

A MODEL OF A SURGING GLACIER

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ABSTRACT. As the cause of a surge we assume a sudden improvement of the sliding conditions at the glacier bed. This improvement has the same effect on the behaviour of the glacier as an equivalent variation of the annual balance. We are therefore able to apply Nye's (1963) theory of the advance and retreat of glaciers to our problem. A simple glacier model discussed in this theory shows as a consequence of an improvement of the sliding conditions, an additional discharge of ice, which can be separated in two phases. The first phase of strong motion yields a transport of ice from the upper to the lower part of the glacier. The ice essentially remains within the previous limits of the glacier. The sequence of motion during this phase corresponds to a surge. The second phase of motion is essentially slower. During this phase the ice accumulated in the lower part of the glacier flows beyond the previous limits, till the whole surface level is lowered.

RÉSUMÉ. *Un modèle de glacier en crue rapide.* Comme cause d'une crue rapide, nous pouvons admettre l'hypothèse d'une soudaine amélioration des conditions de glissement sur le lit du glacier. Cette amélioration a les mêmes effets sur le comportement du glacier qu'une variation équivalente du bilan annuel. Nous pouvons donc appliquer à notre problème la théorie de Nye (1963) sur l'avance et le retrait des glaciers. Un modèle simple de glacier considéré dans la théorie que l'on vient de citer, montre qu'à une amélioration des conditions de glissement répond un débit additionnel de masses de glace qui peut être séparé en deux phases. La première phase de fort mouvement produit un transport des masses de glace du haut en bas du glacier. La glace reste cependant à peu près à l'intérieur des limites antérieures du glacier. La succession des mouvements pendant cette phase correspond à une crue. La seconde phase du mouvement est essentiellement plus lente. Pendant cette phase, la glace accumulée dans la partie basse du glacier, s'étale au-delà des limites antérieures du glacier, jusqu'à ce que le niveau superficiel du glacier soit entièrement abaissé.

ZUSAMMENFASSUNG. *Ein Modell für einen ausbrechenden Gletscher.* Als Ursache eines Gletscherausbruches wird eine plötzliche Verbesserung der Gleitbedingungen am Gletscherbett angenommen. Diese hat dieselbe Wirkung auf das Verhalten des Gletschers, wie eine äquivalente Änderung der jährlichen Massenbilanz. Dadurch ist es möglich, die Theorie der Gletscherschwankungen (Nye, 1963) auch auf das vorliegende Problem anzuwenden. Das in dieser Theorie diskutierte einfache Gletschermodell ergibt als Folge einer Verbesserung der Gleitbedingungen eine in zwei Phasen ablaufende zusätzliche Eisbewegung. Die erste, rasch ablaufende Phase bewirkt eine Umlagerung von Eismassen aus dem oberen Teil des Gletschers in den unteren. Die Eismassen bleiben aber im wesentlichen innerhalb der ursprünglichen Grenzen des Gletschers. Diese Phase entspricht dem Bewegungsablauf während eines Ausbruchs. Die zweite Phase läuft wesentlich langsamer ab. Während dieser fließen die im unteren Gletscherteil angehäuften Eismassen über das frühere Gletscherende hinaus, so dass schliesslich die gesamte Gletscheroberfläche einsinkt.

INTRODUCTION

An exciting and still unsolved phenomenon in glaciology is the occurrence of surges. A glacier stagnant for some tens of years suddenly starts to move with exceptionally high velocity. Great masses of ice are transported from the upper to the lower part of the glacier in one or two years. As a result of this movement, the surface level in the upper part is lowered and the lower part is raised, but the ice does not move beyond the limits of the previously stagnant ice. As a result of the surge, medial moraines are deformed into loops. From this characteristic it can be deduced that a surge is a repeating phenomenon. No external cause can be found for a surge (Post, 1960) and it seems very probable that it is caused by a variation of the sliding conditions at the glacier bed (Weertman, 1962).

In this paper we try to find a theoretical model which explains the sequence of motion during a surge, assuming a sudden improvement of the sliding conditions. As a prototype the surge of Muldrow Glacier described by Post (1960) will be used.

