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CONVEXITY CONDITIONS ON f-RINGS

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Introduction. Let n be a positive integer. An f-ring A is said to satisfy the left n^{th}-convexity property if for any $u, v \in A$ such that $v \geqq 0$ and $0 \leqq u \leqq v^n$, there exists a $w \in A$ such that $u = ,wv$. The right n^{th}-convexity property is defined similarly and an f-ring is said to satisfy the n^{th}-convexity property if it satisfies both the left and the right n^{th}-convexity property. In this paper, we study arbitrary f-rings which satisfy one of the convexity properties.

Those f-rings which satisfy one or more of these properties have been studied by several authors. In [**3**, 1D], L. Gillman and M. Jerison note that any $C(X)$ satisfies the n^{th}-convexity property for all $n \geqq 2$, and in [**3**, 14.25], they give several properties that in $C(X)$ are equivalent to the 1^{st}-convexity property. M. Henriksen proves some results about the ideal theory of an f-ring satisfying the 2^{nd}-convexity property in [**5**] and S. Steinberg studies left quotient rings of f-rings satisfying the left 1^{st}-convexity property in [**13**]. In [**7**, Section 3.4], C. Huijsmans and B. de Pagter use the 2^{nd}-convexity property to prove some results about the ideal theory of uniformly complete archimedean f-algebras and in [**7**, Section 6], [**8**], [**9**] and [**12**], they give several properties that in archimedean f-algebras with identity element are equivalent to the 1^{st}-convexity property.

In Section 2, we derive some basic facts about such f-rings. In Section 3 we give some conditions under which f-rings satisfying a convexity property are guaranteed to have an identity element or to be embeddable in an f-ring with an identity element. Section 4 is concerned with the ideal theory of an f-ring satisfying a convexity property, and in Section 5, we study the relationships between the various convexity properties.

1. Preliminaries. Much of the material given here can also be found in [**1**].

Given an f-ring A and an element $x \in A$, we let

$$A^+ = \{a \in A: a \geqq 0\},\ x^+ = x \vee 0,\ x^- = (-x) \vee 0$$

$$\text{and } |x| = x \vee (-x).$$

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Then $x = x^+ - x^-$ and $|x| = x^+ + x^-$. Recall that the Riesz decomposition property states that if $0 \leqq a \leqq b_1 + b_2 + \ldots + b_m$ with $b_1, \ldots, b_m \in A^+$ then there exists $x_1, \ldots, x_m \in A^+$ such that

$$a = x_1 + x_2 + \ldots + x_m \quad \text{and} \quad x_i \leqq b_i \quad \text{for } i = 1, 2, \ldots, m.$$

By an ideal in A we always mean a ring ideal. Suppose A is a ring and I an ideal of A. Then I is called *semiprime* (*prime*) if whenever $J(J_1, J_2)$ is an ideal such that $J^2 \subset I(J_1J_2 \subset I)$, $J \subset I(J_1 \subset I$ or $J_2 \subset I)$. The ring A is called semiprime (prime) if $\{0\}$ is a semiprime (prime) ideal. We call I *pseudoprime* if $a, b \in A$ and $ab = 0$ implies $a \in I$ or $b \in I$. A subset M of A is called an *m-system* if whenever $a, b \in M$ there exists an $x \in A$ such that $axb \in M$. In [**11**, Section 18] it is shown that I is semiprime if and only if $I = \{a \in A$:every m-system in A which contains a has nonempty intersection with $I\}$.

An ideal I of an f-ring A is said to be an l-ideal if $|x| \leqq |y|, y \in I$ implies $x \in I$. It is well known that the sum of two l-ideals in A is again an l-ideal. Given an f-ring A and an element $x \in A$ there is a smallest (left) l-ideal containing x, and we will denote this by $(\langle x \rangle_l)\langle x \rangle$. The (left) ideal generated by $x \in A$ is denoted by $((x)_l)$, (x). The following may be found in [**2**] or in [**1**].

1.1) If I is an l-ideal, then I is semiprime (prime) if and only if $a^2 \in I$ $(ab \in I)$ implies $a \in I$ $(a \in I$ or $b \in I)$.

An element $a > 0$ of an f-ring is called a *super idempotent* if $a \leqq a^2$, called a *weak order unit* if $a \wedge w = 0$ implies $w = 0$, and called a *super unit* (left super unit) if $ax \geqq x$ and $xa \geqq x$ $(ax \geqq x)$ for all $x \geqq 0$. An f-ring is said to be *l-simple* if $A^2 \neq \{0\}$ and A has no proper l-ideals. Then as can be found in [**1**]:

1.2) A super idempotent in a totally ordered ring is not contained in any proper left or right l-ideal.

1.3) If a is a super idempotent which is also a weak order unit, then a is not contained in any proper l-ideal.

1.4) If A is l-simple, then A is totally ordered and prime, A contains a super unit and has no proper left or right l-ideals.

An f-ring A is called *infinitesimal* if $a^2 \leqq |a|$ for every $a \in A$. Now not every f-ring can be embedded in an f-ring with identity element. The following is established in [**1**].

1.5) If a totally ordered ring is embeddable in an f-ring with identity element, then it is embeddable in a totally ordered ring with identity.

