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DIVISORS ON VARIETIES OVER A REAL CLOSED FIELD

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ABSTRACT. Let X be a projective nonsingular variety over a real closed field R such that the set $X(R)$ of R-rational points of X is nonempty. Let $\mathrm{Cl}_R(X) = \mathrm{Cl}(X)/\Gamma(X)$, where $\mathrm{Cl}(X)$ is the group of classes of linearly equivalent divisors on X and $\Gamma(X)$ is the subgroup of $\mathrm{Cl}(X)$ consisting of the classes of divisors whose restriction to some neighborhood of $X(R)$ in X is linearly equivalent to 0. It is proved that the group $\mathrm{Cl}_R(X)$ is isomorphic to $(Z/2)^s$ for some non-negative integer s. Moreover, an upper bound on s is given in terms of the $Z/2$-dimension of the group cohomology modules of $\mathrm{Gal}(C/R)$, where $C = R(\sqrt{-1})$, with values in the Néron-Severi group and the Picard variety of $X_C = X \times_R C$.

1. **Introduction.** Let k be a commutative field. Let X be a quasi-projective nonsingular variety over k (that is, X is assumed to be a quasi-projective integral scheme over k, which is smooth over k). We let $\mathrm{Div}(X)$ and $\mathrm{Cl}(X)$ denote the group of (Weil) divisors on X and the group of classes of linearly equivalent divisors on X, respectively. Given a divisor D in $\mathrm{Div}(X)$, let $[D]$ denote its class in $\mathrm{Cl}(X)$. Assume that the set $X(k)$ of k-rational points of X is nonempty and put

$$\mathrm{Cl}_k(X) = \mathrm{Cl}(X)/\Gamma(X),$$

where $\Gamma(X)$ is the subgroup of $\mathrm{Cl}(X)$ consisting of all classes $[D]$ in $\mathrm{Cl}(X)$ such that the restriction of D to some neighborhood $X(k)$ in X is linearly equivalent to 0.

Throughout the remaining part of this note R stands for a fixed *real closed field*. Our first result is as follows.

THEOREM 1. *Let X be a quasi-projective nonsingular variety over R with $X(R)$ nonempty. Then the group $\mathrm{Cl}_R(X)$ is isomorphic to $(\mathbf{Z}/2)^s$ for some nonnegative integer s.*

This result is of interest since, in general, the group $\mathrm{Cl}(X)$ is not even finitely generated. For example, this is the case when X is an affine or projective cubic curve over $R = \mathbf{R}$. Let us also mention that $X(R) \neq \emptyset$ implies density of $X(R)$ in X (*cf.* for example [1]).

REMARK. If in Theorem 1, X is projective and $R = \mathbf{R}$, then a more precise result is known. Namely, there exists a canonical monomorphism

$$\phi\colon \mathrm{Cl}_{\mathbf{R}}(X) \longrightarrow H^1\big(X(\mathbf{R}), \mathbf{Z}/2\big)$$

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(*cf.* [3] or [2, Definition 11.3.2, Corollary 12.4.7]). Here $X(\mathbf{R})$ is equipped with the metric topology and $H^1(-,\mathbf{Z}/2)$ stands for the first cohomology group with coefficients in $\mathbf{Z}/2$. The above statement follows also easily from [7] and [13, Theorem 2.2], which concern vector bundles.

In case of an arbitrary real closed field R, we still have the cohomology group $H^1\big(X(R),\mathbf{Z}/2\big)$ suitably defined (*cf.* [2, 6]). This group is, as in the classical case $R = \mathbf{R}$, a finite-dimensional $\mathbf{Z}/2$-vector space. Moreover, one can easily define a canonical homomorphism $\phi_R\colon \mathrm{Cl}_R(X) \longrightarrow H^1\big(X(R),\mathbf{Z}/2\big)$, which coincides with the monomorphism ϕ for $R = \mathbf{R}$. Using Witt's theorem [9], one can show that ϕ_R is a monomorphism if $\dim X = 1$. However, in higher dimensions it is not known whether ϕ_R is injective. For $R = \mathbf{R}$, injectivity is proved by applying the approximation theorem of Weierstrass. ∎

Theorem 1 is an easy consequence of Theorem 2, stated in Section 2 and proved in Section 3. Section 4 deals with the Picard group of some R-algebras and is based on Theorem 1.

