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On the optimal control of a manufacturing firm

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Various existing models of the optimal control of production rates of manufacturing firms are discussed. A new model is derived by considering the combined effects of: the inventory level of the firm, the shipment sent from the firm, the shipment rate, the orders received by the firm, the demand rate, the rate of change of the demand rate, the production rate, the advertising expenditure, and the level of unfilled orders. Further, a new version of shortage cost is introduced. The well-known Pontryagin maximum principle and transversality condition are used to obtain the optimal production rate and the optimal advertising expenditure. A numerical example is given for illustration.

1. Introduction

In dealing with optimal control of production rate of a manufacturing firm, the demand rate of products has often been taken as an exogenous variable [1], [2], [3]. However, this is not so as the demand rate of products may be influenced by many economic factors such as supply situation, price, purchasing power, advertising, and so on. Although the demand rate has also been reported [2], [3] to be related with advertising, this effect has never been combined when dealing with optimal control of production rate.

In this paper a new model of a manufacturing firm is derived by considering the combined effects of: the inventory level of the firm, the

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shipment sent from the firm, the shipment rate, the orders received by the firm, the demand rate, the rate of change of the demand rate, the production, the production rate, the advertising expenditure, and the level of unfilled orders. Regarding other economic factors to be constant, the demand rate is assumed to be related to the advertising done in the previous time period rather than just taken as an exogenous variable.

The expression of shortage cost used by Connors, Teichroew [2] is

$$C_s \int_0^T [\dot{D}(t)-u_1(t)]dt \ , \tag{1}$$

where C_s is the shortage cost per unit of products, T is a preassigned time, $\dot{D}(t)$ the demand rate of products at time t, and $u_1(t)$ the production rate at time t. Since the meaning of this expression disappears when $\dot{D}(t) < u_1(t)$, a modified version of shortage cost is introduced.

To construct our model for a manufacturing firm over a given planned period $[0, T]$, it is assumed that the actual inventory level at time t, $I(t)$, is equal to the production up to the time t, $P(t)$, minus the shipment up to the time t, $S(t)$, plus I_1. That is

$$I(t) = P(t) - S(t) + I_1 \ , \tag{2}$$

where $I_1 = I_0 + S_0$, $I_0 = I(0)$, and $S_0 = S(0)$ is the initial shipment sent from the firm. Further, let

$$\dot{P}(t) = u_1(t) \tag{3}$$

where "$\cdot$" denotes the differential operator d/dt and $u_1(t)$ is clearly the production rate at time t.

It is also assumed that the shipment rate of products at time t, $\dot{S}(t)$, is related to the level of unfilled orders at time t, $U(t)$, by a first-order exponential lag with time-constant α $(\alpha > \frac{1}{4})$, namely

$$\dot{S}(t) = \frac{1}{\alpha} \cdot \nu \cdot \int_0^\infty U(t-\tau)e^{-\tau/\alpha} \cdot d\tau \ , \tag{4}$$

where ν is the desired ratio of the shipment rate and the level of unfilled orders determined by the firm.

Letting

(5) $$\dot{S}(t) = S_1(t)$$

and making a change of variable $y = t - \tau$, we obtain

(6) $$\dot{S}_1(t) = -\frac{1}{\alpha}\left[S_1(t) - \nu \cdot U(t)\right] .$$

The direct interpretation of equation (6) is that for any given level of unfilled orders, there is a corresponding desired shipment rate $\nu \cdot U$ determined by the firm, and that the rate of change of shipment rate is proportional to the excess of the desired shipment rate over the actual shipment rate.

It is further assumed that

(7) $$U(t) = D(t) - S(t) ,$$

where $U(t)$ and $S(t)$ are as defined before and $D(t)$ the order for products received by the firm up to time t .

Now let us write

(8) $$\dot{D}(t) = D_1(t) ,$$

where $D_1(t)$ is clearly the demand rate of products at time t .

