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Representation Equivalent Bieberbach Groups and Strongly Isospectral Flat Manifolds

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Abstract. Let Γ_1 and Γ_2 be Bieberbach groups contained in the full isometry group G of $\mathbb{R}^n$. We prove that if the compact flat manifolds $\Gamma_1\backslash\mathbb{R}^n$ and $\Gamma_2\backslash\mathbb{R}^n$ are strongly isospectral, then the Bieberbach groups Γ_1 and Γ_2 are representation equivalent; that is, the right regular representations $L^2(\Gamma_1\backslash G)$ and $L^2(\Gamma_2\backslash G)$ are unitarily equivalent.

1 Introduction

Let $X = G/K$ be a homogeneous Riemannian manifold, where $G = \mathrm{Iso}(X)$ is the full isometry group of X and where $K \subset G$ is a compact subgroup. Let $\widehat{G}$ denote the unitary dual group of G. Let Γ_1 and Γ_2 be discrete cocompact subgroups of G acting on X without fixed points. The right regular representation R_{Γ_i} on $L^2(\Gamma_i\backslash G)$ splits as a direct sum

$$L^2(\Gamma_i\backslash G) = \sum_{(\pi,H_\pi)\in\widehat{G}} n_{\Gamma_i}(\pi)\, H_\pi, \tag{1.1}$$

where all the multiplicities $n_{\Gamma_i}(\pi)$ of π in $L^2(\Gamma_i\backslash G)$ are finite and only countably many are not zero. The groups Γ_1 and Γ_2 are called *representation equivalent* if the representations R_{Γ_1} and R_{Γ_2} are equivalent, that is, $n_{\Gamma_1}(\pi) = n_{\Gamma_2}(\pi)$ for every $\pi \in \widehat{G}$.

A generalized version of Sunada's theorem (see [Go09, §3] and the references therein) says that if the groups Γ_1 and Γ_2 are representation equivalent, then the compact manifolds $\Gamma_1\backslash X$ and $\Gamma_2\backslash X$ are *strongly isospectral*, that is, for any natural bundle E of X and for any strongly elliptic natural operator D acting on sections of E, the associated operators D_{Γ_1} and D_{Γ_2} acting on sections of the bundles $\Gamma_1\backslash E$ and $\Gamma_2\backslash E$ have the same spectrum.

One may ask whether the converse holds. Actually, little is known about this problem. In [Pe95], H. Pesce proved that the converse is true for spherical space forms ($X = S^n$) and for compact hyperbolic manifolds ($X = H^n$). Our goal is to complete the picture within the class of spaces of constant curvature by extending Pesce's result to the flat case ($X = \mathbb{R}^n$).

In his proof, Pesce only used the isospectrality with respect to certain natural operators. More precisely, for $(\tau, V_\tau) \in \widehat{K}$, one associates the vector bundle $E_\tau = G \times_\tau V_\tau$

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(see §3). Thus, the Laplace operator acting on sections of E_τ induces the operator Δ_{τ,Γ_1} and Δ_{τ,Γ_2} acting on sections of $\Gamma_1\backslash E_\tau$ and $\Gamma_2\backslash E_\tau$ respectively. We will say that the manifolds $\Gamma_1\backslash X$ and $\Gamma_2\backslash X$ are *τ-isospectral* if these operators have the same spectrum.

In our case, the isometry group of $X = \mathbb{R}^n$ is $G = \mathrm{O}(n) \ltimes \mathbb{R}^n$, and a discrete cocompact subgroup of G acting without fixed points is usually called a *Bieberbach group*. We state the theorem in a way analogous to [Pe95].

