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On the Entire Coloring Conjecture

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Abstract. The Four Color Theorem says that the faces (or vertices) of a plane graph may be colored with four colors. Vizing's Theorem says that the edges of a graph with maximum degree Δ may be colored with $\Delta + 1$ colors. In 1972, Kronk and Mitchem conjectured that the vertices, edges, and faces of a plane graph may be simultaneously colored with $\Delta + 4$ colors. In this article, we give a simple proof that the conjecture is true if $\Delta \geq 6$.

1 Introduction

The original graph coloring problem was that of coloring the faces (equivalently the vertices) of a plane graph. This was known as the four color problem, and was solved by Appel and Haken in 1976 (see [1] or [13] for a simplified proof). Vizing [17] proved that if Δ is the maximum degree of a vertex of a graph (we will always use Δ for this purpose), then its edges may be colored with $\Delta + 1$ colors, which is easily seen to be best possible.

The problem of simultaneously coloring sets of elements of a graph began in the mid-1960s with Ringel [12], who conjectured that the vertices and faces of a plane graph may be colored with six colors (solved by Borodin [2]), and with Vizing [17], who conjectured that the vertices and edges of any graph may be colored with $\Delta + 2$ colors (best known general result: [11]). There was also an edge-face coloring conjecture of Melnikov [10] which was recently solved by the authors [14] (see also [19]). For a complete history of problems of this type, see [6].

The most complicated problem of this type for plane graphs is that of coloring all its elements simultaneously, that is its vertices, edges, and faces. In 1972, Kronk and Mitchem [8] called this type of coloring an entire coloring, and conjectured that any plane graph of maximum degree Δ is entirely $(\Delta + 4)$-colorable and proved this conjecture for $\Delta = 3$ [9]. Entire colorings of special classes of graphs had previously been studied in [5] and [7]. The graph K_4 shows that an entire $(\Delta + 3)$-coloring theorem is impossible. In 1989, Borodin [3] proved the conjecture for $\Delta \geq 12$, and recently improved this to $\Delta \geq 7$ [4]. This article gives a simple proof of the entire coloring conjecture for $\Delta \geq 6$, using the discharging method.

To formalize, some terminology is necessary. All graphs in this paper are simple and finite. Let G be a plane graph. We denote the *vertex*, *edge*, and *face* sets of G by $V(G)$, $E(G)$, and $F(G)$, respectively. For convenience, in this article, we use *adjacent* instead of the conventional *incident*. An *entire coloring* of G is a function assigning values (colors) to elements of $V(G) \cup E(G) \cup F(G)$ in such a way that any two distinct adjacent elements receive distinct colors. A plane graph G is *entirely k-colorable* if there is an entire coloring of G with colors $\{1, \ldots, k\}$. The main theorem may then be stated as:

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Theorem 1.1 *Every plane graph with maximum degree $\Delta \geq 6$ is entirely $(\Delta + 4)$-colorable.*

As the proof presented here is so simple, the authors have hope that the discharging method may be used to complete a proof of Kronk and Mitchem's conjecture (that is the remaining cases of $\Delta \in \{4, 5\}$). Already, it has succeeded where others did not expect success.

2 The Structure of a Minimal Counterexample

Suppose that Theorem 1.1 is false. Then we define a *minimal graph* to be a counterexample to Theorem 1.1 with the fewest number of edges. This section will demonstrate some very simple lemmas, which give some insight into the structure of a minimal graph. Each lemma describes small configurations of vertices and faces of low degree, and then proves that a minimal graph cannot have such configurations.

But first, a standard, simple connectivity reduction is useful. A graph is *connected* if there is a path of edges between each pair of its vertices. Further, a graph is *non-separable* if it remains connected after the deletion of any one of its vertices.