Differentiating equation (7) with respect to t and noting that $\dot{S}(t) = S_1(t)$, we have

(9) $$\dot{U}(t) = D_1(t) - S_1(t) .$$

Finally, the effect of the advertisement on the demand rate is assumed to be given by the following equation as considered by Connors, Teichroew [2];

(10) $$\frac{d}{dt}\left[D_1(t)\right] = -\lambda D_1(t) + \gamma \int_0^\infty A(t-\tau) \cdot e^{-\tau} \cdot d\tau ,$$

where $A(t)$ is the advertising expenditure at time t .

Equation (10) assumes that if the firm does no advertising, the demand

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rate of its products at any point in time will decrease at a rate proportional to the demand rate of products at that time. It is also assumed, however, that the demand rate at any point in time increases in proportion to the advertising done in the previous time period.

Differentiating equation (10) with respect to t and using the change of variables, $y = t - \tau$, equation (10) may be rewritten as

(11) $$\frac{d^2D_1(t)}{dt^2} + (1+\lambda)\frac{dD_1(t)}{dt} + \lambda D_1(t) = \gamma \cdot A(t) .$$

Letting

(12) $$D_2(t) = \dot{D}_1(t) ,$$

equation (11) can be written as

(13) $$\dot{D}_2(t) = -\lambda D_1(t) - (1+\lambda)\cdot D_2(t) + \gamma \cdot A(t) .$$

Thus the present model comprises equations (3), (5), (6), (8), (9), (12), and (13) and determines the behaviour of the endogenous variables $P(t)$, $S(t)$, $S_1(t)$, $D(t)$, $U(t)$, $D_1(t)$, and $D_2(t)$ for any assumed behaviour of the exogenous variables $u_1(t)$ and $A(t)$. This model can be controlled through the appropriate choice of variables u_1 (production rate) and A (advertisement decision). For a given production capacity of the firm, the production rate (units of products per week, say) may be increased from normal to a maximum by assigning appropriate amount of overtime.

Assume that the normal working time per week is 40 hours, and that the corresponding production rate is a certain amount of units of products per week. Then the production rate can be increased by granting overtime work. Let p_l be the normal production rate (without any overtime) and p_u the maximum working hours allowed per week for the available facilities of the firm. Thus the natural constraint of the control variable u_1 is such that

$$p_l \leq u_1 \leq p_u .$$

Since the advertising decision can be controlled completely by the firm, it will be considered as the second control variable. This control variable is denoted by u_2 . Let the lower and upper bounds of the amount allocated to advertising by the firm be a_l and a_u , respectively. u_2 can then be chosen to satisfy the inequality

$$a_l \le u_2 \le a_u .$$

For convenience of further references, M denotes the class of all piecewise continuous 2-dimensional vector-valued functions $u = (u_1, u_2)$ satisfying the following inequalities

$$p_l \le u_1 \le p_u$$

and

$$a_l \le u_2 \le a_u .$$

Incorporating the control variables u_1 and u_2 as described above, the model of the manufacturing firm is given by the following system of differential equations:

$$\text{(14)} \qquad \begin{cases} \dfrac{dP(t)}{dt} = u_1(t) , \\ \dfrac{dS(t)}{dt} = S_1(t) , \\ \dfrac{dS_1(t)}{dt} = -\dfrac{1}{\alpha}\left[S_1(t)-\nu\cdot U(t)\right] , \\ \dfrac{dD(t)}{dt} = D_1(t) , \\ \dfrac{dU(t)}{dt} = D_1(t) - S_1(t) , \\ \dfrac{dD_1(t)}{dt} = D_2(t) , \\ \dfrac{dD_2(t)}{dt} = -\lambda D_1(t) - (1+\lambda)D_2(t) + \gamma u_2(t) . \end{cases}$$

2. Statement of the problem

The financial success of a firm can be measured by the profit it makes. For profit maximizing, the problem of the manufacturing firm may be stated as: subject to the dynamic constraint (14) with the initial

