



COMPOSITIO MATHEMATICA

Springer theory via the Hitchin fibration

David Nadler

Compositio Math. **147** (2011), 1635–1670.

[doi:10.1112/S0010437X1100546X](https://doi.org/10.1112/S0010437X1100546X)



FOUNDATION
COMPOSITIO
MATHEMATICA

*The London
Mathematical
Society*





Springer theory via the Hitchin fibration

David Nadler

ABSTRACT

We develop the Springer theory of Weyl group representations in the language of symplectic topology. Given a semisimple complex group G , we describe a Lagrangian brane in the cotangent bundle of the adjoint quotient \mathfrak{g}/G that produces the perverse sheaves of Springer theory. The main technical tool is an analysis of the Fourier transform for constructible sheaves from the perspective of the Fukaya category. Our results can be viewed as a toy model of the quantization of Hitchin fibers in the geometric Langlands program.

1. Introduction

The primary aim of this paper is to develop Springer’s theory [Spr76, Spr78, Spr82] of Weyl group representations in the language of symplectic topology. Let G be a semisimple complex group with Lie algebra \mathfrak{g} , nilpotent cone $\mathcal{N} \subset \mathfrak{g}$, Springer resolution $\mu: \tilde{\mathcal{N}} \rightarrow \mathcal{N}$, and Weyl group W . In a form due to Lusztig [Lus81] and Borho–MacPherson [BM81] (building on the topological setting introduced by Kazhdan–Lusztig [KL80]), Springer theory identifies the group algebra $\mathbb{C}[W]$ with the (degree zero) endomorphisms of the perverse sheaf $S_{\mathcal{N}} = R\mu_*\mathbb{C}_{\tilde{\mathcal{N}}}[\dim_{\mathbb{C}} \mathcal{N}]$. We explain here how one can tell this story via the Fukaya category of the cotangent bundle of the adjoint quotient \mathfrak{g}/G . An initial motivation for this is the relative accessibility of objects of the Fukaya category (smooth Lagrangian submanifolds with structure) versus objects such as perverse sheaves (complexes of sheaves living on a singular variety). For example, the object of the Fukaya category corresponding to $S_{\mathcal{N}}$ (or, to be precise, its Fourier transform) is a regular fiber of a particular instance of the Hitchin fibration. The Weyl group action arises from Hamiltonian isotopies coming from motions of the regular parameter.

The approach to Springer theory adopted here is via the Fourier transform for constructible sheaves. According to Ginzburg [Gin83] and Hotta–Kashiwara [HK84], the Fourier transform of the Springer sheaf $S_{\mathcal{N}}$ can be identified with the intersection cohomology of \mathfrak{g} with coefficients in the regular $\mathbb{C}[W]$ local system over the regular semisimple locus $\mathfrak{g}_{\text{rs}} \subset \mathfrak{g}$. (Here we have identified \mathfrak{g} and its dual \mathfrak{g}^* using the Killing form.) In general, given a finite-dimensional real vector space V , the Fourier transform exchanges conic constructible sheaves on V and its dual V^* . The Fourier transform \mathcal{F}^\wedge of a constructible sheaf \mathcal{F} encodes the Morse groups (or vanishing cycles) of \mathcal{F} along rays through the origin. When V is a complex vector space, the (shifted) Fourier transform exchanges conic perverse sheaves on V and its dual V^* . Its structure is most transparent from the perspective of \mathcal{D} -modules via the Riemann–Hilbert correspondence. The Fourier transform for \mathcal{D} -modules exchanges conic \mathcal{D}_V -modules with conic \mathcal{D}_{V^*} -modules by the elementary change of variables $x \mapsto -\partial_x$, $\partial_x \mapsto x$, where x is a coordinate on V and ∂_x is the dual coordinate on V^* . In other words, it is nothing more than a ‘90° rotation’ of the phase space.

Received 28 November 2010, accepted in final form 4 February 2011, published online 29 July 2011.

2010 Mathematics Subject Classification 53D37 (primary), 32S60, 22E57 (secondary).

Keywords: Fukaya category, Fourier transform, Hitchin fibration.

This journal is © Foundation Compositio Mathematica 2011.