Lemma 2.1 *A minimal graph is non-separable.*

Proof Let G be a minimal graph. Either the lemma is true, or there are subgraphs G_1 and G_2 of G such that $G_1 \cup G_2 = G$, and $|V(G_1) \cap V(G_2)| \leq 1$. Since G is minimal, each of G_1, G_2 has an entire $(\Delta + 4)$-coloring. These entire colorings are easily combined to give an entire $(\Delta + 4)$-coloring of G by permuting the colors to match on the shared face and vertex, and to differ on the edges adjacent to the shared vertex (should it exist). This is a contradiction. ∎

To describe the small configurations, some notation is useful. Let G be a plane graph. A *k-vertex*, an *$(\geq k)$-vertex*, and an *$(\leq k)$-vertex* are vertices of degree k, at least k, and at most k in G, respectively. Let k-face, $(\geq k)$-face, and $(\leq k)$-face be defined similarly. Let an $(n_1, \ldots, n_k)$-face be a k-face f such that there is an ordering $v_1, \ldots, v_k$ of the vertices adjacent to f, which is a cyclic order according to the plane embedding of G, such that for $i \in \{1, \ldots, k\}$, $\deg(v_i) \leq n_i$. Given a face f of non-separable G, and an edge uv which is adjacent to f, let f_{uv} be the face not equal to f which is adjacent to uv. Given an edge uv, let the *weight* of uv be $\deg(u) + \deg(v)$.

Now we are prepared to give the first six configurations. The first simply says that in a minimal graph, no (≤ 3)-vertex is adjacent to a 3-face. For the proof, it is useful to speak of a *partial entire coloring*, that is, where some elements may not have a color, but distinct adjacent colored elements have distinct colors. Given a plane graph G, it is important to notice that an entire coloring of a subgraph of G is not necessarily a partial entire coloring of G.

Lemma 2.2 *No minimal graph has a $(3, \Delta, \Delta)$-face, a $(2, \Delta, \Delta, \Delta)$-face, a $(3, 3, 3, 3, \Delta)$-face, a $(3, \Delta - 1, \Delta, \Delta)$-face, a $(2, \Delta, 3, \Delta, \Delta)$-face, or a $(2, \Delta, 2, \Delta, 3, \Delta)$-face.*

Proof Let G be a minimal graph with a face f which satisfies one of the six conditions in the statement of the lemma. Let e be an edge adjacent to f with minimum weight. Let v be

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an (≤ 3)-vertex adjacent to e. Since G is minimal, $G-e$ has an entire $(\Delta+4)$-coloring. This entire coloring induces a partial entire $(\Delta+4)$-coloring of G with v, e, and f uncolored. Remove the colors from each (≤ 3)-vertex adjacent to f. Also, remove the color from any edge adjacent to f which is also adjacent to an (≤ 3)-vertex.

Now the uncolored elements will easily be colored. If $\deg(f) \geq 4$, then color it first with a color from $\{1,\ldots,\Delta+4\}$ which none of its adjacent elements is colored; this may be done as f is currently adjacent to at most nine colored elements, and $\Delta \geq 6$. Next, if $\deg(f) = 5$, and there is an uncolored edge e' of weight $\Delta+3$ which is not adjacent to an uncolored edge of weight at most $\Delta+2$, color e' differently than its colored adjacent elements, of which there are at most $\Delta+3$. Next, color, in a decreasing order of weight, the remaining uncolored edges adjacent to f; each is adjacent to at most $\Delta+3$ colored elements when it is colored. Finally, color each (≤ 3)-vertex adjacent to f, as well as f itself if $\deg(f) = 3$; each is adjacent to at most nine elements. Thus G is entirely $(\Delta+4)$-colorable, a contradiction. ■

The proof of the next lemma is different only in the manner in which a smaller graph is found. It is not useful to simply delete an edge for this lemma, as no face of small degree is nearby.

Lemma 2.3 *No minimal graph has a 2-vertex u adjacent to an (≤ 3)-vertex v.*

Proof Let G be a minimal graph with u and v as in the statement. Let w be the neighbor of u distinct from v. Let $H = G - u + vw$, with the edge vw embedded in the plane where the path wuv was. Note that v is not adjacent to w in G, by Lemmas 2.1 and 2.2, and thus, H is simple. Since G is minimal, H has an entire $(\Delta+4)$-coloring. This induces a partial entire $(\Delta+4)$-coloring of G with only uw, uv, and u uncolored. These elements may be colored in that order to give an entire $(\Delta+4)$-coloring of G, a contradiction. ■

The proofs of the remaining lemmas follow the same vein, but some elements may need recoloring in a little less straightforward manner.