Theorem 1.1 *Let Γ_1 and Γ_2 be Bieberbach groups contained in the full isometry group G of $\mathbb{R}^n$. The following assertions are equivalent:*

(i) Γ_1 *and* Γ_2 *are representation equivalent;*
(ii) $\Gamma_1\backslash\mathbb{R}^n$ *and* $\Gamma_2\backslash\mathbb{R}^n$ *are strongly isospectral;*
(iii) $\Gamma_1\backslash\mathbb{R}^n$ *and* $\Gamma_2\backslash\mathbb{R}^n$ *are τ-isospectral for every $\tau \in \widehat{K}$.*

As we mentioned above, (i)⇒(ii) is well known and (ii)⇒(iii) holds trivially. It suffices to show (iii)⇒(i) (see page 362). The techniques used here are similar to those in [LMR12], where the p-spectrum of any constant curvature space form is determined in terms of the multiplicities in (1.1).

2 Preliminaries

2.1 Irreducible Representations of Orthogonal Groups

In this subsection we describe the unitary dual group of the orthogonal group $\mathrm{O}(n)$. Furthermore, we recall the branching laws to $\mathrm{O}(n-1)$.

We write $n = 2m$ if n is even or $n = 2m + 1$ if n is odd. We fix the Cartan subalgebra of $\mathfrak{so}(n,\mathbb{C})$ as

$$\mathfrak{h} = \left\{ H = \sum_{j=1}^{m} i h_j (E_{2j-1,2j} - E_{2j,2j-1}) : h_j \in \mathbb{C} \right\}.$$

For $H \in \mathfrak{h}$, set $\varepsilon_j(H) = h_j$ for $1 \le j \le m$. The highest weight theorem gives a one-to-one correspondence between the irreducible representations of $\mathrm{SO}(n)$ and the elements in $\mathcal{P}(\mathrm{SO}(n))$, that is, the dominant analytically integral linear functionals on $\mathfrak{h}$. The correspondence being that Λ is the highest weight of the representation. We have

$$\mathcal{P}\big(\mathrm{SO}(2m+1)\big) = \left\{ \sum_{i=1}^{m} a_i\varepsilon_i : a_i \in \mathbb{Z}\ \forall i,\ a_1 \ge a_2 \ge \cdots \ge a_m \ge 0 \right\},$$

$$\mathcal{P}\big(\mathrm{SO}(2m)\big) = \left\{ \sum_{i=1}^{m} a_i\varepsilon_i : a_i \in \mathbb{Z}\ \forall i,\ a_1 \ge \cdots \ge a_{m-1} \ge |a_m| \right\}.$$

For $\Lambda \in \mathcal{P}(\mathrm{SO}(n))$, let $(\tau_\Lambda, V_\Lambda)$ denote the irreducible representation of $\mathrm{SO}(n)$ with highest weight Λ.

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We now describe the irreducible representations of the full orthogonal group $\mathrm{O}(n)$ in terms of the irreducible representations of $\mathrm{SO}(n)$. We let

$$g_0 = \begin{cases} -\mathrm{Id}_n & \text{if } n \text{ is odd,} \\ \left[\begin{smallmatrix} \mathrm{Id}_{n-1} & \\ & -1 \end{smallmatrix}\right] & \text{if } n \text{ is even,} \end{cases} \qquad \text{thus } \mathrm{O}(n) = \mathrm{SO}(n) \cup g_0\,\mathrm{SO}(n).$$

It suffices to define the representations of $\mathrm{O}(n)$ on each connected component.

We first consider n odd. For $\Lambda \in \mathcal{P}(\mathrm{SO}(2m+1))$ and $\delta = \pm 1$ we define a representation $(\tau_{\Lambda,\delta}, V_\Lambda)$ of $\mathrm{O}(2m+1)$ by setting

$$\tau_{\Lambda,\delta}(g)(v) = \begin{cases} \tau_\Lambda(g)(v) & \text{if } g \in \mathrm{SO}(2m+1), \\ \delta\,\tau_\Lambda(g_0 g)(v) & \text{if } g \in g_0\,\mathrm{SO}(2m+1). \end{cases}$$

Clearly $\tau_{\Lambda,\delta}|_{\mathrm{SO}(2m+1)} \cong \tau_\Lambda$. These representations are irreducible, and every irreducible representation can be constructed in this way, thus

$$\mathrm{O}\widehat{(2m+1)} = \left\{ \tau_{\Lambda,\delta} : \Lambda \in \mathcal{P}\big(\mathrm{SO}(2m+1)\big),\ \delta \in \{\pm 1\} \right\}.$$