GENERAL EQUATIONS

We consider a two-dimensional model of a glacier. As in the theory of advance and retreat of glaciers, we need two equations. First the continuity equation and secondly a relation between the discharge $q(x,t)$ and the ice thickness $h(x,t)$ and its derivative $\partial h(x,t)/\partial x$. This relation can be deduced from a general flow law which implies the flow law of the ice and

the sliding conditions at the glacier bed. As in the present problem we are concerned with changes of the sliding conditions, we have to consider q as a function of the parameters α_i of the general flow law too. We make the assumption, that the mathematical form of the general flow law always remains the same.

If we regard only perturbations (suffix 1) from a steady state (suffix 0) we may write the two equations:

$$\frac{\partial q_1}{\partial x} + \frac{\partial h_1}{\partial t} = a_1, \quad (1)$$

$$q_1 = \left(\frac{\partial q}{\partial h}\right)_0 h_1 + \left(\frac{\partial q}{\partial(\partial h/\partial x)}\right)_0 \frac{\partial h_1}{\partial x} + \sum_{i=1}^n \left(\frac{\partial q}{\partial \alpha_i}\right)_0 \alpha_{i1} \quad (2)$$

where $a_1(x, t)$ is the deviation from a mean annual balance a_0 . x is measured positive from the upper end of the glacier along the surface and h is perpendicular to the x -direction.

Eliminating h_1 and q_1 respectively we obtain from Equations (1) and (2) the equations:

$$\frac{\partial q_1}{\partial t} = -\left(\frac{\partial q}{\partial h}\right)_0 \frac{\partial q_1}{\partial x} - \left(\frac{\partial q}{\partial(\partial h/\partial x)}\right)_0 \frac{\partial^2 q_1}{\partial x^2} + \sum_{i=1}^n \left(\frac{\partial q}{\partial \alpha_i}\right)_0 \frac{\partial \alpha_{i1}}{\partial t} + \left(\frac{\partial q}{\partial h}\right)_0 a_1 + \left(\frac{\partial q}{\partial(\partial h/\partial x)}\right)_0 \frac{\partial a_1}{\partial x}, \quad (3)$$

$$\begin{aligned} \frac{\partial h_1}{\partial t} = & -\frac{\partial}{\partial x} \left(\frac{\partial q}{\partial h}\right)_0 h_1 - \left[\left(\frac{\partial q}{\partial h}\right)_0 + \frac{\partial}{\partial x} \left(\frac{\partial q}{\partial(\partial h/\partial x)}\right)_0\right] \frac{\partial h_1}{\partial x} - \left(\frac{\partial q}{\partial(\partial h/\partial x)}\right)_0 \frac{\partial^2 h_1}{\partial x^2} - \\ & - \sum_{i=1}^n \frac{\partial}{\partial x} \left(\frac{\partial q}{\partial \alpha_i}\right)_0 \alpha_{i1} - \sum_{i=1}^n \left(\frac{\partial \alpha}{\partial \alpha_i}\right)_0 \frac{\partial \alpha_{i1}}{\partial x} + a_1. \end{aligned} \quad (4)$$

The third term on the right side of Equation (3) represents the influence of a change in the sliding conditions on q_1 . It is equivalent to a variation of the annual balance a_1' which can be found from the first-order differential equation

$$\left(\frac{\partial q}{\partial(\partial h/\partial x)}\right)_0 \frac{\partial a_1'}{\partial x} + \left(\frac{\partial q}{\partial h}\right)_0 a_1' - \sum_{i=1}^n \left(\frac{\partial q}{\partial \alpha_i}\right)_0 \frac{\partial \alpha_{i1}}{\partial t} = 0. \quad (5)$$

In the same manner we can replace the fourth term on the right side of Equation (4) by an equivalent variation of the annual balance a_1'' given by

$$a_1'' + \sum_{i=1}^n \frac{\partial}{\partial x} \left(\frac{\partial q}{\partial \alpha_i}\right)_0 \alpha_{i1} + \sum_{i=1}^n \left(\frac{\partial q}{\partial \alpha_i}\right)_0 \frac{\partial \alpha_{i1}}{\partial x} = 0. \quad (6)$$

With Equations (5) and (6) we have reduced our problem to the response of glaciers to climatic changes. We are therefore able to apply the theory of advance and retreat of glaciers (Nye, 1963) with only small modifications.