There is a dictionary between constructible sheaves on a manifold and the Fukaya category of its cotangent bundle (see [Nad09, NZ09]). The main technical result of this paper describes how the Fourier transform for constructible sheaves on V and V^* appears from the perspective of the Fukaya category of $V \times V^*$. We can identify $V \times V^*$ as a symplectic target with T^*V and also with T^*V^* . (If x, ξ denote coordinates on T^*V , v, λ coordinates on $V \times V^*$, and y, η coordinates on T^*V^* , then our identifications take the form $x = v = -\eta$, $\xi = \lambda = y$.) In this way, one can associate to a Lagrangian brane $L \hookrightarrow V \times V^*$ constructible sheaves $\pi_V(L)$ and $\pi_{V^*}(L)$ living on V and V^* , respectively. We introduce a class of Lagrangian branes in $V \times V^*$, called balanced branes, such that the conic limits of $\pi_V(L)$ and $\pi_{V^*}(L)$ are Fourier transforms of each other. Thus, for balanced branes, the operation underlying the Fourier transform is again nothing more than a ‘90° rotation’ of the target. With this theory in hand, we construct the Fourier transform of the Springer sheaf $S_{\mathcal{N}}$ by identifying the corresponding balanced brane in $T^*\mathfrak{g}$.

The primary motivation for this paper is the study of the Hitchin fibration within the framework of the geometric Langlands program (see Beilinson–Drinfeld [BD] and Kapustin–Witten [KW07]). Fix a smooth projective complex curve C . Let $\mathfrak{g}^*//G = \text{Spec}(\text{Sym } \mathfrak{g})^G$ be the affine coadjoint quotient, $(\mathfrak{g}^*//G)_{\omega_C}$ its twist by the canonical bundle of C , and $\mathbb{B}_G(C) = \Gamma(C, (\mathfrak{g}^*//G)_{\omega_C})$ the Hitchin base. Consider the moduli $\text{Bun}_G(C)$ of G -bundles over C , its cotangent bundle $T^*\text{Bun}_G(C)$ with fibers

$$T^*\mathcal{P} \text{Bun}_G(C) = \Gamma(C, \mathfrak{g}^*_{\mathcal{P}} \otimes \omega_C) \quad \text{for } \mathcal{P} \in \text{Bun}_G(C),$$

where $\mathfrak{g}^*_{\mathcal{P}}$ denotes the \mathcal{P} -twist of \mathfrak{g}^* , and the Hitchin fibration

$$\mathcal{H} : T^*\text{Bun}_G(C) \rightarrow \mathbb{B}_G(C), \quad \mathcal{H}(\mathcal{P}, \Phi) = \bar{\Phi}$$

induced by the morphism $\mathfrak{g}^*_{\mathcal{P}} \otimes \omega_C \rightarrow (\mathfrak{g}^*//G)_{\omega_C}$. One of the main goals of the geometric Langlands program is the quantization of the Hitchin fibers. Given a parameter $\mathfrak{b} \in \mathbb{B}_G(C)$, with Hitchin fiber $\mathcal{L}_{\mathfrak{b}} = \mathcal{H}^{-1}(\mathfrak{b}) \subset T^*\text{Bun}_G(C)$, we seek a \mathcal{D} -module on $\text{Bun}_G(C)$ whose ‘support’ is equal to $\mathcal{L}_{\mathfrak{b}}$. In physical terms, we would like to understand the structure of the A-brane wrapping $\mathcal{L}_{\mathfrak{b}}$.

In the toy case when C is a cuspidal elliptic curve, the moduli of semistable G -bundles (equivalently, bundles whose pullbacks to the normalization \mathbb{P}^1 are trivializable) reduces to \mathfrak{g}/G , and the Hitchin fibration reduces to a form of the moment map. To simplify the discussion, let us use the Killing form to identify \mathfrak{g} and its dual \mathfrak{g}^* , and fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Then, under the resulting induced identifications

$$T^*(\mathfrak{g}/G) \simeq \{(x, \xi) \in \mathfrak{g} \times \mathfrak{g} \mid [x, \xi] = 0\}/G, \quad \mathfrak{g}^*//G \simeq \mathfrak{h}/W,$$

the restriction of the Hitchin fibration takes the form

$$\mathcal{H} : T^*(\mathfrak{g}/G) \rightarrow \mathfrak{h}/W, \quad \mathcal{H}(x, \xi) = \bar{\xi},$$

where $\bar{\xi} \in \mathfrak{h}/W$ denotes the class of $\xi \in \mathfrak{g}$. As an application of our main result, we show that for a regular parameter $\lambda \in \mathfrak{h}/W$, the semistable Hitchin fiber $\mathcal{L}_{\lambda} = \mathcal{H}^{-1}(\lambda) \subset T^*(\mathfrak{g}/G)$ is the balanced brane corresponding to the Fourier transform of the Springer sheaf $S_{\mathcal{N}}$. Motions of the regular parameter provide a braid group action that descends to the usual Weyl group action of Springer theory.

1.1 Outline

Here is a further outline of the contents and arguments of the paper.