1.6) An f-ring is embeddable in an f-ring with identity if and only if it is a subdirect product of totally ordered rings embeddable in f-rings with identity.

1.7) Every infinitesimal or prime f-ring is embeddable in an f-ring with identity.

1.8) A totally ordered ring is embeddable in an f-ring with identity if and only if every super idempotent is a super unit.

1.9) If an f-ring is embeddable in an f-ring with identity, its idempotent elements are in its center.

The following is probably well known, and a proof is given for the sake of completeness.

1.10) Let A be an f-ring with an identity and $a \in A$. Then a left inverse of a is also a right inverse of a.

Proof. Suppose b is a left inverse of a. Then $(ab)^2 = ab$ and so ab is in the center of A. Then $ab = abba = baba = 1$.

2. In this section, we will define some convexity conditions on an f-ring and look at some of the basic properties of an f-ring which satisfies one of these conditions.

Definition. Let $n \geqq 1$. An f-ring is said to satisfy the *left n^{th}-convexity property* if for any $u, v \in A$ such that $v \geqq 0$ and $0 \leqq u \leqq v^n$, there exists a $w \in A$ such that $u = wv$. The right n^{th}-convexity property is defined similarly. An f-ring is said to satisfy the *n^{th}-convexity property* if it satisfies both the left and the right n^{th}-convexity property. If A satisfies the left n^{th}-convexity property, then we may assume that the element w satisfies $0 \leqq w \leqq v^{n-1}$ (by replacing the element w, if necessary, by $(w \wedge v^{n-1}) \vee 0)$.

Examples of f-rings satisfying the n^{th}-convexity property for $n \geqq 2$ include a totally ordered division ring, and $C(X)$, the f-ring of all real-valued continuous functions defined on a topological space. C. Huijsmans and B. de Pagter have shown that a uniformly complete f-algebra with identity element satisfies the n^{th}-convexity property for all $n \geqq 2$ [**7**, 3.11].

The next result has been proved for uniformly complete f-algebras with identity element in [**7**, 3.11]. The proof is similar and therefore omitted.

2.1 LEMMA. *Let $n \geqq 2$. If A is a semiprime f-ring satisfying the left n^{th}-convexity property, then the w of the previous definition is unique.*

The condition that A be semiprime cannot be dropped, as shown next.

2.2 *Example.* Let $n \geqq 2$ and $\mathbf{R}'[x]$ denote the ring of polynomials with real coefficients and zero constant term. Order $\mathbf{R}'[x]$ by defining $p(x) = p_1x + \ldots + p_mx^m > 0$ if the first non-zero coefficient of $p(x)$ (i.e., the coefficient of the least power of x) is greater than 0. Let

$$A = \mathbf{R}'[x]/\langle x^n \rangle.$$

Then A is a totally ordered ring satisfying the n^{th}-convexity property. In A,

$$0 \leqq u = 0 \leqq (x)^n = 0;$$

$$0 \leqq w = 0 \leqq w' = x^{n-1} \leqq x^{n-1} \quad \text{and}$$

$$u = wx = w'x.$$

So w and w' both satisfy the result in Lemma 2.1.

2.3 THEOREM. *Let $n \geqq 1$ and A be an f-ring which satisfies the left n^{th}-convexity property. Any l-homomorphic image of A satisfies the left n^{th}-convexity property.*

Proof. Suppose $\phi:A \rightarrow B$ is an l-homomorphism onto B, $\phi(v) \geqq 0$ and $0 \leqq \phi(u) \leqq [\phi(v)]^n$ in B. Then

$$0 \leqq (0 \vee u) \wedge (0 \vee v)^n \leqq (0 \vee v)^n$$

in A. Then there is $w \in A$ such that

$$(0 \vee u) \wedge (0 \vee v)^n = w(0 \vee v).$$

Thus,

$$\phi[(0 \vee u) \wedge (0 \vee v)^n] = \phi[w(0 \vee v)].$$

So $\phi(u) = \phi(w)\phi(v)$.

2.4 THEOREM. *Let A be an f-ring satisfying the left* 1^{st}*-convexity property. Then*

(1) *For each $f \in A$, there is an $r \in A$ such that $f = r|f|$ and $|f| = rf$.*
(2) *Every left ideal I in A is a left l-ideal.*
(3) *For all f, $g \in A$, $(f, g)_l = (|f| + |g|)_l$.*
(4) *For all f, $g \in A$, $(f, g)_l = (|f| \vee |g|)_l$.*
(5) *If A is semiprime, then every left ideal I is an intersection of pseudoprime left ideals.*

If A also contains a left super unit a, then (2), (3) *and* (4) *are equivalent to the left* 1^{st}*-convexity property.*

Before proving this, we note that (1) through (4) have been shown for $C(X)$ in [**3**, 14.25] and subsequently shown for an archimedean f-algebra with identity element in [**7**, 6.3, 6.5]. Part (5) has been shown for $C(X)$ in [**4**, 4.7 and 6.2]. S. Steinberg gives a related result in [**13**, 3.2]. The proofs mimic the proofs given in [**7**] and [**4**], so we confine ourselves to showing the last statement.

Proof. We show that (2), (3) and (4) imply the left 1^{st}-convexity property under the additional hypothesis that A contains a left super unit a.

Suppose $0 \leqq u \leqq v$. Then $0 \leqq u \leqq av$.