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condition

(15) $x_0 = \{P(0) = 0,\ S(0) = S_0,\ S_1(0) = S_{1_0},\ D(0) = D_0,$

$$U(0) = D_0 - S_0,\ D_1(0) = D_{1_0},\ D_2(0) = D_{2_0}\} ,$$

find a control vector $u = (u_1, u_2) \in M$ over the planned period $[0, T]$ that will maximize the profit functional

(16) $$J(u) = C_1 \int_0^T D_1(t)dt - C_2 \int_0^T [u_1(t) - p_l]dt$$

$$- C_3 \int_0^T I(t)dt - C_4 \int_0^T \max\{[D(t) - (P(t) + I_0)], 0\}dt - C_5 \int_0^T u_2(t)dt ,$$

where

C_1 = revenue per unit of product,

C_2 = overtime production cost per unit of product,

C_3 = inventory holding cost per unit of product for per unit of time,

C_4 = shortage cost per unit of product,

C_5 = advertising cost per unit of time,

I_0 = the initial inventory level,

and

$$I = P - S + I_0 + S_0 .$$

3. Determination of optimal production rate and optimal advertising decision

As it has been previously stated that the production rate and the advertising decision can be controlled by the firm, one may now ask: "What is the best policy the firm shall follow so that it can maximize its profit over the planned period?" This problem can be solved by applying the well-known Pontryagin Maximum Principle and Transversality conditions [5]. Noting that

(17) $$\max J(u) = \min\{-J(u)\} ,$$

(18) $$H = C_1 D_1 - C_2\left(u_1 - p_l\right) - C_3(p-s) - C_4 \max\left\{D - \left(P + I_0\right), 0\right\}$$
$$- C_5 u_2 + u_1 \psi_1 + S_1 \psi_2 - \frac{1}{\alpha}\left(S_1 - \nu \cdot U\right)\psi_3$$
$$+ D_1 \psi_4 + \left(D_1 - S_1\right)\psi_5 + D_2 \psi_6 - \lambda D_1 \psi_7 - (1+\lambda) D_2 \psi_7 + \gamma u_2(t) \psi_7 ,$$

where H is the hamiltonian function and the vector

$$\psi = \left(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7\right)$$

is the solution of the following system, adjoint to the system (14),

(19) $$\begin{cases} \dfrac{d\psi_1(t)}{dt} = +C_3 + C_4 \cdot f(t) , \\ \dfrac{d\psi_2(t)}{dt} = -C_3 , \\ \dfrac{d\psi_3(t)}{dt} = -\psi_2(t) + \dfrac{1}{\alpha}\psi_3(t) + \psi_5(t) , \\ \dfrac{d\psi_4(t)}{dt} = C_4 \cdot g(t) , \\ \dfrac{d\psi_5(t)}{dt} = -\dfrac{\nu}{\alpha}\psi_3(t) , \\ \dfrac{d\psi_6(t)}{dt} = -C_1 - \psi_4(t) - \psi_5(t) + \lambda\psi_7(t) , \\ \dfrac{d\psi_7(t)}{dt} = -\psi_6(t) + (1+\lambda)\psi_7(t) , \end{cases}$$

with

$$f(t) = \begin{cases} -1 & \text{if } D(t) > P(t) + I_0 , \\ 0 & \text{if } D(t) \le P(t) + I_0 , \end{cases}$$

and

$$g(t) = \begin{cases} 1 & \text{if } D(t) > P(t) + I_0 , \\ 0 & \text{if } D(t) \le P(t) + I_0 . \end{cases}$$

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Maximizing the hamiltonian function H globally with respect to u_1 and u_2, and denoting them by u_1^* and u_2^*, we obtain

$$u_1^*(t) = \begin{cases} p_l & \text{if } -C_2 + \psi_1(t) < 0 , \\ p_u & \text{if } -C_2 + \psi_1(t) > 0 , \\ \text{undetermined if} & -C_2 + \psi_1(t) = 0 \end{cases} \tag{20}$$

and

$$u_2^*(t) = \begin{cases} a_l & \text{if } -C_5 + \gamma\psi_7(t) < 0 , \\ a_u & \text{if } -C_5 + \gamma\psi_7(t) > 0 , \\ \text{undetermined if} & -C_5 + \gamma\psi_7(t) = 0 . \end{cases} \tag{21}$$

