustments of the methods and ideas of this paper will not work. For example, one of the crutial points was the introduction of the gauged evolutions, which cannot be defined for pairs $(\mathcal{L}, \mathcal{M})$ if $\mathcal{M}$ has order larger than $\mathcal{L}$.

https://doi.org/10.4153/CJM-2012-026-7 Published online by Cambridge University Press

References

[1] P. Cassidy and M. F. Singer, *A Jordan-Hölder theorem for differential algebraic groups*. J. Algebra **328**(2011), 190–217.
[2] G. Darboux, *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal*. Deuxième partie, Chelsea Publishing Co., Bronx, N.Y., 1972.
[3] M. Fels and P. J. Olver, *Moving coframes. I. A practical algorithm*. Acta Appl. Math. **51**(1998), no. 2, 161–213.
[4] D. Gomez-Ullate, N. Kamran, and R. Milson, *Two-step Darboux transformations and exceptional Laguerre polynomials*. J. Math. Anal. Appl. **387**(2012), no. 1, 410–418.
[5] I. M. Krichever, *Algebraic-geometric n-orthogonal curvilinear coordinate systems and the solution of associativity equations*. Funct. Anal. Appl. **31**(1997), no. 1, 25–39. http://dx.doi.org/10.1007/BF02466001
[6] E. L. Mansfield, *A practical guide to the invariant calculus*. Cambridge Monographs on Applied and Computational Mathematics, 26 Cambridge University Press, Cambridge, 2010.
[7] V. B. Matveev and M. A. Salle, *Darboux transformations and solitons*. Springer Series in Nonlinear Dynamics, Springer-Verlag, Berlin, 1991.
[8] P. J. Olver, *Differential invariant algebras*. Symmetries and related topics in differential and difference equations, Contemp. Math. 549, American Mathematical Society, Providence, RI, pp. 95–121.
[9] P. J. Olver and J. Pohjanpelto, *Maurer-Cartan forms and the structure of Lie pseudo-groups*. Selecta Math. **11**(2005), no. 1, 99–126.
[10] E. Shemyakova, *Laplace transformations as the only degenerate Darboux transformations of first order.* Programming and Computer Software **38**(2012), no. 2, 105–108. http://dx.doi.org/10.1134/S0361768812020053
[11] S. P. Tsarev, *Factorization in categories of systems of linear partial differential equations*. arxiv:0801.1341.
[12] ______, *Generalized Laplace transformations and integration of hyperbolic systems of linear partial differential equations*. In: ISSAC '05: Proceedings of the 2005 international symposium on Symbolic and algebraic computation, ACM, New York, 2005, pp. 325–331.
[13] P. J. Vassiliou, *Method for solving the multidimensional n-wave resonant equations and geometry of generalized Darboux-Manakov-Zakharov systems*. Stud. Appl. Math. **126**(2011), no. 3, 203–243.

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