2. **The main theorem.** Let X be a projective nonsingular variety over R with $X(R)$ nonempty. Let C denote the algebraic closure of R, that is, $C = R(\sqrt{-1})$. Then $X_C = X \times_R C$ is a nonsingular variety over C. The Galois group $G = \{1,\sigma\}$ of C over R acts on $\mathrm{Div}(X_C)$ as follows. Let $\sigma_X\colon X_C \longrightarrow X_C$ be the involution corresponding to σ. Given $D = \sum k_i D_i$ in $\mathrm{Div}(X_C)$, where the k_i are integers and D_i are prime divisors, one sets $D^\sigma = \sum k_i \sigma_X(D_i)$. This action induces actions of G on $\mathrm{Cl}(X_C)$ and the Néron-Severi group $\mathrm{NS}(X_C)$ of X_C. Thus $\mathrm{Div}(X_C)$, $\mathrm{Cl}(X_C)$ and $\mathrm{NS}(X_C)$ can be regarded as G-modules. If P is the Picard variety of X, then $P(C) = \mathrm{Mor}_R(\mathrm{Spec}\, C, P)$ is also a G-module.

Recall that if M is a (right) G-module, then the second cohomology group $H^2(G,M)$ is the $\mathbf{Z}/2$-vector space defined by

$$H^2(G,M) = M^G/\{m + m^\sigma \mid m \in M\},$$

where m^σ is the image of m under the action of σ and $M^G = \{m \in M \mid m^\sigma = m\}$.

We can now state our main result.

THEOREM 2. *Let X be a projective nonsingular variety over R with $X(R)$ nonempty. Then the group $\mathrm{Cl}_R(X)$ is isomorphic to $(\mathbf{Z}/2)^s$ for some nonnegative integer s. Moreover, $H^2\big(G,\mathrm{NS}(X_C)\big)$ and $H^2\big(G,P(C)\big)$, where P is the Picard variety of X, are finite-dimensional $\mathbf{Z}/2$-vector spaces and*

$$s \leq \dim_{\mathbf{Z}/2} H^2\big(G,\mathrm{NS}(X_C)\big) + \dim_{\mathbf{Z}/2} H^2\big(G,P(C)\big).$$

We should mention that Theorem 2 with $R = \mathbf{R}$ is related to [12, p. 58]. A proof of Theorem 2 will be postponed to Section 3. Here we show only how to derive Theorem 1 from Theorem 2.

PROOF OF THEOREM 1. By Hironaka's resolution of singularities theorem [8], we may assume that X is an open subvariety of some projective nonsingular variety Y over

R. Clearly, the inclusion morphism $X \hookrightarrow Y$ induces an epimorphism $\mathrm{Cl}_R(Y) \longrightarrow \mathrm{Cl}_R(X)$ and hence Theorem 1 follows from Theorem 2. ∎

3. **Proof of the main theorem.** We begin with some preliminary results.

LEMMA 1. *Let X be a quasi-projective variety over R with $X(R)$ nonempty. Let N be a neighborhood of $X(R)$ in X. Then there exists an affine neighborhood U of $X(R)$ in N.*

PROOF. We may assume that X is a locally closed subvariety of projective space $\mathbf{P}^n_R$ for some n. Let Y be the closure of X in $\mathbf{P}^n_R$. Then N can be written as $N = Y \setminus V(H_1, \ldots, H_k)$, where $H_1, \ldots, H_k$ are homogeneous polynomials in $R[X_0, \ldots, X_n]$ and $V(H_1, \ldots, H_k)$ denotes the closed subspace of $\mathbf{P}^n_R$ determined by the zeros of the H_i, $1 \leq i \leq k$. Select nonnegative integers $d_1, \ldots, d_k$ such that

$$H = \sum_{i=1}^{k} (X_0^2 + \cdots + X_n^2)^{d_i} H_i^2$$

is a homogeneous polynomial. By construction, $U = Y \setminus V(H)$ is a neighborhood of $X(R)$ in N. It is obvious that U is affine. ∎

Recall that R (being real closed) is an ordered field and the order on R is uniquely determined. The open intervals $(a,b) = \{x \in R \mid a < x < b\}$, with $a, b \in R$, $a < b$, form a base of open sets of a topology on R, called the *order topology*.