Lemma 2.4 *No minimal graph has a $(4, j, k)$-face with $j \leq k$, and $j + k \leq 2\Delta - 2$.*

Proof Assume that there is a minimal graph G with a face f adjacent to a 4-vertex u, a j-vertex v, and a k-vertex w such that $j + k \leq 2\Delta - 2$. Let u_1, u_2 be the ends of the edges incident with u and not f. Let $v_1,\ldots,v_{j-2}$ and $w_1,\ldots,w_{k-2}$ be defined similarly. Since G is minimal, $G - uv$ has an entire $(\Delta+4)$-coloring. Removing the colors from u and uw induces a partial entire coloring of G such that only u, uv, uw, f are not colored. Color u differently than its at most nine colored neighbors.

Either uw may be immediately colored, or $k = \Delta$, and without loss of generality, $ww_1,\ldots,ww_{\Delta-2}, vw, w, f_{uw}, u, uu_1, uu_2$ are respectively colored $1,\ldots,\Delta+4$. Either u may be recolored, so that uw may be colored $\Delta+2$, or $\Delta = 6$, and v is colored one of $1,\ldots,\Delta-2$. In this case, vw may be recolored (one of $\Delta+1,\ldots,\Delta+4$), and uw may be colored $\Delta-1$.

Either uv may be immediately colored, or $j = k = \Delta-1$, and without loss of generality, $vv_1,\ldots,vv_{\Delta-3}, v, f_{uv}, vw, u, uu_1, uu_2, uw$ are respectively colored $1,\ldots,\Delta+4$. Either uw

may be recolored, so that uv may be colored $\Delta + 4$, or the colors of f_{uw}, w, $ww_1, \ldots, ww_{\Delta-3}$ are $1, \ldots, \Delta - 1$. In this case, vw may be recolored (one of $\Delta + 1, \ldots, \Delta + 3$), and uv may be colored Δ.

Since f may be immediately colored, this contradicts the minimality of G. ■

Lemma 2.5 *No minimal graph has two adjacent* 3*-faces* $f = uvw$ *and* $g = uvx$ *such that* $\deg(u) \leq 5$*, and there is a* $y \in \{u, w\}$ *such that* $\deg(y) = 4$.

Proof Let G be a minimal graph with f, g, u, v, w, x, y as in the statement. Since G is minimal, $G - vy$ has an entire $(\Delta + 4)$-coloring. This induces a partial entire coloring χ of G with only y, vy, and f uncolored. Remove the color from g, and if $w = y$, from uv and uw as well. Color y differently than its at most nine colored adjacent elements. If $w = y$, color vw differently than its at most $\Delta + 3$ colored adjacent elements.

Either uv may be immediately colored, or it has $\Delta + 4$ colored adjacent elements, all of which have distinct colors. Clearly, one of $\{1, \ldots, \Delta + 4\} \setminus \{\chi(ux), \chi(vx)\}$ does not appear among the colors of x, its adjacent edges, or g_{ux}, g_{vx}. Thus, there is an $e \in \{ux, vx\}$ such that e may be recolored so that uv may be colored $\chi(e)$. Color as indicated to give a partial entire $(\Delta + 4)$-coloring ψ.

If $w = y$, then either uw may be immediately colored, or $\Delta = 6$, and uw has ten colored adjacent elements, all of which have distinct colors. At this point, one of $\{1, \ldots, 10\} \setminus \{\psi(uv), \psi(vw)\}$ does not appear among the colors of w, its adjacent edges, of f_{vw}. Thus, there is an $e \in \{uv, vw\}$ to recolor so that uw may be colored $\psi(e)$.

As f and g may be immediately colored, this gives an entire $(\Delta + 4)$-coloring of G, a contradiction. ■

The proof for the final configuration is the most complicated, and yet still quite simple.