SPECIAL MODEL

For the description of the glacier flow we adopt a power law (Nye, 1959).

$$q = hU \left(\frac{\tau}{T}\right)^n, \quad (7)$$

Where τ is the shear stress at the glacier bed and as a first approximation is given by

$$\tau = \rho gh \left(\beta - \frac{\partial h}{\partial x}\right), \quad (8)$$

β is the inclination of the glacier bed, ρ the density of the ice, and g the gravity acceleration.

At any datum state of the glacier we are able to choose the coefficients U and T in such a manner that q becomes independent of n . If we do this, U is consequently the mean velocity of the ice and T the basal shear stress at this datum state.

For $(\partial q/\partial h)_0$ which represents the velocity of the kinematic waves, and $[\partial q/\partial(\partial h/\partial x)]_0$ which is the diffusion coefficient, we adopt the same polynomials as discussed and used by Nye (1963).

$$\left(\frac{\partial q}{\partial h}\right)_0 = (n+1) \frac{q_0}{h_0} = \frac{x}{\sigma} \left(1 - \frac{x}{l}\right), \tag{9}$$

$$\left(\frac{\partial q}{\partial(\partial h/\partial x)}\right)_0 = \frac{n\rho g h_0 q_0}{\tau_0 h_0} = -\frac{x^2}{\sigma} \left(1 - \delta - \frac{x}{l}\right), \tag{10}$$

where $l(1-\delta)$ is the length L of the glacier and σ is a natural time unit. For a glacier of $L \approx 10$ km a rough estimate gives $\sigma \approx 10$ year and $\delta \approx 0.01$. This estimate of σ is only valid if the exponent n is about 3 or 4. If we vary $n \rightarrow n_1$ we have to multiply σ by the factor $(n+1)/(n_1+1)$.

If we choose U and T in Equation (7) in such a manner that $(\partial q/\partial n)_0 = 0$, $\Sigma(\partial q/\partial \alpha_i)_0$ reduces using Equations (7) and (9) to

$$\left(\frac{\partial q}{\partial T}\right)_0 = -\frac{nh_0 q_0}{T_0 h_0} = -\frac{nh_0 x}{(n+1) T_0 \sigma} \left(1 - \frac{x}{l}\right). \tag{11}$$

We adopt for h_0/T_0 the constant value $(h_0/T_0)_{x=L/2}$. Following from Equation (8), this simplification means a uniform surface inclination of the whole glacier. Therefore T_0 tends to zero for $x \rightarrow 0$ and $(1-\delta)l$ in the same manner as h_0 . From Equations (10) and (11) we see that $[\partial q/\partial(\partial h/\partial x)]_0$ and $(\partial q/\partial T)_0$ should have the same analytical form. But as we need only the approximate form of the coefficients, the difference between the adopted polynomials will not falsify the results.

SOLUTION

After having determined the coefficients of Equation (3) we look for a solution which satisfies the following conditions: $a_1 = 0$ for any time; for $t < 0$ we assume a steady state with $h_1, \partial h_1/\partial x, \partial T_1/\partial t = 0$ and an arbitrary n ; for the small time interval $0 \leq t < \Delta t$ we take a constant $\partial T_1/\partial t = T_1/\Delta t$. For $t \geq \Delta t$ we take $\partial T_1/\partial t = 0$.

In our model T_0 tends to zero for $x \rightarrow 0$ or $(1-\delta)l$. As we shall assume T_1 constant and negative, this will lead to a basal shear stress in the direction of the glacier flow, which is physically meaningless. But as this discrepancy is restricted to the highest and lowest part of the glacier it can hardly affect the solution of Equation (3) for the main part of the glacier.