Let X be a quasi-projective variety over R with $X(R)$ nonempty. Suppose that X is a locally closed subvariety of $\mathbf{P}^n_R$ for some n. Then $X(R)$ is a semi-algebraic subset of $\mathbf{P}^n_R(R)$. The order topology on R determines a topology on $\mathbf{P}^n_R(R)$, which in turn induces a topology on $X(R)$. This topology on $X(R)$ is called the *order topology*. Recall that $X(R)$ can be written as $X(R) = S_1 \cup \cdots \cup S_k$, where the S_i are pairwise disjoint semi-algebraic subsets of $X(R)$, which are open and closed in the order topology on $X(R)$, and S_i cannot be represented as a union of two semi-algebraic, closed, disjoint, nonempty subsets. Moreover, the S_i are uniquely determined up to permutation. They are called the *semi-algebraic connected components* of $X(R)$. The above constructions do not depend on the choice of the embedding of X in $\mathbf{P}^n_R$. All these facts, and others which will be used in the proof of Lemma 2 below, can be found in [2] [4] [5].

LEMMA 2. *Let A be an abelian variety over R. Let c be the number of semi-algebraic connected components of $A(R)$. Then considering $A(C)$ as a G-module and setting $2A(R) = \{x + x \mid x \in A(R)\}$, one has*

$$\dim_{\mathbf{Z}/2} H^2\big(G, A(C)\big) \leq \dim_{\mathbf{Z}/2} A(R)/2A(R)$$
$$\text{order}\,\big(A(R)/2A(R)\big) \leq c.$$

PROOF. The first inequality is obvious by virtue of the definition of $H^2(G, -)$. Below we prove the second inequality.

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Since $A(R)$ is nonempty, it follows that $A(R)$ is dense in A (*cf.* for example [1]). Hence $2A(R) = 2_A\big(A(R)\big)$, where $2_A\colon A \longrightarrow A$ is the isogeny multiplication by 2, is also dense in A. By a theorem of Seidenberg and Tarski [2], $2A(R)$ is a semi-algebraic subset of $A(R)$. The last two facts imply that $2A(R)$ has a nonempty interior in the order topology on $A(R)$ (*cf.* [2, Proposition 2.8.12]) and hence, using translations on $A(R)$, one easily sees that $2A(R)$ is open in the order topology on $A(R)$. By [2, Theorem 2.5.8], $2A(R)$ is also closed in the order topology on $A(R)$.

Let S be a semi-algebraic connected component of $A(R)$. Let x be a point in $A(R)$ and let $f_x\colon A(R) \longrightarrow A(R)$ be the mapping defined by $f_x(y) = y - x$ for y in $A(R)$. It follows from the properties of $2A(R)$ discussed above that the set

$$S_x = S \cap f_x^{-1}\big(2A(R)\big) = \{y \in S \mid y - x \in 2A(R)\}$$

is semi-algebraic, and open and closed in the order topology on $A(R)$. Thus $S = S_x$, which shows that

$$\operatorname{order}\big(A(R)/2A(R)\big) \leq c. \qquad \blacksquare$$

PROOF OF THEOREM 2. The short exact sequence of groups

$$0 \longrightarrow P(C) \longrightarrow \mathrm{Cl}(X_C) \longrightarrow \mathrm{NS}(X_C) \longrightarrow 0$$

gives rise to an exact sequence of $\mathbf{Z}/2$-vector spaces

$$H^2\big(G, P(C)\big) \longrightarrow H^2\big(G, \mathrm{Cl}(X_C)\big) \longrightarrow H^2\big(G, \mathrm{NS}(X_C)\big)$$

and hence

$$\dim_{\mathbf{Z}/2} H^2\big(G, \mathrm{Cl}(X_C)\big) \leq \dim_{\mathbf{Z}/2} H^2\big(G, \big(\mathrm{NS}(X_C)\big) + \dim_{\mathbf{Z}/2} H^2\big(G, P(C)\big).$$