(2). Now $av \in Av$ and Av is a left l-ideal. Hence $u \in Av$.

(3). By (3),

$$(u, av - u)_l = (u + av - u)_l = (av)_l \subset Av.$$

So $u \in Av$.

(4). By (4),

$$(u, av)_l = (u \vee av)_l = (av)_l \subset Av.$$

So $u \in Av$.

Part (1) and (5) of this theorem are not equivalent to the left 1^{st}-convexity property (even when an identity element is present) since every totally ordered prime ring with identity element satisfies (1) and (5).

2.5 THEOREM. *Let $n \geqq 2$. If A is an f-ring with the left n^{th}-convexity property, then for every $v \geqq 0$, $\langle v^n \rangle_l \subset (v)_l$. If A has a left super unit a, then these properties are equivalent.*

Proof. Let $u \in \langle v^n \rangle_l$. Then $|u| \leqq sv^n + lv^n$ for some $s \in A^+$, $l \in \mathbf{N}$. It is easily seen that

$$0 \leqq u^+ \leqq (sv + (l \vee 1)v)^n.$$

So there is $w \in A$ such that

$$u^+ = w(sv + (l \vee 1)v).$$

Thus $u^+ \in (v)_l$. Similarly, $u^- \in (v)_l$.

Suppose now that a is a left super unit and that for every $v \geqq 0$, $\langle v^n \rangle_l \subset (v)_l$. Assume $v \geqq 0$ and $0 \leqq u \leqq v^n$. Then

$$0 \leqq u \leqq (av)^n \quad \text{and} \quad u \in \langle (av)^n \rangle_l \subset (av)_l \subset Av.$$

3. We now consider two general questions: under what conditions are we assured that an f-ring satisfying one of the convexity conditions contains an identity element, and what can be said about f-rings with one of the convexity conditions and having an identity element?

The following has also been proven by Steinberg in [**13**, 3.1(d)].

3.1 THEOREM. *If A is a prime f-ring satisfying the left 1^{st}-convexity property, then A has an identity element.*

Proof. Let $u > 0$. Then $0 < u \leqq u$ and there exists $w \in A$ such that $u = wu$. Now, $(w^2 - w)u = 0$. Since A is prime, $w^2 = w$. This implies that w is a right identity element for Aw and a left identity element for wA. Also, by 1.7 and 1.9, w is in the center of A. Hence $Aw = wA$ and w is an identity element for $Aw = wA$. We now show $A = Aw$. Note w

is a weak order unit in A. Also, by 2.4.2, Aw is an l-ideal. So 1.3 implies $Aw = wA = A$.

By considering the direct sum of an infinite number of copies of $\mathbf{R}$, we see that the condition that our f-ring be prime cannot be omitted from this theorem. The condition that A satisfy the left 1^{st}-convexity property cannot be weakened to the condition that A satisfy the left n^{th}-convexity property for any $n \geqq 2$ as shown next.

3.2 *Example*. Totally order $\mathbf{R}[x]$ by defining

$$\sum_{i=0}^{m} a_i x^i > 0 \quad \text{if } a_m > 0.$$

In the quotient ring of $\mathbf{R}[x]$, define

$$\deg \frac{p(x)}{q(x)} = \deg p(x) - \deg q(x).$$

Now let A be the subring

$$A = \left\{ \frac{p(x)}{q(x)} : p(x), q(x) \in \mathbf{R}[x] \text{ and } \deg \frac{p(x)}{q(x)} < 0 \right\}.$$

Order A so that

$$\frac{p(x)}{q(x)} \geqq 0 \quad \text{if } p(x)q(x) \geqq 0 \text{ in } \mathbf{R}[x].$$

Then A is a totally ordered prime f-ring satisfying the n^{th}-convexity property for all $n \geqq 2$ but without identity element.

3.3 THEOREM. *Let $n \geqq 1$ and A be a totally ordered ring satisfying the n^{th}-convexity property. If A contains a super idempotent a, then A contains an identity element.*

Two portions of the proof will be separated out and stated as a lemma.

3.4 LEMMA. *Let $n \geqq 1$ and A be an f-ring satisfying the n^{th}-convexity property. If A has a super idempotent a, then*

(1) *The left ideal Aa has a right identity element and the right ideal aA has a left identity element.*

(2) *Aa is a left l-ideal and aA is a right l-ideal.*

Proof. (1) If $n = 1$, then $0 < a \leqq a$ and if $n > 1$, then $0 \leqq a \leqq a^2 \leqq a^n$. In either case, there exist $w, w' \in A$ such that $a = wa$ and $a = aw'$. Then w' is a right identity for Aa and w a left identity for aA.

(2) In order to show that Aa is a left l-ideal it is sufficient to prove that $0 \leqq u \leqq |ba|$ for some $b \in A$ implies that $u \in Aa$. If $0 \leqq u \leqq |ba| = |b|a$, then

$$0 \leqq u \leqq (|b|a + a)^n.$$

So there is $x \in A$ such that

$$u = x(|b|a + a).$$

Hence $u \in Aa$. Similarly, aA is a right l-ideal.

We proceed to the proof of Theorem 3.3.