The even case is more complicated (see [LMR12, Subsection 2.2] for more details). Set $\overline{\Lambda} = \sum_{i=1}^{m-1} a_i\,\varepsilon_i - a_m\varepsilon_m$ if $\Lambda = \sum_{i=1}^{m} a_i\,\varepsilon_i \in \mathcal{P}(\mathrm{SO}(2m))$. For $\Lambda \in \mathcal{P}(\mathrm{SO}(2m))$ satisfying $\Lambda = \overline{\Lambda}$ (*i.e.*, $a_m = 0$) and $\delta \in \{\pm 1\}$, one associates $\tau_{\Lambda,\delta} \in \widehat{\mathrm{O}(2m)}$ on the vector space V_Λ. Again we have that $\tau_{\Lambda,\delta}|_{\mathrm{SO}(2m)} \cong \tau_\Lambda$. The parameter δ depends on a certain intertwining operator T_Λ (see [Pe95, p. 372] and [LMR12, (2.7)]). In the case $\Lambda \neq \overline{\Lambda}$ (i.e. $a_m \neq 0$), there is a single representation $\tau_{\Lambda,0} \in \widehat{\mathrm{O}(2m)}$ defined on the vector space $V_\Lambda \oplus V_{\overline{\Lambda}}$, which satisfies $\pi_{\Lambda,0}|_{\mathrm{SO}(2m)} \cong \pi_\Lambda \oplus \pi_{\overline{\Lambda}}$. Hence,

$$\widehat{\mathrm{O}(2m)} = \left\{ \tau_{\Lambda,\delta} : \Lambda = \sum_{i=1}^{m} a_i\,\varepsilon_i \in \mathcal{P}\big(\mathrm{SO}(2m)\big),\ a_m = 0,\ \delta \in \{\pm 1\} \right\}.$$
$$\bigcup \quad \left\{ \tau_{\Lambda,0} : \Lambda = \sum_{i=1}^{m} a_i\,\varepsilon_i \in \mathcal{P}\big(\mathrm{SO}(2m)\big),\ a_m > 0 \right\}.$$

We shall use the notation $\tau_{\Lambda,\delta}$ in both cases, with the understanding that either $\delta = \pm 1$ or $\delta = 0$ according to $a_m = 0$ or $a_m \neq 0$ respectively.

One can check that $\tau_{\Lambda,\delta} \simeq \tau_{\Lambda,-\delta} \otimes \det \simeq \tau^*_{\Lambda,\delta}$ for any $\tau_{\Lambda,\delta} \in \widehat{\mathrm{O}(n)}$.

We conclude this subsection by stating the branching laws from $\mathrm{O}(n)$ to $\mathrm{O}(n-1)$.

Theorem 2.1 *Let $\tau_{\Lambda,\delta} \in \widehat{\mathrm{O}(2m)}$ with $\Lambda = \sum_{i=1}^{m} a_i\,\varepsilon_i$ and $\delta \in \{0, \pm 1\}$. If $a_m > 0$ (resp. $a_m = 0$), then $\tau_{\Lambda,\delta}|_{\mathrm{O}(2m-1)} = \sum\ \sigma_{\mu,\kappa}$, where the sum is over all $\mu = \sum_{i=1}^{m-1} b_i\varepsilon_i$ such that*

$$a_1 \geq b_1 \geq a_2 \geq b_2 \geq \cdots \geq a_{m-1} \geq b_{m-1} \geq a_m$$

and any $\kappa \in \{\pm 1\}$ (resp. a single value of $\kappa \in \{\pm 1\}$).