Note that $\dim_{\mathbf{Z}/2} H^2\big(G, \mathrm{NS}(X_C)\big) < \infty$, the Néron-Severi group $\mathrm{NS}(X_C)$ being finitely generated [10]. Moreover, by Lemma 2, $\dim_{\mathbf{Z}/2} H^2\big(G, P(C)\big) < \infty$. Thus in order to complete the proof of Theorem 2, it suffices to find an epimorphism of $H^2\big(G, \mathrm{Cl}(X_C)\big)$ onto $\mathrm{Cl}_R(X)$ or, equivalently, to construct an epimorphism

$$\phi\colon \mathrm{Cl}(X_C)^G \longrightarrow \mathrm{Cl}_R(X)$$

such that

$$\phi([D + D^\sigma]) = 0 \tag{1}$$

for all D in $\mathrm{Div}(X_C)$.

We proceed as follows. First recall that the canonical projection $\pi\colon X_C = X \times_R C \longrightarrow X$ induces a monomorphism $\pi^*\colon \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(X_C)$, whose image is equal to $\mathrm{Cl}(X_C)^G$ (*cf.* [11, V. 20]). We define $\phi\colon \mathrm{Cl}(X_C)^G \longrightarrow \mathrm{Cl}_R(X)$ to be the composition of $(\pi^*)^{-1}\colon \mathrm{Cl}(X_C)^G \longrightarrow \mathrm{Cl}(X)$ and the canonical projection $\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}_R(X) = \mathrm{Cl}(X)/\Gamma(X)$ (*cf.* Section 1). By construction, ϕ is an epimorphism. Now it remains to prove (1), where without any loss

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of generality we may assume that D is a prime divisor. We precede the proof of (1) by some preliminary remarks.

Recall that X_C endowed with its canonical descent datum relative to C/R can be identified with X (*cf.* [11, V. 20]). Let $\sigma_X: X_C \longrightarrow X_C$ be the involution corresponding to σ in G. We regard $X(C) = \mathrm{Mor}_R(\mathrm{Spec}\, C, X)$ as the set of closed points in X_C. Then $X(C)^G = \{x \in X(C) \mid \sigma_X(x) = x\}$ corresponds to the subset $X(R)$ of X. In particular, by Lemma 1, for each neighborhood N of $X(C)^G$ in X_C, there exists an affine neighborhood U of $X(C)^G$ in N such that $\sigma_X(U) = U$ (observe that $N \cap \sigma_X(N)$ is a neighborhood of $X(C)^G$).

Let O be the structure sheaf of X_C. Given an open subset V of X_C, we identify elements of $O(V)$ with morphisms from V into affine line $\mathbf{A}^1_C$. If f is an element of $O(V)$, then f^σ denotes the element of $O\big(\sigma_X(V)\big)$ defined by $f^\sigma = \sigma_1 \circ f \circ \big(\sigma_X | \sigma_X(V)\big)$, where $\sigma_1: \mathbf{A}^1_C \longrightarrow \mathbf{A}^1_C$ is the involution corresponding to σ. Observe that if $\sigma_X(V) = V$ and $f = f^\sigma$, then $f(x)$ is in R for all x in $V \cap X(C)^G$, where R is considered as a subset of $\mathbf{A}^1_C(C) = C$. Furthermore, if $\sigma_X(V) = V$ and g is any element of $O(V)$, then $(gg^\sigma)(x) \geq 0$ for all x in $V \cap X(C)^G$.