Proof. By the lemma, Aa and aA have a right and left identity element, and are left and right l-ideals respectively. By 1.2, a is not contained in any proper left or right l-ideal of A. But, $a \in Aa$ and $a \in aA$. So $Aa = aA = A$ and A has a right and a left identity element.

At least for $n \geqq 2$, the hypothesis that A contain a super idempotent cannot be omitted from this theorem since the totally ordered f-ring A of Example 3.2 satisfies the n^{th}-convexity property for all $n \geqq 2$ and does not contain a super idempotent or identity element. The hypothesis that A satisfy the n^{th}-convexity property cannot be weakened to the hypothesis that it satisfy the left or right n^{th}-convexity property, as shown by the following example.

3.5 *Example.* Let $A = \{ae + bz{:}a, b \in \mathbf{R}\}$. Carry out addition coordinatewise and multiplication using the rules $e^2 = e$, $ze = z$ and $z^2 = ez = 0$. Order A lexicographically so that

$$ae + bz \geqq ce + dz \quad \text{if } a > c \text{ or } a = c,\ b \geqq d.$$

Then A is a totally ordered f-ring satisfying the left n^{th}-convexity property for all $n \geqq 2$, but not the right n^{th}-convexity property for any n. It even has an idempotent (namely $e + z$), but not an identity element. Moreover, in [**10**, III 3.4], D. Johnson shows that A is not even embeddable in a totally ordered ring with identity element.

3.6 Corollary. *Let $n \geqq 1$ and A be an f-ring satisfying the n^{th}-convexity property. Then*

(1) *A is a subdirect sum of totally ordered rings, each of which is either infinitesimal or contains an identity element.*

(2) *A is embeddable in an f-ring with identity element.*

Proof. A is a subdirect sum of totally ordered rings, say A_i, each of which is either infinitesimal or contains a super idempotent. The latter implies that A_i has an identity element. The former implies A_i is embeddable in a totally ordered ring with identity by 1.7.

Example 3.5 illustrates that the condition that A satisfy the n^{th}-convexity property cannot be weakened to satisfy the left or right n^{th}-convexity property.

3.7 THEOREM. *Let $n \geqq 1$ and A be an f-ring satisfying the n^{th}-convexity property. If A contains a left or a right super unit a, then A has an identity element.*

Proof. Suppose A is a left super unit. By Lemma 3.4, Aa and aA possess a right and left identity element respectively. We will show $Aa = aA = A$. To that end, we first show a is a right super unit. Suppose A is a subdirect product of the totally ordered rings A_i. By Theorems 2.3 and 3.3, each A_i has an identity element 1_i. Since a is a left super unit, $1_i \leqq a_i 1_i = a_i$ for each i (where a_i denotes the projection of a onto A_i). Then for each $x \geqq 0$ and each i, $x_i = x_i 1_i \leqq x_i a_i$ and hence $x \leqq xa$. So a is a right as well as a left super unit. Now Aa and aA is a left and right l-ideal respectively, by Lemma 3.4. For any $b \geqq 0$, $0 \leqq b \leqq ba$ and $ba \in Aa$. So $b \in Aa$. Thus $A = Aa$. Similarly, $aA = A$.

Again Example 3.5 shows that the hypothesis in this theorem cannot be weakened to include "A satisfies the left n^{th}-convexity property."

The previous theorem is the key to the proof of the following.

3.8 THEOREM. *Let $n \geqq 1$ and A be an f-ring satisfying the n^{th}-convexity property. If A is l-simple, then A is a division ring.*

Proof. By 1.4, A contains a super unit and so the previous theorem implies A has an identity element 1. Also by 1.4, A is totally ordered and has no proper left or right l-ideals. Let $v > 0$. Then $\langle v \rangle_l = A$. In particular, $1 \in \langle v \rangle_l$ and so $1 \leqq av$ for some $a \in A$. Thus, $0 \leqq 1 \leqq (av)^n$, and so there is $w \in A$ such that $1 = wav$. So wa is a left inverse for v. By 1.10, wa is also a right inverse.

Next we show that an f-ring satisfying a convexity property is what M. Henriksen and D. Johnson have called closed under bounded inversion [**6**].

3.9 THEOREM. *Let $n \geqq 1$ and A be an f-ring satisfying the left n^{th}-convexity property. If A has an identity element and if $0 \leqq u \leqq v$ and u^{-1} exists in A, then v^{-1} exists in A.*

Proof. If $0 \leqq u \leqq v$ and u^{-1} exists, then $0 \leqq 1 \leqq u^{-1}v$ and so $0 \leqq 1 \leqq (u^{-1}v)^n$. There is a $w \in A$ such that $1 = wu^{-1}v$. Hence wu^{-1} is a left inverse for v. Then 1.10 implies wu^{-1} is the inverse for v.

4. Ideals and l-ideals in an f-ring satisfying a convexity condition will be considered in this section. Recall that Theorem 2.4 states that in an f-ring satisfying the left 1^{st}-convexity property, every (left) ideal is a (left) l-ideal. In light of this, most of the content of this chapter will be redundant in case the 1^{st}-convexity property is satisfied. We will be most concerned with ideals in f-rings satisfying the (left) n^{th}-convexity property for some $n \geqq 2$.