In §2, we develop for our purposes some foundational material on the constructible derived category $D_c(X)$ of a real analytic manifold X . We explain how to form the standard differential graded (dg) category $\text{Sh}_c(X)$ with cohomology category $D_c(X)$. We then collect background material on the Fourier transform for constructible sheaves. To formulate our main theorem, we need to take certain limits of constructible sheaves. Let V be a real finite-dimensional vector space with dilation action $\alpha^t : V \rightarrow V$ for scalars $t \in \mathbb{R}^+$. For \mathcal{F} a constructible sheaf on V , we construct a sheaf $\Upsilon(\mathcal{F})$ that formalizes the notion of the conic limit ‘ $\lim_{t \rightarrow 0} \alpha_*^t(\mathcal{F})$ ’. We describe its sections over certain cones in V as needed in our main theorem.

In §3, we review the Fukaya category of the cotangent bundle T^*X of a compact real analytic manifold X following [Nad09, NZ09]. We review the microlocalization quasi-equivalence

$$\mu_X : \text{Sh}_c(X) \xrightarrow{\sim} F(T^*X)$$

from constructible sheaves to the triangulated envelope of the Fukaya category. We will not need the full import of this result, only that μ_X is a quasi-embedding. But, we will need the main ingredient in the proof: the invariance of Floer calculations under non-characteristic motions explained in [Nad09, §3.7]. This is the fact that Hamiltonian isotopies of non-compact branes during which no critical event occurs near infinity lead to quasi-isomorphic calculations.

Section 4 contains the main technical work of the paper. Given a real finite-dimensional vector space V with dual V^* , we study non-compact Lagrangian branes $L \hookrightarrow V \times V^*$. We describe how such an object L gives rise to constructible sheaves $\pi_V(L)$ and $\pi_{V^*}(L)$ on the respective factors V and V^* . We isolate a class of Lagrangian branes, which we call balanced branes, to which our main theorem applies. Recall that for \mathcal{F} a constructible sheaf on a vector space, $\Upsilon(\mathcal{F})$ denotes its conic limit ‘ $\lim_{t \rightarrow 0} \alpha_*^t(\mathcal{F})$ ’. Our main result, Theorem 4.4.2, is the following.

THEOREM 1.1.1. *Let $L \hookrightarrow V_1 \times V_2$ be a balanced brane. The Fourier transform and its inverse exchange the conic limits of the corresponding constructible sheaves*

$$(\Upsilon(\pi_V(L)))^\wedge \simeq \Upsilon(\pi_{V^*}(L)), \quad (\Upsilon(\pi_{V^*}(L)))^\vee \simeq \Upsilon(\pi_V(L)).$$

In its most succinct form, the proof of the theorem takes the following shape. One observes that a pair of constructible sheaves are exchanged by the Fourier transform if and only if certain invariants of the sheaves are exchanged. Namely, the sections of the Fourier transform of a sheaf over an open convex cone are equal to the sections of the original sheaf within the closed polar cone. One shows that for a pair of sheaves coming from a single brane, these invariants can be understood as a single invariant within the Fukaya category. Namely, for a given pair of an open convex cone and its closed polar cone, there is a single brane such that pairing with it realizes both spaces of sections. The argument involves a delicate application of the invariance of Floer calculations under non-characteristic motions. To help the reader, we include a version of the argument in the case $\dim V = 1$, where one can gently acclimatize to some of the intricacies involved.

Finally, in §5, we apply the preceding theory to the adjoint quotient \mathfrak{g}/G . We first give a brief synopsis of Springer theory in the language of perverse sheaves. We then review the standard formalism for working with the cotangent bundle of a stack such as \mathfrak{g}/G . We then introduce the Hitchin fibration in the case of the cotangent bundle $T^*(\mathfrak{g}/G)$. Finally, we show that its regular fibers define balanced branes in $T^*(\mathfrak{g}/G)$. We use our main result to deduce that these branes give rise to the perverse sheaves of Springer theory.

2. Constructible sheaves

In this section, we first review background material on the constructible derived category and then recall a differential graded model of it. Finally, we review the Fourier–Sato transform.

2.1 Derived category

In this section, we briefly recall the construction of the constructible derived category of a real analytic manifold. For a comprehensive treatment of this topic, the reader could consult the book of Kashiwara–Schapira [KS94].

Let X be a topological space. Let $\text{Top}(X)$ be the category whose objects are open sets $U \hookrightarrow X$ and morphisms are inclusions $U_0 \hookrightarrow U_1$ of open sets:

$$\text{hom}_{\text{Top}(X)}(U_0, U_1) = \begin{cases} pt & \text{when } U_0 \hookrightarrow U_1, \\ \emptyset & \text{when } U_0 \not\hookrightarrow U_1. \end{cases}$$

Let ‘Vect’ be the abelian category of complex vector spaces.