Lemma 2.6 *No minimal graph has a* 5*-vertex adjacent to more than three* 3*-faces.*

Proof Assume there is a minimal graph G with a 5-vertex u adjacent to 3-faces $f = uvw$, $g = uwx$, $h = uxy$, and $i = uyz$. Since G is minimal, $G - ux$ has an entire $(\Delta + 4)$-coloring, which induces a partial entire coloring χ of G with u, ux, and g uncolored. Also, remove the colors from f, h, i, and uw. Color u differently from its at most nine colored adjacent elements.

Either uw may be colored immediately, or its $\Delta + 4$ adjacent elements are all colored differently. In this case, there is an $e \in \{uv, vw\}$ to recolor so that uw may be colored $\chi(e)$. This yields a partial entire $(\Delta + 4)$-coloring ψ with only ux, f, g, h, and i uncolored.

Either ux may be colored immediately, or all $\Delta + 4$ colors appear among the colors of its at most $\Delta + 5$ adjacent elements. Suppose $\psi(uy) = \psi(wx)$. In this case, either uw may be recolored so that ux may be colored $\psi(uw)$, or wx may be recolored differently from uy. Thus, without loss of generality, $\psi(uy) \neq \psi(wx)$, and $\psi(uw) \neq \psi(xy)$. It follows that there is a $t \in \{w, y\}$ such that each of $\psi(tu)$ and $\psi(tx)$ appears only once among the colors of the adjacent elements of ux. Thus, there is an $e \in \{tu, tx\}$ to recolor so that ux may be colored $\psi(e)$.

As f, g, h, and i may be immediately colored, this gives an entire $(\Delta + 4)$-coloring of G, a contradiction. ■

3 Discharging

This section uses the discharging method, together with the lemmas of Section 2, to prove that there is no minimal graph. The basis of the discharging method is the following simple lemma, which follows from Euler's formula for connected graphs, that $|V(G)| - |E(G)| + |F(G)| = 2$, and the simple fact that for each $S \in \{V(G), F(G)\}$, $\sum_{x\in S} \deg(x) = 2|E(G)|$.

Lemma 3.1 *For any connected plane graph,*

$$\sum_{x\in V(G)\cup F(G)} \bigl(4 - \deg(x)\bigr) = 8.$$

For $x \in V(G) \cup F(G)$, let $\mathrm{ch}(x) = 4 - \deg(x)$, and let $\mathrm{ch}'(x)$ be defined by modifying the charge function ch according to the following discharging rules:

R1. For each 3-face x, send $1/3$ from x to each (≥ 5)-vertex adjacent to it.
R2. For each 3-face x, if it is adjacent to a (≥ 4)-face y, then for each (≥ 6)-vertex v adjacent to both x and y, send $1/6$ from x to y, and then from y to v (so that this rule does not affect y).
R3. For each 2-vertex x, send 1 from x to each (≥ 5)-face adjacent to it.
R4. For each 3-vertex x, send $1/3$ from x to each (≥ 5)-face adjacent to it.
R5. For each 3-vertex x, if it is adjacent to a 4-face f, then for each (≥ 6)-vertex v adjacent to both x and f, send $1/6$ from x to f, and then from f to v (so that this rule does not affect f).

Now we are ready to prove Theorem 1.1.

Proof Suppose the theorem is false, so that there is a minimal graph G. By Lemma 2.1, G is connected. Let $x \in V(G) \cup F(G)$ be given. In order to contradict Lemma 3.1, we will prove that, no matter what x is, $\mathrm{ch}'(x) \leq 0$.

By Lemma 2.1, x is not an (≤ 1)-vertex.

Assume x is a 2-vertex ($\mathrm{ch}(x) = 2$). By Lemma 2.2, x is not adjacent to an (≤ 4)-face, and so x sends out 1 to each face adjacent to it, no rule sends charge into x, and $\mathrm{ch}'(x) = 0$.

Assume x is a 3-vertex. By Lemma 2.2, x is not adjacent to a 3-face, and if it is adjacent to a 4-face f, there are two (≥ 6)-vertices adjacent to both x and f. Thus, x sends $1/3$ to each face adjacent to it either by R4 or R5, and $\mathrm{ch}'(x) = 0$.