C. Huijsmans and B. de Pagter have proved the following for uniformly complete semiprime f-algebras [7, 4.7, 4.9].

4.1 THEOREM. *Let* $n \geqq 1$ *and* A *be an* f*-ring satisfying the left* n^{th}*-convexity property. Then*

(1) *If* I *is an idempotent left ideal,* I *is a left* l*-ideal.*

(2) *If* I *is a semiprime ideal, then* I *is an* l*-ideal.*

Proof. (1) Suppose $0 \leqq u \leqq |v|$ and $v \in I$. Since $I = I^2 = I^{2n}$,

$$v = \sum_{k=1}^{m} r_{1,k}r_{2,k} \dots r_{2n,k}$$

for some $r_{1,k}, r_{2,k}, \dots, r_{2n,k} \in I$, $0 \leqq k \leqq m$. Then

$$u \leqq |v| \leqq \sum_{k=1}^{m} [{r_{1,k}}^{2n} + \dots + {r_{2n,k}}^{2n}].$$

By the Riesz decomposition property,

$$u = \sum_{k=1}^{m} (p_{1,k} + p_{2,k} + \dots + p_{2n,k})$$

for some $p_{1,k}, \dots, p_{2n,k}$, $0 \leqq k \leqq m$ with $0 \leqq p_{1,k} \leqq {r_{1,k}}^{2n}, \dots,$ $0 \leqq p_{2n,k} \leqq {r_{2n,k}}^{2n}$. Then there exist $q_{1,k}, \dots, q_{2n,k}$ such that

$$p_{1,k} = q_{1,k}{r_{1,k}}^2, \dots, p_{2n,k} = q_{2n,k}{r_{2n,k}}^2.$$

Hence

$$u = \sum_{k=1}^{m} (q_{1,k}{r_{1,k}}^2 + \dots + q_{2n,k}{r_{2n,k}}^2) \in I.$$

(2) Suppose $0 \leqq u \leqq |v|$ and $v \in I$, and let M be an m-system in A which contains u. We will show $M \cap I \neq \phi$. There is some $x_1 \in A$ such that $ux_1u \in M$. There also is some $x_2 \in A$ such that $(ux_1u)x_2u \in M$. Continuing this, we find $x_1, \dots, x_{2n} \in A$ such that

$$ux_1ux_2 \dots ux_{2n}u \in M.$$

Then

$$0 \leqq (ux_1ux_2 \dots ux_{2n}u)^+ \leqq |v|\,|x_1|\,|v|\,|x_2| \dots |v|\,|x_{2n}|\,|v|.$$

For $i = 1, 3, \dots, 2n - 1$,

$$|v|\,|x_i|\,|v| \leqq v^2|x_i| + |x_i|v^2$$

since on each totally ordered coordinate of A, the projection of $|v|\,|x_i|$ is less than or equal to that of $|x_i|\,|v|$, or vice versa. Thus,

$$\begin{aligned}
0 &\leqq (ux_1ux_2 \ldots ux_{2n}u)^+ \\
&\leqq (v^2|x_1| + |x_1|v^2)\,|x_2|\,(v^2|x_3| + |x_3|v^2)\,|x_4| \\
&\quad \ldots (v^2|x_{2n-1}| + |x_{2n-1}|v^2)\,|x_{2n}|\,|v| \\
&\leqq [\,(v^2|x_1| + |x_1|v^2)\,|x_2| + (v^2|x_3| + |x_3|v^2)\,|x_4| + \ldots \\
&\quad + (v^2|x_{2n-1}| + |x_{2n-1}|v^2)\,|x_{2n}|\,|v|\,]^n.
\end{aligned}$$

So there exists a $w \in A$ such that

$$\begin{aligned}
&(ux_1ux_2 \ldots ux_{2n}u)^+ \\
&= w[\,(v^2|x_1| + |x_1|v^2)\,|x_2| + \ldots \\
&+ (v^2|x_{2n-1}| + |x_{2n-1}|v^2)\,|x_{2n}|\,|v|\,].
\end{aligned}$$

Therefore

$$(ux_1ux_2 \ldots ux_{2n}u)^+ \in I.$$

Similarly

$$(ux_1ux_2 \ldots ux_{2n}u)^- \in I.$$

We have shown that any m-system M which contains u meets I. Therefore, $u \in I$.

As an easy consequence of part (2), we can characterize semiprime and prime ideals in an f-ring with a convexity condition in the same way that D. Johnson has characterized semiprime and prime l-ideals in f-rings (1.1 or [**10**, 1.4]) and in the same way that they are characterized in commutative rings.

4.2 THEOREM. *Let $n \geqq 1$ and A be an f-ring satisfying the left n^{th}-convexity property. If I is an ideal then I is (semiprime) prime if and only if $(a^2 \in I)$ $ab \in I$ implies $(a \in I)$ $a \in I$ or $b \in I$.*

Proof. Suppose I is prime. By the last theorem, I is an l-ideal. So by 1.1 $ab \in I$ implies $a \in I$ or $b \in I$. Now suppose I is an ideal satisfying $ab \in I$ implies $a \in I$ or $b \in I$. Using 1.1, we need only show I is an l-ideal. Suppose $0 \leqq u \leqq |v|$ and $v \in I$. Then $0 \leqq u^{2n} \leqq v^{2n}$. So there is $w \in A$ such that $u^{2n} = wv^2$. Thus $u^{2n} \in I$. So $u \in I$.