The derived category of sheaves of complex vector spaces on X is traditionally defined via the following sequence of constructions.

(1) *Presheaves.* Presheaves on X are functors $\mathcal{F} : \text{Top}(X)^\circ \rightarrow \text{Vect}$, where $\text{Top}(X)^\circ$ denotes the opposite category. Given an open set $U \hookrightarrow X$, one writes $\mathcal{F}(U)$ for the sections of \mathcal{F} over U and, given an inclusion $U_0 \hookrightarrow U_1$ of open sets, one writes $\rho_{U_0}^{U_1} : \mathcal{F}(U_1) \rightarrow \mathcal{F}(U_0)$ for the corresponding restriction map.

(2) *Sheaves.* Sheaves on X are presheaves $\mathcal{F} : \text{Top}(X)^\circ \rightarrow \text{Vect}$, which are locally determined in the following sense. For any open set $U \hookrightarrow X$ and covering $\mathfrak{U} = \{U_i\}$ of U by open subsets $U_i \hookrightarrow U$, there is a complex of vector spaces

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\delta} \prod_i \mathcal{F}(U_i) \xrightarrow{\delta_0} \prod_{i,j} \mathcal{F}(U_i \cap U_j),$$

where $\delta = \prod_i \rho_{U_i}^U$ and $\delta_0 = \prod_{i,j} (\rho_{U_i \cap U_j}^{U_i} - \rho_{U_i \cap U_j}^{U_j})$. A sheaf is a presheaf for which $\ker(\delta) = \ker(\delta_0)/\text{im}(\delta) = 0$ for all open sets and coverings of open sets.

Sheaves on X form an abelian category and thus one can continue with the following sequence of general homological constructions.

(3) *Complexes.* Let $C(X)$ be the abelian category of complexes of sheaves on X with morphisms the degree zero chain maps. Given a complex of sheaves \mathcal{F} , one writes $H(\mathcal{F})$ for the (graded) cohomology sheaf of \mathcal{F} .

(4) *Homotopy category.* Let $K(X)$ be the homotopy category of sheaves on X with objects complexes of sheaves and morphisms homotopy classes of maps. This is a triangulated category whose distinguished triangles are isomorphic to the standard mapping cones.

(5) *Derived category.* The derived category $D(X)$ of sheaves on X is defined to be the localization of $K(X)$ with respect to homotopy classes of quasi-isomorphisms (maps inducing isomorphisms on cohomology). Acyclic objects form a null system in $K(X)$ and thus $D(X)$ inherits the structure of triangulated category.

With the derived category $D(X)$ in hand, one can define many variants by imposing topological and homological conditions on objects.

(6) *Bounded derived category.* The bounded derived category $D^b(X)$ is defined to be the full subcategory of $D(X)$ of bounded complexes.

Two standard equivalent descriptions are worth keeping in mind: first, there is the more flexible description of $D^b(X)$ as the full subcategory of $D(X)$ of complexes with bounded cohomology; second, there is the computationally useful description of $D^b(X)$ as the homotopy category of complexes of injective sheaves with bounded cohomology.

(7) *Constructibility.* Assume that X is a real analytic manifold. Fix an analytic–geometric category \mathcal{C} in the sense of [vM96]. For example, one could take $\mathcal{C}(X)$ to be the subanalytic subsets of X as described in [BM88].

Let $\mathcal{S} = \{S_\alpha\}$ be a Whitney stratification of X by \mathcal{C} -submanifolds $i_\alpha : S_\alpha \hookrightarrow X$. An object \mathcal{F} of $D(X)$ is said to be \mathcal{S} -constructible if the restrictions $i_\alpha^* H(\mathcal{F})$ of its cohomology sheaf to the strata of \mathcal{S} are finite rank and locally constant.

The \mathcal{S} -constructible derived category $D_{\mathcal{S}}(X)$ is the full subcategory of $D(X)$ of \mathcal{S} -constructible objects. The constructible derived category $D_c(X)$ is the full subcategory of $D(X)$ of objects which are \mathcal{S} -constructible for some Whitney stratification \mathcal{S} .

Note that if the stratification \mathcal{S} is finite (for example, if X is compact), then the finite-rank condition implies that all \mathcal{S} -constructible objects have bounded cohomology. In other words, within $D(X)$, every object of $D_{\mathcal{S}}(X)$ is isomorphic to an object of $D^b(X)$.