Let us now return to the proof of (1). One can find affine open sets V_i and elements f_i in $O(V_i)$, $1 \leq i \leq k$, such that $X(C)^G$ is contained in $M = V_1 \cup \cdots \cup V_k$ and $D = (f_i)$ as divisors on V_i. Let U be an affine neighborhood of $X(C)^G$ in M and let $U_i = U \cap V_i \cap \sigma_X(V_i)$ for $1 \leq i \leq k$. Then the U_i form an open cover of U and $\sigma(U_i) = U_i$. Since U and the U_i are affine, one can find g_i in $O(U)$ such that $D = (g_i)$ as divisors on U_i and $g_j = \alpha_{ij} g_i$ for some α_{ij} in $O(U_i)$, $1 \leq i \leq k$, $1 \leq j \leq k$. Note that

(2) $$D + D^\sigma = (g_i g_i^\sigma) \text{ as divisors on } U_i.$$

We claim that if h is the element of $O(U)$ defined by

(3) $$h = \sum_{i=1}^{k} g_i g_i^\sigma,$$

then there is a neighborhood U' of $X(C)^G$ in U such that

(4) $$\sigma_X(U') = U' \text{ and } D + D^\sigma = (h) \text{ as divisors on } U'.$$

Indeed, let x be a point in $X(C)^G$. Then x is in U_i for some i, $1 \leq i \leq k$. By renaming the indices, we may assume that $i = 1$. Then putting $\alpha_j = \alpha_{ij}$, we have $g_j = \alpha_j g_1$ on U_1, and substituting into (3), we obtain

(5) $$h = g_1 g_1^\sigma + \sum_{j=2}^{k} g_j g_j^\sigma = g_1 g_1^\sigma \Big(1 + \sum_{j=2}^{k} \alpha_j \alpha_j^\sigma\Big) \text{ on } U_1.$$

Since $(\alpha_j \alpha_j^\sigma)(x) \geq 0$ in R for $2 \leq j \leq k$, it follows that

$$1 + \sum_{j=2}^{k} \alpha_j \alpha_j^\sigma$$

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is an invertible element in the stalk $\mathcal{O}_x$. Hence, by virtue of (5), $(h) = (g_1 g_1^\sigma)$ as divisors on some neighborhood of x in U. Applying (2), we see that (4) follows.

Since $h = h^\sigma$, it follows from (4) that (1) holds, which completes the proof of Theorem 2. ∎

4. **The Picard group of some algebras over R.** Let A be a finitely generated R-algebra with no zero divisors. Assume that the set $\mathrm{Max}_R(A)$ of maximal ideals of A with residue field R is nonempty, and that the localization of A with respect to every maximal ideal in $\mathrm{Max}_R(A)$ is a regular local ring. Let A_R denote the localization of A with respect to the multiplicatively closed subset consisting of all elements in A not contained in any maximal ideal in $\mathrm{Max}_R(A)$.

THEOREM 3. *With the notation as above, the Picard group* $\mathrm{Pic}(A_R)$ *of* A_R *is isomorphic to* $(\mathbf{Z}/2)^s$ *for some nonnegative integer* s.

PROOF. Let $Y = \mathrm{Spec}\, A$. Observe that there is a neighborhood X of $Y(R)$ in Y, which is a nonsingular variety over R. Hence, by Theorem 1, $\mathrm{Cl}_R(X)$ is isomorphic to $(\mathbf{Z}/2)^s$ for some nonnegative integer s.

Consider the ring $\mathcal{R}(X)$ defined by

$$\mathcal{R}(X) = \lim \operatorname{inj} \mathcal{O}_X(U),$$

where $\mathcal{O}_X$ is the structure sheaf of X and U runs through the set of all affine neighborhoods of $X(R) = Y(R)$ in X (*cf.* Lemma 1). One easily sees that $\mathcal{R}(X)$ is canonically isomorphic to A_R. Moreover, since $\mathrm{Pic}\big(\mathcal{O}(U)\big)$ is canonically isomorphic to $\mathrm{Cl}(U)$, U being affine, one obtains that $\mathrm{Pic}\big(\mathcal{R}(X)\big)$ is isomorphic to $\mathrm{Cl}_R(X)$. Thus the proof is complete. ∎

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