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ON THE QUADRATIC RESIDUES (MOD p) IN THE INTERVAL $(0, p/4)$

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ABSTRACT. A short proof is given of a result of Burde giving the parity of the number of quadratic residues (mod p) in the interval $(0, p/4)$, where $p \equiv 1 \pmod 4$ is prime.

Let $p \equiv 1 \pmod 4$ be a prime. We define (unique) integers a and b by

$$(1) \qquad p = a^2 + b^2, \qquad a \equiv 1 \pmod 4, \qquad \mathrm{b} \equiv \left(\frac{p-1}{2}\right)!\, a \pmod p.$$

Clearly we have

$$(2)(\mathrm{i}) \qquad p \equiv 2a - 1 \pmod{16}, \qquad b \equiv 0 \pmod 4, \quad \text{if} \quad p \equiv 1 \pmod 8,$$

and

$$(2)(\mathrm{ii}) \qquad p \equiv 2a + 3 \pmod{16}, \qquad b \equiv (\mathrm{mod}\ 4), \quad \text{if} \quad p \equiv 5 \pmod 8.$$

Let $N(p)$ denote the number of quadratic residues (mod p) in the interval $(0, p/4)$. Burde [2: Theorems 1 and 2] has shown (with slightly different notation) that

$$(3)(\mathrm{i}) \qquad N(p) \equiv 0 \pmod 2 \Leftrightarrow b \equiv 0 \pmod 8, \quad \text{if} \quad p \equiv 1 \pmod 8,$$

and

$$(3)(\mathrm{ii}) \qquad N(p) \equiv 0 \pmod 2 \Leftrightarrow b \equiv 6 \pmod 8, \quad \text{if} \quad p \equiv 5 \pmod 8.$$

We give a very short proof of this result. We have

$$(4) \qquad N(p) = \tfrac{1}{2} \sum_{0<k<p/4} \left(1 + \left(\frac{k}{p}\right)\right).$$

Now, by a result of Dirichlet [4: p. 152] (or see [3: p. 101]), we have

$$(5) \qquad \sum_{0<k<p/4} \left(\frac{k}{p}\right) = \tfrac{1}{2} h(-4p),$$

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where $h(-4p)$ denotes the class number of the imaginary quadratic field $Q(\sqrt{-p})$ (of discriminant $-4p$). Hence, by (4) and (5), we have

$$(6) \qquad 8N(p) = p - 1 + 2h(-4p).$$

Now Gauss [5: p. 380] (see also Yamamoto [6: Lemma 3], Barkan [1: p. 828]) (Note: Gauss's k is related to $h(-4p)$ by $2k = h(-4p)$.) has shown that

$$(7) \qquad h(-4p) \equiv -a + b + 1 (\text{mod } 8),$$

so by (6) and (7) we have

$$(8) \qquad 8N(p) \equiv p - 2a + 2b + 1 \ (\text{mod } 16).$$

Hence, from (2) and (8), we obtain

$$4N(p) \equiv \begin{cases} b \ (\text{mod } 8), & \text{if} \quad p \equiv 1 \ (\text{mod } 8), \\ b + 2 \ (\text{mod } 8), & \text{if} \quad p \equiv 5 \ (\text{mod } 8), \end{cases}$$

which completes the proof of Burde's result.

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