2.2 Differential graded category

The derived category $D(X)$ is naturally the cohomology category of a differential graded (dg) category $\text{Sh}(X)$. To define it, we will return to the sequence of homological constructions listed above and perform some modest changes. Two principles guide such definitions: (1) structures (such as morphisms and higher morphisms) should be defined at the level of complexes not their cohomologies and (2) properties (such as constructibility) should be imposed at the level of cohomologies rather than complexes. The first principle ensures that we will not lose important information, while the second ensures we will have sufficient flexibility. As an example of the latter, we prefer the realization of the bounded derived category $D^b(X)$ as the full subcategory of $D(X)$ of complexes with bounded cohomologies rather than of strictly bounded complexes.

The reader could consult [Dri04, Kel06] for background on dg categories; in particular, a discussion of the construction of dg quotients.

Recall that sheaves on X form an abelian category. The following sequence of homological constructions can be performed on any abelian category.

(1) *Dg category of complexes.* Let $C_{\text{dg}}(X)$ be the dg category with objects complexes of sheaves and morphisms the usual complexes of maps between complexes. In particular, the degree zero cycles in such a morphism complex are the usual degree zero chain maps, which are the morphisms of the ordinary category $C(X)$.

(2) *Dg derived category.* The dg derived category $\text{Sh}(X)$ is defined to be the dg quotient of $C_{\text{dg}}(X)$ by the full subcategory of acyclic objects. This is a triangulated dg category whose cohomology category $H(\text{Sh}(X))$ is canonically equivalent (as a triangulated category) to the usual derived category $D(X)$.

One can cut out full triangulated dg subcategories of $\text{Sh}(X)$ by specifying full triangulated subcategories of its cohomology category $H(\text{Sh}(X)) \simeq D(X)$.

(3) *Bounded dg derived category.* The bounded dg derived category $\mathrm{Sh}^b(X)$ is defined to be the full dg subcategory of $\mathrm{Sh}(X)$ of objects projecting to $D^b(X)$.

(4) *Constructibility.* Assume that X is a real analytic manifold and fix an analytic–geometric category \mathcal{C} . The constructible dg derived category $\mathrm{Sh}_c(X)$ is the full dg subcategory of $\mathrm{Sh}(X)$ of objects projecting to $D_c(X)$. For a Whitney stratification \mathcal{S} of X , the \mathcal{S} -constructible dg derived category $\mathrm{Sh}_{\mathcal{S}}(X)$ is the full dg subcategory of $\mathrm{Sh}(X)$ of objects projecting to $D_{\mathcal{S}}(X)$.

The formalism of Grothendieck’s six (derived) operations $f^*, f_*, f_!, f^!, \mathcal{H}om, \otimes$ can be lifted to the constructible dg derived category $\mathrm{Sh}_c(X)$ (see for example [Dri04] for a general discussion of deriving functors in the dg setting). In our case, one concrete approach is to recognize that the natural map $C_{\mathrm{dg},c}(\mathcal{I}\mathrm{nj}(X)) \rightarrow \mathrm{Sh}_c(X)$ from the dg category $C_{\mathrm{dg},c}(\mathcal{I}\mathrm{nj}(X))$ of complexes of injective sheaves with constructible cohomology is a quasi-equivalence. With this in hand, one can define derived functors by evaluating their naive versions on $C_{\mathrm{dg},c}(\mathcal{I}\mathrm{nj}(X))$. Since we will only consider derived functors, we will denote them by the above unadorned symbols.

Throughout the remainder of this paper, we fix an analytic–geometric category \mathcal{C} . All subsets will be \mathcal{C} -subsets unless otherwise stated.

2.3 Fourier transform for sheaves

We recall here the Fourier–Sato transform following [KS94, § 3.7]. We will describe the general parameterized version over a base manifold X , though our application will involve only the absolute version over $X = pt$.

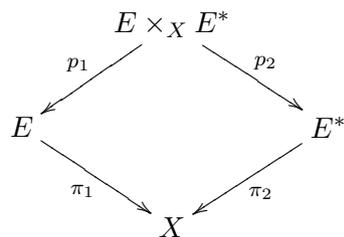
When working with constructible complexes on a non-compact manifold E , it is often technically convenient to fix a relative compactification $i : E \hookrightarrow \overline{E}$ and work with constructible complexes on \overline{E} . In the case of a vector bundle $E \rightarrow X$, we will always take the relative spherical compactification

$$\overline{E} = ((E \times \mathbb{R}^{\geq 0}) \setminus (X \times \{0\})) / \mathbb{R}_+.$$

By a constructible complex on E , we will mean a complex \mathcal{F} on E such that its extension $i_*\mathcal{F}$ to \overline{E} is constructible. Throughout this section, we will abuse notation and write $\mathrm{Sh}_c(E)$ for the full dg subcategory of such complexes. Note in particular that for the natural dilation \mathbb{R}^+ -action on E , any \mathbb{R}^+ -equivariant complex on E that is constructible in the usual sense is also constructible in the above extended sense.