Since the problem under consideration is a free-end-point problem, it follows from the Transversality Condition that the boundary condition for the adjoint system (19) is

$$\begin{aligned} &\psi_1(T) = 0 , \quad \psi_2(T) = 0 , \quad \psi_3(T) = 0 , \quad \psi_4(T) = 0 , \\ &\psi_5(T) = 0 , \quad \psi_6(T) = 0 , \quad \psi_7(T) = 0 . \end{aligned} \tag{22}$$

Incorporating the control variables u_1^* and u_2^* into the system equation (14) and its adjoint system equation (19), we obtain a two-point-boundary-value-problem. This two-point-boundary-value-problem consists of the following system (23) with the boundary condition (24) and the adjoint system (19) with the boundary condition (22).

$$(23)\quad \begin{cases} \dfrac{dP(t)}{dt} = \dfrac{p_l}{2}\left[1-\operatorname{sign}\left(-C_2+\psi_1(t)\right)\right] + \dfrac{p_u}{2}\left[1+\operatorname{sign}\left(-C_2+\psi_1(t)\right)\right] , \\ \dfrac{dS(t)}{dt} = S_1(t) , \\ \dfrac{dS_1(t)}{dt} = -\dfrac{1}{\alpha}\left[S(t)-\nu . U(t)\right] , \\ \dfrac{dD(t)}{dt} = D_1(t) , \\ \dfrac{dU(t)}{dt} = D_1(t) - S_1(t) , \\ \dfrac{dD_1(t)}{dt} = D_2(t) , \\ \dfrac{dD_2(t)}{dt} = -\lambda \cdot D_1(t) - (1+\lambda)D_2(t) + \dfrac{\gamma \cdot a_l}{2}\left[1-\operatorname{sign}\left(-C_5+\gamma\cdot\psi_7(t)\right)\right] \\ \qquad\qquad + \dfrac{\gamma\cdot a_u}{2}\left[1+\operatorname{sign}\left(-C_5+\gamma\cdot\psi_7(t)\right)\right] , \end{cases}$$

with the boundary conditions

$$(24)\quad \begin{matrix} P_0(0) = 0 , \quad S(0) = S_0 , \quad S_1(0) = S_{1_0} , \quad D(0) = D_0 , \\ U(0) = D_0 - S_0 , \quad D_1(0) = D_{1_0} , \quad D_2(0) = D_{2_0} . \end{matrix}$$

The optimality problem is now reduced to the problem of solving the above two-point-boundary-value-problem. It is clear that there are seven initial conditions and seven terminal conditions in the problem which cannot be solved by ordinary methods of integration of differential equations. The problem can be solved by using the Davidson-Fletcher-Powell method [4] together with the Fibonacci Search Technique. The flow chart of this method can be found in [7]. The numerical results presented in the next section are computed by using Subroutine BVP [6].

4. Computational results

The numerical values given to the coefficients and parameters of the problem are:

$$C_1 = 20.0 , \quad C_2 = 3.0 , \quad C_3 = 1.0 , \quad C_4 = 5.0 , \quad C_5 = 2.0 ,$$

$$\alpha = 0.4 , \quad \gamma = 0.22 , \quad \lambda = 0.05 , \quad \nu = 1.0 , \quad I_0 = 0.7 .$$

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Initial State:

$$x_0 \trianglelefteq \{P_0 = 0.0,\ S_0 = 1.3,\ S_{1_0} = 0.5,\ D_0 = 1.5,$$

$$U_0 = 0.2,\ D_{1_0} = 0.05,\ D_{2_0} = 0.001\} ;$$

Control Constraints:

$$0.1 \le u_1 \le 0.45 ,$$

$$1.2 \le u_2 \le 2.0 ;$$

Time: $[0, 5]$;