Theorem 2.2 *Let $\tau_{\Lambda,\delta} \in \widehat{\mathrm{O}(2m+1)}$, where $\Lambda = \sum_{i=1}^m a_i\,\varepsilon_i$ and $\delta \in \{\pm 1\}$. Then $\tau_{\Lambda,\delta}|_{\mathrm{O}(2m)} = \sum \sigma_{\mu,\kappa}$, where the sum is over all $\mu = \sum_{i=1}^m b_i\varepsilon_i$ such that*

$$a_1 \geq b_1 \geq a_2 \geq b_2 \geq \cdots \geq a_{m-1} \geq b_{m-1} \geq a_m \geq b_m \geq 0$$

and, a single value of $\kappa \in \{\pm 1\}$ if $b_m = 0$ or $\kappa = 0$ if $b_m > 0$.

Note that in both cases the branching is multiplicity free, that is, the multiplicity $[\sigma_{\mu,\kappa} : \tau_{\Lambda,\delta}] := \dim_{\mathrm{O}(n-1)}(W_\sigma, V_\tau)$ is always equal to 0 or 1.

2.2 Unitary Dual of $\mathrm{Iso}(\mathbb{R}^n)$

Here we describe the unitary irreducible representations of $\mathrm{O}(n) \ltimes \mathbb{R}^n \simeq \mathrm{Iso}(\mathbb{R}^n)$ (see [LMR12, §4] for more details). We write any element $g \in \mathrm{O}(n) \ltimes \mathbb{R}^n$ as $g = BL_b$, where $B \in \mathrm{O}(n)$ is called the rotational part, and L_b denotes translation by $b \in \mathbb{R}^n$. From now on, we fix the following notation:

$$\begin{aligned} G &= \mathrm{O}(n) \ltimes \mathbb{R}^n, \\ K &= \mathrm{O}(n), \\ M &= \left\{ \begin{pmatrix} B & \\ & \det(B) \end{pmatrix} : B \in \mathrm{O}(n-1) \right\}. \end{aligned} \tag{2.1}$$

An element $(\tau, W_\tau) \in \widehat{K}$ induces a representation $\widetilde{\tau}$ of G on W_τ given by

$$\widetilde{\tau}(BL_b)(w) = \tau(B)(w).$$

In other words, $\widetilde{\tau} = \tau \otimes \mathrm{Id}_{W_\tau}$. Clearly, $\widetilde{\tau}$ is finite dimensional, unitary, and irreducible.

We identify $\widehat{\mathbb{R}^n}$ with $\mathbb{R}^n$ via the correspondence $\alpha \mapsto \xi_\alpha(\,\cdot\,) = e^{2\pi i\langle \alpha,\,\cdot\,\rangle}$ for $\alpha \in \mathbb{R}^n$. Under the notation given by (2.1), given $r > 0$ and $(\sigma, V_\sigma) \in \widehat{M}$, we consider the induced representation of G given by

$$(\pi_{\sigma,r}, H_{\sigma,r}) := \mathrm{Ind}_{M \ltimes \mathbb{R}^n}^{K \ltimes \mathbb{R}^n}(\sigma \otimes \xi_{re_n}).$$

It is well known that $\pi_{\sigma,r}$ is unitary and irreducible.

Finally, a full set of representatives of $\widehat{G}$ is given by

$$\widehat{G} = \{\widetilde{\tau} : \tau \in \widehat{K}\} \cup \{\pi_{\sigma,r} : \sigma \in \widehat{M},\ r > 0\}.$$

3 Main Theorem

In this section we prove Theorem 1.1. We still use the notation in (2.1) for the groups G, K, and M, and $\{e_1, \dots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$.

We recall some notions on homogeneous vector bundles on $\mathbb{R}^n$ and compact flat manifolds (see [Wa73, §5.2] or [LMR12, Subsection 2.1]). Let (τ, V_τ) be a unitary representation of K of finite dimension. The homogeneous vector bundle $E_\tau =$