2.3.1 Definition of Fourier transform. Let $\pi_1 : E \rightarrow X$ be a real finite-rank vector bundle and let $\pi_2 : E^* \rightarrow X$ be the dual vector bundle. For $e \in E$ and $e^* \in E^*$, let $\langle e, e^* \rangle \in \mathbb{R}$ denote the natural pairing.

Consider the Cartesian diagram of vector bundles.



Consider the closed subset of non-positive pairs

$$\kappa : K = \{(e, e^*) \in E \times_X E^* \mid \langle e, e^* \rangle \leq 0\} \hookrightarrow E \times_X E^*$$

and define the integral kernel

$$\mathcal{K} = \kappa_! \mathbb{C}_K \in \text{Sh}_c(E \times_X E^*).$$

Define integral transforms by the formulas

$$\Phi_{\mathcal{K}} : \text{Sh}_c(E) \rightarrow \text{Sh}_c(E^*), \quad \Psi_{\mathcal{K}} : \text{Sh}_c(E^*) \rightarrow \text{Sh}_c(E),$$

$$\mathcal{F}^\wedge = \Phi_{\mathcal{K}}(\mathcal{F}) = p_{2!}(\mathcal{K} \otimes p_1^* \mathcal{F}) = p_{2!} \kappa_! \kappa^* p_1^* \mathcal{F},$$

$$\mathcal{G}^\vee = \Psi_{\mathcal{K}}(\mathcal{G}) = p_{1*} \text{Hom}(\mathcal{K}, p_2^! \mathcal{G}) = p_{1*} \kappa_* \kappa^! p_2^! \mathcal{G}.$$

By the usual formalism (independent of the particular kernel \mathcal{K}), the functors form an adjoint pair $(\Phi_{\mathcal{K}}, \Psi_{\mathcal{K}})$.

2.3.2 Conic sheaves. Let $\mathbb{R}^+ \subset \mathbb{R}^\times$ denote the multiplicative group of positive real numbers. For a compact manifold X with an \mathbb{R}^+ -action, let $\alpha : \mathbb{R}^+ \times X \rightarrow X$ be the action map, X/\mathbb{R}^+ the quotient stack, and $p : X \rightarrow X/\mathbb{R}^+$ the natural projection.

We refer to $\text{Sh}_c(X/\mathbb{R}^+)$ as the \mathbb{R}^+ -equivariant constructible dg derived category. Since \mathbb{R}^+ is contractible, \mathbb{R}^+ -equivariance is a property not a structure. Thus, pullback (or, in other words, the forgetful functor) $p^* : \text{Sh}_c(X/\mathbb{R}^+) \rightarrow \text{Sh}_c(X)$ identifies $\text{Sh}_c(X/\mathbb{R}^+)$ with the full subcategory of $\text{Sh}_c(X)$ of \mathbb{R}^+ -equivariant objects. We will often refer to \mathbb{R}^+ -equivariant objects as conic objects.

We have the induction functors

$$\gamma = p_![-1] : \text{Sh}_c(X) \rightarrow \text{Sh}_c(X/\mathbb{R}^+), \quad \Gamma = p_* : \text{Sh}_c(X) \rightarrow \text{Sh}_c(X/\mathbb{R}^+).$$

By the usual formalism, γ is a left adjoint to the forgetful functor p^* and Γ is a right adjoint. (For the former, we have used that \mathbb{R}^+ is smooth, one dimensional, and has a canonical ‘positive’ orientation.)

We will next describe an alternative way to construct conic objects by taking the limit at zero of ordinary ones. To informally describe this, let $\alpha^t : X \rightarrow X$ denote the action $\alpha : \mathbb{R}^+ \times X \rightarrow X$ evaluated at $t \in \mathbb{R}^+$. Then, given an object $\mathcal{F} \in \text{Sh}_c(X)$, we would like to make sense of the limit $\lim_{t \rightarrow 0} \alpha_*^t \mathcal{F}$ as an object of $\text{Sh}_c(X)$.