Initial guess for $\psi^0 \trianglelefteq \left[\psi_1^0, \psi_2^0, \ldots, \psi_7^0\right]$	$E(\psi^0)$ Mean Squared Error	Value of ψ^0 obtained by BVP [6] for best error	$E(\psi^0)$ Best Mean Squared Error
1.0		2.000×10^1	
1.0		5.000	
1.0		4.010×10^{-1}	
1.0	8.176×10^5	-2.500×10^1	2.748×10^{-10}
1.0		3.999	
1.0		-4.032×10^1	
1.0		3.747×10^1	

Profit $J(u)$: 3.012×10^1

The optimal control policies and optimal trajectories for the above example are shown in Figures 1 and 2 respectively (pages 121 and 122).

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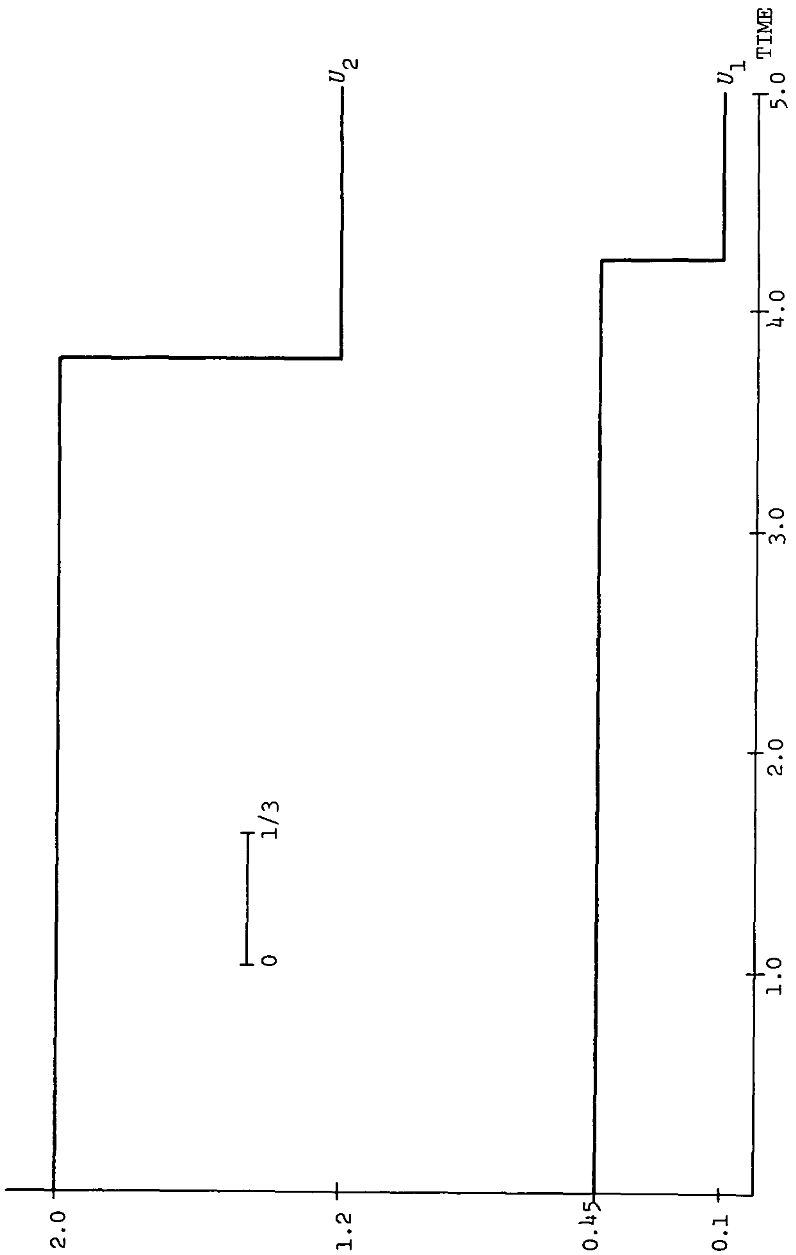