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$G \times_\tau V_\tau$ of X is constructed as $G \times V_\tau / \sim$, where $(x, v) \sim (xk, \tau(k^{-1})v)$ for every $k \in K$. The group G acts on E_τ by $g \cdot [x, v] = [gx, v]$, where $[x, v]$ denotes the class of equivalence of (x, v). The space of smooth sections $\Gamma^\infty(E_\tau)$ of E_τ is in correspondence with the set $C^\infty(G; \tau)$, the smooth functions $f : G \to V_\tau$ such that $f(xk) = \tau(k^{-1})f(x)$ for every $k \in K$ and $x \in G$. The element

$$C := e_1^2 + \cdots + e_n^2 \in U(\mathfrak{g})$$

defines a differential operator Δ_τ on $\Gamma^\infty(E_\tau)$. For example, if $\tau = \mathbf{1}$, the trivial representation of K, then

$$\Delta_{\mathbf{1}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

is just the Laplace operator on $\mathbb{R}^n$. Furthermore, the element C commutes with every irreducible representation of G contained in $C^\infty(G; \tau)$, thus by Schur's lemma, C acts by an scalar $\lambda(\pi, C)$ on V_π for every $\pi \in \widehat{G}$ such that $V_\pi \subset C^\infty(G; \tau)$.

The quotient $\Gamma \backslash E_\tau$ is a homogeneous vector bundle over the compact flat manifold $\Gamma\backslash\mathbb{R}^n$, and again, the element C defines a differential operator $\Delta_{\tau,\Gamma}$ acting on the sections of $\Gamma\backslash E_\tau$. Given Γ_1 and Γ_2, two Bieberbach groups in G, the spaces $\Gamma_1\backslash\mathbb{R}^n$ and $\Gamma_2\backslash\mathbb{R}^n$ are said to be τ-*isospectral* if Δ_{τ,Γ_1} and Δ_{τ,Γ_2} have the same spectrum.

We now determine the τ-spectrum for any $\tau \in \widehat{K}$ as in [LMR12]. Recall that $n_\Gamma(\pi)$ ($\pi \in \widehat{G}$) denotes the multiplicity of π in $L^2(\Gamma\backslash G)$ as we stated in (1.1). We shall use the following notation:

$$\widehat{G}(0) = \{\widetilde{\tau} : \tau \in \widehat{K}\}, \widehat{G}(\sigma) = \{\pi_{\sigma,r} : r > 0\} \text{for } \sigma \in \widehat{M},$$

thus

$$\widehat{G} = \widehat{G}(0) \cup \bigcup_{\sigma\in\widehat{M}} \widehat{G}(\sigma).$$

Theorem 3.1 *Let $\tau \in \widehat{K}$ and $\lambda \in \mathbb{R}$. The multiplicity $d_\lambda(\tau, \Gamma)$ of λ in the spectrum of $\Delta_{\tau,\Gamma}$ is given by*

$$d_\lambda(\tau, \Gamma) = \begin{cases} 0 & \text{if } \lambda < 0, \\ n_\Gamma(\widetilde{\tau}) & \text{if } \lambda = 0, \\ \sum_{\sigma\in\widehat{M} : [\sigma : \tau|_M]>0} n_\Gamma(\pi_{\sigma,\sqrt{\lambda}/2\pi}) & \text{if } \lambda > 0. \end{cases} \tag{3.1}$$

Proof For any locally symmetric space we have that (see [LMR12, Prop. 2.4])

$$d_\lambda(\tau, \Gamma) = \sum_{\substack{\pi\in\widehat{G} \\ \lambda(C,\pi)=\lambda}} n_\Gamma(\pi)\, [\tau^* : \pi|_K].$$

Note that the sum is already over the elements in

$$\widehat{G}_{\tau^*} = \widehat{G}_\tau := \left\{ \pi \in \widehat{G} : [\tau : \pi|_K] > 0 \right\},$$

since $\tau^* \simeq \tau$. Since $\widetilde{\tau}_0|_K = \tau_0$ for any $\tau_0 \in \widehat{K}$, it follows that $\widehat{G}_\tau \cap \widehat{G}(0) = \{\widetilde{\tau}\}$. On the other hand, $[\tau : \pi_{\sigma,r}|_K] = [\tau : \operatorname{Ind}_M^K(\sigma)] = [\sigma : \tau|_M]$ by Frobenius reciprocity. Then

$$\widehat{G}_\tau = \{\widetilde{\tau}\} \cup \bigcup_{\substack{\sigma \in \widehat{M} \\ [\sigma : \tau|_M] > 0}} \widehat{G}(\sigma).$$

The branching rules given in Theorems 2.1 and 2.2 give a complete description of $\widehat{G}_\tau$ in terms of highest weights. Moreover, they also ensure that $[\tau : \pi|_K] = 1$ for every $\pi \in \widehat{G}_\tau$.