To make this precise, let $\mathbb{R}^{\geq 0} \subset \mathbb{R}$ denote the non-negative real numbers. Consider the diagram

$$X \xleftarrow{\alpha} X \times \mathbb{R}^+ \xrightarrow{i_+} X \times \mathbb{R}^{\geq 0} \xleftarrow{j_0} X \times \{0\} \simeq X$$

where α is the action map and i_+ and j_0 are the obvious inclusions. There is a natural \mathbb{R}^+ -action on the above diagram where \mathbb{R}^+ acts trivially on the X at the left, diagonally on $X \times \mathbb{R}^+$ and $X \times \mathbb{R}^{\geq 0}$, and by the action α on the X at the right. Thus, we can pass to the quotient diagram

$$X \xleftarrow{\alpha} (X \times \mathbb{R}^+)/\mathbb{R}^+ \xrightarrow{i_+} (X \times \mathbb{R}^{\geq 0})/\mathbb{R}^+ \xleftarrow{j_0} X \times \{0\}/\mathbb{R}^+ \simeq X/\mathbb{R}^+.$$

Now we define the limit functor to be the composition

$$\Upsilon = j_0^* i_{+*} \alpha^* : \text{Sh}_c(X) \rightarrow \text{Sh}_c(X/\mathbb{R}^+).$$

Let us relate this construction to the usual nearby cycles functor. Suppose that X is a complex manifold and the \mathbb{R}^+ -action on X extends to a \mathbb{C}^\times -action $\alpha : X \times \mathbb{C}^\times \rightarrow X$. Then we can use the action map α together with the nearby cycles

$$R\psi : \text{Sh}_c(X \times \mathbb{C}^\times) \rightarrow \text{Sh}_c(X),$$

with respect to the canonical projection $X \times \mathbb{C} \rightarrow \mathbb{C}$, to define a functor

$$R\psi \circ \alpha^*[1] : \text{Sh}_c(X) \rightarrow \text{Sh}_c(X).$$

In fact, since the action map α is naturally \mathbb{C}^\times -equivariant for the trivial action on X and diagonal action on $X \times \mathbb{C}^\times$, and the nearby cycles $R\psi$ makes sense in the equivariant setting, the functor $R\psi \circ \alpha^*[1]$ canonically lifts to a functor $\text{Sh}_c(X) \rightarrow \text{Sh}_c(X/\mathbb{C}^\times)$.

With the usual conventions (so that $R\psi$ preserves perverse sheaves), the limit construction and nearby cycles construction are equivalent:

$$\Upsilon \simeq R\psi \circ \alpha^*[1].$$

2.3.3 Properties of Fourier transform. Consider the natural \mathbb{R}^+ -action on the vector bundles E, E^* . Since $K \subset E \times_X E^*$ is $\mathbb{R}^+ \times \mathbb{R}^+$ -invariant, and so $\mathcal{K} = \kappa_! \mathbb{C}_K$ is $\mathbb{R}^+ \times \mathbb{R}^+$ -equivariant, the Fourier transforms $\Phi_{\mathcal{K}}, \Psi_{\mathcal{K}}$ land in the full subcategories of conic objects.

Here are two main properties of the Fourier transforms.

(1) [KS94, Theorem 3.7.9] The restrictions of the Fourier transforms $\Phi_{\mathcal{K}}, \Psi_{\mathcal{K}}$ to the full subcategories $\text{Sh}_c(E/\mathbb{R}^+), \text{Sh}_c(E^*/\mathbb{R}^+)$ of conic objects are inverse equivalences.

(2) [KS94, Proposition 10.3.18] Suppose that X is a complex manifold and E is a complex vector bundle. Then the restrictions of the shifted Fourier transforms $\Phi_{\mathcal{K}}[\dim_{\mathbb{C}} E], \Psi_{\mathcal{K}}[-\dim_{\mathbb{C}} E]$ to the full subcategories $\text{Perv}(E/\mathbb{R}^+), \text{Perv}(E^*/\mathbb{R}^+)$ of conic perverse sheaves are inverse equivalences.

Finally, the Fourier transforms are compatible with equivariantization: by standard identities among functors, there are canonical quasi-isomorphisms

$$\Phi_{\mathcal{K}}(\gamma(\mathcal{F})) \simeq \Phi_{\mathcal{K}}(\mathcal{F}), \quad \Psi_{\mathcal{K}}(\Gamma(\mathcal{F})) \simeq \Psi_{\mathcal{K}}(\mathcal{F}).$$

2.3.4 Characterizing calculations. It will be useful to recall the explicit result of evaluating the Fourier transform on basic objects. In particular, in the proof of our main theorem, we will need the following elementary calculation.

Recall that small open convex subsets $b : B \hookrightarrow E$ generate the topology of E . To identify an object $\mathcal{F} \in \text{Sh}_c(E)$, it suffices to know its sections $\Gamma(B, \mathcal{F}) \simeq \text{hom}_{\text{Sh}_c(E)}(b_! \mathbb{C}_B, \mathcal{F})$ over small open convex subsets, along with the restriction maps among the sections.