Remark. Let A be an f-ring. The fact that if I is an l-ideal in A, then I is prime if and only if I is pseudoprime and semiprime, has been proved by H. Subramanian [**14**] for commutative f-rings with identity element and by C. Huijsmans and B. de Pagter [**7**] for archimedean f-rings. However, the proof that is given by Huijsmans and de Pagter does not require the archimedean condition. So, a direct consequence of this and of Theorem 4.1 (2) might be stated as: if I is an ideal in an f-ring satisfying the left n^{th}-convexity property for some $n \geqq 1$, then I is a prime ideal if

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and only if I is pseudoprime and semiprime. Examples do exist showing that the left n^{th}-convexity property is necessary here.

Next, we use M. Henriksen's notion of a square dominated ideal [**5**] to show a relationship between semiprime and idempotent ideals. An ideal I of an f-ring A is called square dominated whenever the l-ideal $\langle I \rangle$ generated by I satisfies

$$\langle I \rangle = \{a \in A : |a| \leqq x^2 \text{ for some } x \in A \text{ such that } x^2 \in I\}.$$

Let I be an l-ideal of a semiprime f-ring A. In [**5**, 3.4], M. Henriksen proves $I = \langle I^2 \rangle$ if and only if I is semiprime and square dominated.

4.3 THEOREM. *Let $n \geqq 1$ and A be a semiprime f-ring satisfying the left n^{th}-convexity property. An ideal I is idempotent if and only if I is semiprime and square dominated.*

Proof. Since an f-ring satisfying the left 1^{st}-convexity property also satisfies the left 2^{nd}-convexity property, we may assume $n \geqq 2$. Suppose $I = I^2$. Then I is an l-ideal by 4.1 and so by Henriksen's result, I is semiprime and square dominated.

Now suppose I is semiprime and square dominated. Again I is an l-ideal and so

$$I = \langle I \rangle = \{a \in A : |a| \leqq x^2 \text{ for some } x \in A \text{ such that } x^2 \in I\}.$$

Let $a \in I$. Let m be the largest integer such that $2^m \leqq n$. Now $0 \leqq a^+ \leqq x_1^2$ for some $x_1 \in A^+$ with $x_1^2 \in I$. Since I is semiprime, $x_1 \in I$. As before, $0 \leqq x_1 \leqq x_2^2$ for some $x_2 \in A^+$ with $x_2^2 \in I$. Again $x_2 \in I$. So

$$0 \leqq a^+ \leqq x_1^2 \leqq x_2^4.$$

Continuing this, we get

$$0 \leqq a^+ \leqq x_m^{2^m} \leqq x_{m+1}^{2^{m+1}} \quad \text{with } x_m, x_{m+1} \in I^+.$$

If $2^m = n$, then $0 \leqq a^+ \leqq x_m^n$ and there is a $w \in A$ such that

$$0 \leqq w \leqq x_m^{n-1} \quad \text{and} \quad a^+ = wx_m.$$

Then since I is an l-ideal, $w \in I$ and so $a^+ \in I^2$. If $2^m \neq n$, then

$$\begin{aligned} 0 \leqq a^+ \leqq x_m^{2^m} &= x_m^{2^{m+1}-n} x_m^{n-2^m} \\ &\leqq x_m^{2^{m+1}-n} x_{m+1}^{2n-2^{m+1}} \leqq (x_m \vee x_{m+1})^n. \end{aligned}$$

So there is $w \in A$ such that

$$0 \leqq w \leqq (x_m \vee x_{m+1})^{n-1} \quad \text{and} \quad a^+ = w(x_m \vee x_{m+1}).$$

Thus $a^+ \in I^2$. Similarly, $a^- \in I^2$.

We now consider products and sums of ideals. The product theorems and corresponding example will reflect a difference between the 2^{nd}-convexity condition and the n^{th}-convexity condition for $n \geqq 3$.

4.4 THEOREM. *Let A be an f-ring and I, J l-ideals of A. Then*

(1) *If A satisfies the left 2^{nd}-convexity property, then I^2 is also an l-ideal.*

(2) *If A satisfies the 2^{nd}-convexity condition, then IJ is also an l-ideal.*

Proof. (1) Suppose $0 \leqq u \leqq |v|$ and $v \in I^2$. Then $|v| \leqq p^2$ for some $p \in I^+$. Hence $0 \leqq u \leqq p^2$, and there is $w \in A$ such that $u = wp$ and $0 \leqq w \leqq p$. Now $w \in I$ and therefore $u = wp \in I^2$.

(2) Suppose $0 \leqq u \leqq |v|$ with $v \in IJ$. Then there is $p \in I^+$, $q \in J^+$ such that $|v| \leqq pq$. So

$$0 \leqq u \leqq (p \vee q)^2 \leqq p^2 + q^2.$$

The Riesz decomposition property implies $u = r + s$ for some r, s such that $0 \leqq r \leqq p^2$ and $0 \leqq s \leqq q^2$. So there are $w_1, w_2 \geqq 0$ such that $r = pw_1$ and $s = w_2q$. Hence

$$u = pw_1 + w_2q.$$

Let $r_1 = w_1 \wedge q$ and $r_2 = w_2 \wedge p$. Then $r_1 \in J$, $r_2 \in I$ and $pr_1 + r_2q \in IJ$. Now,

$$\begin{aligned} pr_1 + r_2q &= p(w_1 \wedge q) + (w_2 \wedge p)q \\ &= pw_1 \wedge pq + w_2q \wedge pq. \end{aligned}$$

But $pw_1 \leqq u \leqq pq$ and $w_2q \leqq pq$. So

$$pr_1 + r_2q = pw_1 + w_2q = u.$$

So $u \in IJ$.