The variation of the annual balance equivalent to the adopted variation of the sliding conditions is found by Equation (5) to be

$$a_1' = -\frac{nh_0 T_1}{(n+1) T_0 \Delta t}.$$

With a_1 and the coefficients given by Equations (9), (10) and (11), the solution of Equation (3) for the time interval $0 \leq t < \Delta t$ is given by (Nye, 1963)

$$q_1 = -\frac{nh_0 T_1}{(n+1) T_0 \Delta t} \left[\{1 - \exp(-t/\sigma)\}x + \frac{1}{1-2\delta} \{ \exp(-t/\sigma) - \exp(-2\delta t/\sigma) \} \frac{x^2}{l} \right].$$

We expand $\exp(-t/\sigma)$ and $\exp(-2\delta t/\sigma)$ into series and, taking $\Delta t \rightarrow 0$, we find for q_1 at $t = 0$

$$q_1 = -\frac{nh_0 T_1 x}{(n+1) \sigma T_0} \left(1 - \frac{x}{l}\right). \tag{12}$$

A sudden change of T_0 to $T_0 + T_1$ yields therefore a q_1 given by Equation (12).

For $t \geq \Delta t$ we have to solve the homogenous differential equation (3) with the initial condition (12). The solution is

$$q_1 = -\frac{nh_0 T_1}{(n+1)\sigma T_0} \left[x \left(1 - \frac{x}{l} \right) \exp(-t/\sigma) + \frac{x^2}{l} \frac{2\delta}{1-2\delta} (\exp(-2\delta t/\sigma) - \exp(-t/\sigma)) \right]. \quad (13)$$

The solution for h_1 is found from Equations (13) and (1):

$$h_1 = \frac{nh_0 T_1}{(n+1)T_0} \left[1 - \left(1 - \frac{x}{l} \frac{2}{1-2\delta} \right) \exp(-t/\sigma) - \frac{l}{x} \frac{2}{1-2\delta} \exp(-2\delta t/\sigma) \right] \quad (14)$$

DISCUSSION

From Equation (13) we see that a sudden change of T_0 to $T_0 + T_1$ (T_1 negative) causes an additional discharge q_1 which can be separated into two phases. The first phase, represented by the first term in Equation (13), describes a discharge which reaches its maximum in the middle of the glacier and vanishes completely at the head and nearly at the snout. The motion caused by this discharge yields a transport of ice from the upper part of the glacier to the lower part and decays from a maximum at the beginning with a time constant σ . No essential transport of ice beyond the previous limits of the glacier takes place.

The second phase of motion, represented by the second term in Equation (13), becomes important after the first has decayed. It increases quadratically toward the end of the glacier and therefore causes an advance. Its magnitude is smaller by the factor 2δ , but, decaying with the time constant $\sigma/2\delta$, it continues correspondingly longer. $\sigma/2\delta$ is known as the long time constant from the theory of advance and retreat of glaciers.

We find a similar behaviour in h_1 . During the first phase of rapid motion the surface level lowers in the upper part of the glacier and rises in the lower part. During the second phase, decaying with the long time constant $\sigma/2\delta$, the surface level in the lower part drops too, till a constant decrease of ice thickness $h_1 = nh_0 T_1 / (n+1) T_0$ over the whole glacier is reached. The resulting diminution of the length of the glacier is of no physical significance for $|T_1| > T_0$ at the extreme ends of the glacier as mentioned in the previous section.

Qualitatively the first phase of motion is very similar to a surge. To fit our model quantitatively we assume $T_1 \approx \frac{1}{4} T_0$ to get the right magnitude of the surface level variation. σ , normally about 10 years, must be more than 10-times smaller during a surge to yield the very fast motion observed. This corresponds to a very high value of the exponent n . Normally no advance is produced by a surge. The first phase of motion of our theoretical model meets this observation. The second phase, transporting ice beyond the steady-state end of the glacier, is weaker by the factor 2δ than the first phase. One could imagine that this second phase is suppressed if a consolidation of the sliding conditions takes place after the first phase of strong motion is over.

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