FIGURE 1

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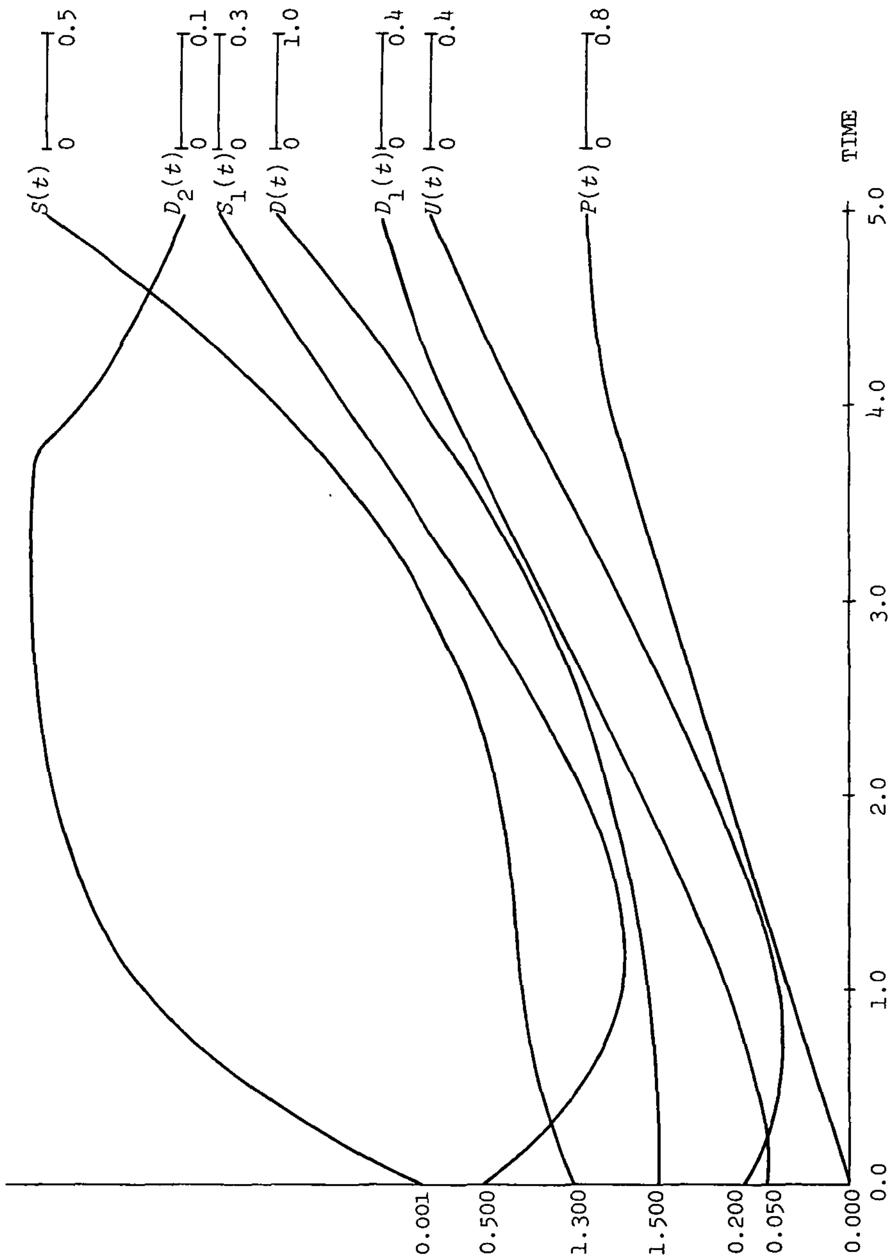

FIGURE 2

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