Finally, by Schur's lemma, the element C acts by a scalar $\lambda(C, \pi)$ on each H_π. We have (see [LMR12, Lem. 4.2])

$$\lambda(C, \pi) = \begin{cases} 0 & \text{for } \pi \in \widehat{G}(0), \\ -4\pi^2 r^2 & \text{for } \pi = \pi_{\sigma,r} \in \widehat{G}(\sigma), \end{cases}$$

which concludes the proof. ■

We are now in a position to prove the main theorem.

Proof of Theorem 1.1 We have to prove that

$$n_{\Gamma_1}(\pi) = n_{\Gamma_2}(\pi) \tag{3.2}$$

for every $\pi \in \widehat{G}$, by assuming that $d_\lambda(\tau, \Gamma_1) = d_\lambda(\tau, \Gamma_2)$ for every $\lambda \in \mathbb{R}$ and every $\tau \in \widehat{K}$. From (3.1) for the eigenvalue $\lambda = 0$, it follows that (3.2) holds for every $\pi \in \widehat{G}(0)$.

It remains to prove that, for any $\sigma \in \widehat{M}$, (3.2) holds for every $\pi \in \widehat{G}(\sigma)$. We shall do this by the repeated application of the following lemmas. We write $n = 2m$ if n is even and $n = 2m+1$ if n is odd. For $\mu_1 = \sum_{i=1}^m b_i\varepsilon_i$ and $\mu_2 = \sum_{i=1}^m c_i\varepsilon_i$ in $\mathcal{P}(\mathrm{SO}(n))$ with $b_m, c_m \geq 0$, we write $\mu_1 < \mu_2$ if $c_1 - b_1 \geq c_2 - b_2 \geq \cdots \geq c_m - b_m \geq 0$, and set $\ell(\mu) = p$ if $b_p \neq 0$ and $b_i = 0$ for all $i > p$.

Lemma 3.2 *Let Γ_1 and Γ_2 be Bieberbach groups in G and let $\mu_0 \in \mathcal{P}(\mathrm{SO}(n-1))$. If $\Gamma_1\backslash\mathbb{R}^n$ and $\Gamma_2\backslash\mathbb{R}^n$ are $\tau_{\mu_0,\delta}$-isospectral and $n_{\Gamma_1}(\pi) = n_{\Gamma_2}(\pi)$ for every $\pi \in \bigcup_{\mu<\mu_0} \widehat{G}(\sigma_{\mu,\kappa})$, then $n_{\Gamma_1}(\pi) = n_{\Gamma_2}(\pi)$ for every $\pi \in \widehat{G}(\sigma_{\mu_0,\kappa})$.*

Proof Write $\mu_0 = \sum_{i=1}^m b_i\varepsilon_i \in \mathcal{P}(\mathrm{SO}(n-1))$, with the convention that $b_m = 0$ if n is even, since $n - 1 = 2m - 1$. Let $\Lambda = \sum_{i=1}^m b_i\varepsilon_i \in \mathcal{P}(\mathrm{SO}(n))$. Theorems 2.1 and 2.2 ensure that $[\sigma_{\mu,\kappa} : \tau_{\Lambda,\delta}|_M] > 0$ if and only if $\mu = \sum_{i=1}^m c_i\varepsilon_i$ satisfies

$$b_1 \geq c_1 \geq b_2 \geq c_2 \geq \cdots \geq b_m \geq c_m \geq 0,$$

and for a single value of $\kappa \in \{0, \pm 1\}$. Now, by (3.1) we have that

$$d_{4\pi^2r^2}(\tau_{\Lambda,\pm\delta}, \Gamma_i) = n_{\Gamma_i}(\pi_{\sigma_{\mu_0,\pm\kappa_0},r}) + \sum_{\substack{[\sigma_{\mu,\kappa} : \tau_{\Lambda,\delta}] > 0 \\ \mu \neq \mu_0}} n_{\Gamma_i}(\pi_{\sigma_{\mu,\pm\kappa_\mu},r})$$