Part (2) has been shown by C. Huijsmans and B. de Pagter [**7**, 4.13] under more restricted circumstances, and our proof mimics theirs.

This result need not hold in case A satisfies the (left) n^{th}-convexity property for $n \geqq 3$. In this case, no power of I larger than 1 need be an l-ideal as shown by the following (inspired by [**7**, 3.13]).

4.5 *Example.* Let $n \geqq 3$ and A be the set of all real-valued functions defined on [0, 1] of the form ri^{n-2}, where $r \in C([0, 1])$ and $i(x) = x$ for all $x \in [0, 1]$. With respect to the usual pointwise operations A is an f-ring satisfying the n^{th}-convexity property, but does not satisfy the m^{th}-convexity property for any $m < n$. For $m < n$, A does not satisfy the m^{th}-convexity property since $u:[0, 1] \to \mathbf{R}$ defined by

$$u(x) = x^{n-1}\left|\sin \frac{1}{x}\right|$$

is in A, and $0 \leqq u \leqq i^m$, but $u \neq wi$ for any $w \in A$. Now A is an l-ideal in itself and for any $m > 1$, A^m is not an l-ideal since

$$0 \leqq i^{m(n-2)}\left|\sin \frac{1}{x}\right| \leqq i^{m(n-2)},\ i^{m(n-2)} \in A^m \quad \text{and}$$

$$i^{m(n-2)}\left|\sin \frac{1}{x}\right| \notin A^m.$$

The next example shows that the conclusion of part (2) of this theorem is not valid under the weaker hypothesis that A satisfy the left 2^{nd}-convexity property.

4.6 *Example*. Let

$$A = \{ax_1 + bx_2 + cx_3 : a, b, c \in \mathbf{R}\}.$$

Carry out addition coordinatewise and multiplication using the rules

$$x_1^2 = x_1,\ x_2x_1 = x_1x_2 = x_2,\ x_3x_1 = x_3 \quad \text{and}$$

$$x_1x_3 = x_2^2 = x_3x_2 = x_2x_3 = x_3^2 = 0.$$

Order A lexicographically by defining

$$ax_1 + bx_2 + cx_3 \geqq 0$$

if either $a > 0$, or $a = 0$, $b > 0$, or $a = b = 0$ and $c \geqq 0$. It is easily seen that A is a totally ordered ring. The ring A satisfies the left but not the right 2^{nd}-convexity property. Let

$$I = \{ax_1 + bx_2 + cx_3 \in A : a = 0\}.$$

Then I is an l-ideal. But

$$AI = \{ax_1 + bx_2 + cx_3 \in A : a = c = 0\}$$

is not an l-ideal in A.

Let A be an f-ring. In [**5**, 3.10, 3.11], it is proved that the sum of two square dominated semiprime l-ideals is semiprime, and if one of the ideals is prime, then the sum is also prime. Recall that an f-ring is said to be *square-root closed* if for every $a \in A$ there is an $x \in A$ such that $x^2 = |a|$. Under more restricted conditions, part (2) of the following theorem has also been shown in [**7**].

4.7 THEOREM. *Let $n \geqq 1$ and A be an f-ring satisfying the left n^{th}-convexity property. Then*

(1) *The sum of two idempotent ideals is idempotent.*

(2) *If A is square-root closed, then the sum of any two semiprime ideals is semiprime, and if one of the summands is prime, then the sum is also prime.*

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Proof. (1) We may assume $n \geqq 2$. Suppose I, J are idempotent ideals. Then $I = I^n$ and $J = J^n$ and both are l-ideals. Let $a \in I + J$. We will show $a \in (I + J)^2$. Now $a = b + c$ for some $b \in I = I^n$, $c \in J = J^n$. Also, $|b| \leqq p^n$ for some $p \in I^+$ and $|c| \leqq q^n$ for some $q \in J^+$. So

$$a^+ \leqq p^n + q^n \leqq (p + q)^n.$$

So there is $w \in A$ such that

$$0 \leqq w \leqq (p + q)^{n-1} \quad \text{and} \quad a^+ = w(p + q).$$

Since $I + J$ is an l-ideal, $w \in I + J$ and hence $a^+ \in (I + J)^2$. Similarly, $a^- \in (I + J)^2$.

(2) Suppose I and J are semiprime ideals. Then they are l-ideals. Any semiprime l-ideal in a square-root closed f-ring is square dominated and so the theorem follows from Henriksen's result.

When A satisfies the left n^{th}-convexity property for some $n \geqq 3$, the hypothesis that A is square-root closed in part (2) cannot be weakened to the hypothesis that A is square dominated, as shown next. We will make use of the following example which also appears in [**7**, 4.16].