Assume x is a 4-vertex. As the rules do not affect x, $\mathrm{ch}'(x) = \mathrm{ch}(x) = 0$.

Assume x is a 5-vertex. By Lemma 2.6, x is adjacent to at most three 3-faces. The only charge sent into x is then at most 1 from R1, and $\mathrm{ch}'(x) \leq 0$.

Assume x is a k-vertex, for $k \geq 6$. The charge sent into x is at most $1/3$ from each adjacent face, by either R1 or a combination of R2 and/or R5. Thus, $\mathrm{ch}'(x) \leq \mathrm{ch}(x) + k/3 = (12 - 2k)/3 \leq 0$.

Assume x is a 3-face. By Lemma 2.2, x is adjacent to three (≥ 4)-vertices. If x is adjacent to three (≥ 5)-vertices, then x sends out 1 by R1. Otherwise, by Lemma 2.4, x is adjacent to two (≥ 5)-vertices, one of which is an (≥ 6)-vertex. In this case, by Lemma 2.5, x is not adjacent to a 3-face. It follows that x sends out $2/3$ by R1 and $1/3$ by R2. In either case, $\mathrm{ch}'(x) \leq 0$.

Assume x is a 4-face. As the rules do not affect x, $\mathrm{ch}'(x) = \mathrm{ch}(x) = 0$.

Assume x is a 5-face. If x is adjacent to a 2-vertex, then it receives 1 from it by R3, but by Lemmas 2.2 and 2.3, x is adjacent to no other (≤ 3)-vertex, and thus receives no more charge. If x is not adjacent to a 2-vertex, then by Lemma 2.2, x is adjacent to at most three 3-vertices, each of which sends $1/3$ by R4. In either case, $\mathrm{ch}'(x) \leq 0$.

Assume x is a 6-face. If x is adjacent to two 2-vertices, then it receives 2 from them by R3, but by Lemmas 2.2 and 2.3, x is adjacent to no other (≤ 3)-vertices. If x is adjacent to exactly one 2-vertex, then it is adjacent to at most three 3-vertices by Lemma 2.3, and x receives 1 by R3 and at most 1 by R4. If x is adjacent to no 2-vertices, then it receives at most $1/3$ from each of its adjacent 3-vertices by R4, and no more. In any case, $\mathrm{ch}'(x) \leq 0$.

Assume x is a k-face, for $k \geq 7$. For $i \in \{2, 3\}$, let v_i be the number of i-vertices adjacent to x. By R3 and R4, $\mathrm{ch}'(x) = 4 - k + v_2 + (v_3/3)$. If $v_3 = 0$, then by Lemma 2.3 (and since $k \geq 7$), $v_2 \leq k - 4$, and $\mathrm{ch}'(x) \leq 0$ in this case. If $v_3 > 0$, then Lemma 2.3 gives $2v_2 + 1 + v_3 \leq k$, or $(v_3/3) \leq (k - 2v_2 - 1)/3$. Substituting above, $\mathrm{ch}'(x) \leq (11 - 2k + v_2)/3$. Consider $2v_2 + 1 + v_3 \leq k$ again; solving for v_2 and using $v_3 \geq 1$ gives $v_2 \leq \frac{k}{2} - 1$, and substituting that above gives $\mathrm{ch}'(x) \leq (20 - 3k)/6$. Since $k \geq 7$, $\mathrm{ch}'(x) \leq 0$.

Since x was arbitrary, this gives

$$\sum_{x \in V(G) \cup F(G)} \mathrm{ch}'(x) \leq 0.$$

As the charge was only moved around, and the sum of it never changed, this contradicts Lemma 3.1. ■

The class of planar graphs of maximum degree six is an important class of graphs, and it is important to study their structure. Vizing's total coloring conjecture [17] for planar graphs is open only for $\Delta = 6$ [15]. Also, Vizing's planar graph conjecture [18] is open only for $\Delta = 6$ [16]. Hopefully, the results in this paper may provide a starting place for research on at least these two conjectures.

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