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for every $r > 0$. It is clear that if $[\sigma_{\mu,\kappa} : \tau_{\Lambda,\delta}|_M] > 0$, then $\mu = \mu_0$ or $\mu < \mu_0$. Hence $n_{\Gamma_1}(\pi_{\sigma_{\mu_0,\pm\kappa_0},r}) = n_{\Gamma_2}(\pi_{\sigma_{\mu_0,\pm\kappa_0},r})$ for every $r > 0$, since we are assuming that $d_{4\pi^2 r^2}(\tau_{\Lambda,\pm\delta}, \Gamma_1) = d_{4\pi^2 r^2}(\tau_{\Lambda,\pm\delta}, \Gamma_2)$ ($\tau_{\mu_0,\kappa}$-isospectrality) and

$$n_{\Gamma_1}(\pi_{\sigma_{\mu,\pm\kappa_\mu},r}) = n_{\Gamma_2}(\pi_{\sigma_{\mu,\pm\kappa_\mu},r})$$

for every $\mu < \mu_0$ and every $r > 0$. ■

For example, by applying Lemma 3.2 to $\mu_0 = 0$, we obtain that (3.2) holds for every $\pi \in \widehat{G}(\sigma_{0,\kappa})$, which is the same result as [Pe96, Prop. 3.2 (c)].

Lemma 3.3 *Let Γ_1 and Γ_2 be Bieberbach groups in G and let $1 \le p < m$. If $\Gamma_1\backslash\mathbb{R}^n$ and $\Gamma_2\backslash\mathbb{R}^n$ are $\tau_{\mu,\delta}$-isospectral for every $\mu \in \mathcal{P}(\mathrm{SO}(n))$ such that $\ell(\mu) = p+1$ and $n_{\Gamma_1}(\pi) = n_{\Gamma_2}(\pi)$ for every $\pi \in \widehat{G}(\sigma_{\mu,\kappa})$ such that $\ell(\mu) \le p$, then $n_{\Gamma_1}(\pi) = n_{\Gamma_2}(\pi)$ for every $\pi \in \widehat{G}(\sigma_{\mu,\kappa})$ such that $\ell(\mu) = p+1$.*

Proof We begin by considering the first case $\mu_0 = \varepsilon_1 + \cdots + \varepsilon_{p+1} \in \mathcal{P}(\mathrm{SO}(n))$. Every $\mu \in \mathcal{P}(\mathrm{SO}(n))$ such that $\mu < \mu_0$ satisfies $\ell(\mu) \le p$, thus $n_{\Gamma_1}(\pi) = n_{\Gamma_2}(\pi)$ for every $\pi \in \cup_{\mu<\mu_0} \widehat{G}(\sigma_{\mu,\kappa})$. Hence $n_{\Gamma_1}(\pi) = n_{\Gamma_2}(\pi)$ for every $\pi \in \widehat{G}(\sigma_{\mu_0,\kappa})$ by Lemma 3.2.

We continue in this fashion obtaining that $n_{\Gamma_1}(\pi) = n_{\Gamma_2}(\pi)$ for every $\pi \in \widehat{G}(\sigma_{\mu_0,\kappa})$ for any $\mu_0 = \sum_{i=1}^{p} b_i\varepsilon_i + \varepsilon_{p+1} \in \mathcal{P}(\mathrm{SO}(n))$, since the ordering $<$ is complete. We now proceed by induction on b_{p+1}, and the proof is complete. ■

By proceeding by induction on p with repeated applications of Lemma 3.3, we have that (3.2) holds for every $\pi \in \widehat{G}(\sigma_{\mu,\kappa})$ for any $\mu \in \mathcal{P}(\mathrm{SO}(n))$. This completes the proof. ■

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