4.8 *Example.* In $C([0, 1])$, denote by i the function $i(x) = x$, by e the function $e(x) = 1$, and let $w = \sqrt{i}$. Let

$$A = \{f \in C([0, 1]): f = ae + bw + g;\ g \in \langle i \rangle,\ a, b \in \mathbf{R}\}$$

with the pointwise operations. A tedious but straightforward calculation shows that A satisfies the n^{th}-convexity property for all $n \geqq 3$. Moreover, A is square dominated, as every $f \in A$ satisfies $|f| \leqq a^2 e$ for appropriate $a \in \mathbf{R}$. Since $\sqrt{w} \notin A$, the f-ring A is not square-root closed. Let

$$I = \left\{f \in A: f\left(\frac{1}{2n-1}\right) = 0 \text{ for } n = 1, 2, \dots\right\} \quad \text{and}$$

$$J = \left\{f \in A: f\left(\frac{1}{2n}\right) = 0 \text{ for } n = 1, 2, \dots\right\}.$$

Then I and J are semiprime l-ideals, but as is shown in [**7**, 4.16], $I + J$ is not semiprime.

5. Some of our examples and theorems have shown that there is a difference between the various convexity conditions, and at least the 1^{st} and 2^{nd}-convexity conditions are significantly stronger than the other convexity conditions. In this section, we give some relationships between these conditions.

First, note that any f-ring A satisfying the left 1^{st}-convexity condition also satisfies the left n^{th}-convexity condition for all $n \geqq 2$.

5.1 THEOREM. *Let $n \geqq 2$ and A be an f-ring satisfying the left n^{th}-convexity property. If A is infinitesimal or if A has a left super unit, then A also satisfies the left m^{th}-convexity property for every $m > n$.*

Proof. Suppose $v \geqq 0$ and $0 \leqq u \leqq v^m$. If A is infinitesimal, then $0 \leqq u \leqq v^m \leqq v^n$ and so by the left n^{th}-convexity property, there is a $w \in A$ such that $u = wv$. Now assume A has a left super unit $a > 0$. If n divides m, then

$$0 \leqq u \leqq (v^{m/n})^n$$

implies that

$$u = (wv^{(m/n)-1})v$$

for appropriate $w \in A$. If not, then

$$0 \leqq u \leqq ((v \vee a)v^{\lfloor m/n \rfloor})^n$$

and so there is a $w \in A$ such that

$$u = w(v \vee a)v^{\lfloor m/n \rfloor}.$$

5.2 THEOREM. *Let A be a semiprime f-ring with identity element. Then A satisfies the left 1^{st}-convexity property (and hence the left m^{th}-convexity property for all $m \geqq 1$) if and only if A satisfies the left or right n^{th}-convexity property for some $n \geqq 1$ and every finitely generated left ideal in A is principle.*

Proof. Necessity follows from Theorem 2.4. Now suppose A satisfies the left n^{th}-convexity property for some $n \geqq 1$ and that every finitely generated left ideal is principal. Suppose also that $0 \leqq u \leqq v$ and $v \neq 0$. For some $d \neq 0$, $(u, v)_l = (d)_l$. So there are $p, q, r, s \in A$ such that $u = pd$, $v = qd$ and $ru + sv = d$. Then

$$[(rp + sq) - 1]d = 0.$$

Let

$$\begin{aligned} I &= \{f \in A : fd = 0\} \\ &= \{f \in A : |f| \wedge |d| = 0\} = \{f \in A : df = 0\}. \end{aligned}$$

Then I is a semiprime l-ideal by 4.2. Note that A/I satisfies the left n^{th}-convexity property by 2.3. Now

$$((rp + sq) - 1) + I = 0 + I$$

and hence

$$(rp + sq) + I = 1 + I.$$

Now $|p| - |p| \wedge |q| \in I$. So

$$|p| + I = |p| \wedge |q| + I = (|p| + I) \wedge (|q| + I).$$

Hence $|p| + I \leqq |q| + I$. Thus

$$1 + I = (rp + sq) + I \leqq (|r| + |s|)|q| + I.$$

Therefore $[(|r| + |s|)|q| + I]^{-1}$ exists in A/I by 3.9. Hence

$$[(|r| + |s|)|q| + I]^{-1}((|r| + |s|) + I)$$

is a left inverse of $|q| + I$ in A/I. Then there is a $q' \in A$ such that

$$q' + I = [(|r| + |s|)|q| + I]^{-1}((|r| + |s|) + I)$$

and

$$(q' + I)(|q| + I) = 1 + I.$$

Then

$$(|p|q')v = |p|q'|qd| = |p|(1 + a)|d|$$

for some $a \in I$. Thus

$$(|p|q')v = |p|(1 + a)|d| = |p|(|d| + a|d|) = |p|\,|d| = u.$$

Not both of the conditions that A be semiprime and have an identity element can be left out of the hypothesis of this theorem as shown next.

5.3 *Example.* Let $A = \{ae + bz{:}a, b \in \mathbf{Q}\}$, with addition, multiplication and ordering defined as in Example 3.5. Then A satisfies the left n^{th}-convexity property for all $n \geqq 2$, and every finitely generated left ideal is principal. But A does not satisfy the left 1^{st}-convexity property.

Example 4.8 shows that the condition that every finitely generated left ideal be principal is not redundant.

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