By a cone $u : U \hookrightarrow E$, we will mean an \mathbb{R}^+ -invariant subset (or, in other words, the inverse image of a subset $U/\mathbb{R}^+ \hookrightarrow E/\mathbb{R}^+$). To identify an object $\mathcal{F} \in \text{Sh}_c(E/\mathbb{R}^+)$, it suffices to know its sections $\Gamma(U, \mathcal{F}) \simeq \text{hom}_{\text{Sh}_c(E)}(u_! \mathbb{C}_U, \mathcal{F})$ over open convex cones, along with the restriction maps among the sections.

We have the following useful description (see [KS94, Proposition 3.7.12]) of the sections of the Fourier transform over convex open subsets.

For $u : U \hookrightarrow E^*$ a subset, define the closed polar cone

$$v : U^\circ = \{e \in E \times_X \pi_2(U) \mid \langle e, e^* \rangle \geq 0 \text{ for all } e^* \in \{\pi_1(e)\} \times_X U\} \hookrightarrow E.$$

Now if $u : U \hookrightarrow E^*$ is an open convex cone with polar cone $v : U^\circ \hookrightarrow E$, then (see [KS94, Proposition 3.7.12]) there is a quasi-isomorphism

$$\pi_{2*} u^* \Phi_{\mathcal{K}}(\mathcal{F}) \simeq \pi_{1*} v^! \mathcal{F} \quad \text{for all } \mathcal{F} \in \text{Sh}_c(E/\mathbb{R}^+).$$

Note that U is open, so $u^* \simeq u^!$, and U° is closed, so $u_* \simeq u_!$. By adjunction, we can interpret this as a quasi-isomorphism of morphism complexes

$$\mathrm{hom}_{\mathrm{Sh}_c(E^*)}(u_! \mathbb{C}_U, \Phi_{\mathcal{K}}(\mathcal{F})) \simeq \mathrm{hom}_{\mathrm{Sh}_c(E)}(v_* \mathbb{C}_{U^\circ}, \mathcal{F}).$$

In particular, one can deduce the above identification from the calculation $\Phi_{\mathcal{K}}(v_* \mathbb{C}_{U^\circ}) \simeq u_! \mathbb{C}_U$ (see [KS94, Lemma 3.7.10]).

Furthermore, an inclusion of open convex cones $U_0 \hookrightarrow U_1 \hookrightarrow E^*$ leads to an inclusion morphism $u_{0!} \mathbb{C}_{U_0} \rightarrow u_{1!} \mathbb{C}_{U_1}$ and an inclusion morphism $v_{1!} \omega_{U_1^\circ} \rightarrow v_{0!} \omega_{U_0^\circ}$. In turn, for all $\mathcal{F} \in \mathrm{Sh}_c(E/\mathbb{R}^+)$, these induce a commutative (at the level of cohomology) square.

$$\begin{array}{ccc} \mathrm{hom}_{\mathrm{Sh}_c(E^*)}(u_{1!} \mathbb{C}_{U_1}, \Phi_{\mathcal{K}}(\mathcal{F})) & \xrightarrow{\sim} & \mathrm{hom}_{\mathrm{Sh}_c(E)}(v_{1*} \mathbb{C}_{U_1^\circ}, \mathcal{F}) \\ \downarrow & & \downarrow \\ \mathrm{hom}_{\mathrm{Sh}_c(E^*)}(u_{0!} \mathbb{C}_{U_0}, \Phi_{\mathcal{K}}(\mathcal{F})) & \xrightarrow{\sim} & \mathrm{hom}_{\mathrm{Sh}_c(E)}(v_{0*} \mathbb{C}_{U_0^\circ}, \mathcal{F}) \end{array}$$

In conclusion, we see that for $\mathcal{F} \in \mathrm{Sh}_c(E/\mathbb{R}^+)$, the Fourier transform $\Phi_{\mathcal{K}}(\mathcal{F})$ is characterized by the above calculations.

Before moving on, it will be convenient to go one step further and note the following. Suppose that $u : U \hookrightarrow E^*$ is an open convex cone but does not contain any fiber of E^* . Then the interior $\mathrm{int}(v) : \mathrm{int}(U^\circ) \hookrightarrow E$ of the closed polar cone $v : U^\circ \hookrightarrow E$ is an open convex submanifold. We have a quasi-isomorphism $v_* \mathbb{C}_{U^\circ} \simeq \mathrm{int}(v)_* \mathbb{C}_{\mathrm{int}(U^\circ)}$ and thus a quasi-isomorphism

$$\mathrm{hom}_{\mathrm{Sh}_c(E^*)}(u_! \mathbb{C}_U, \Phi_{\mathcal{K}}(\mathcal{F})) \simeq \mathrm{hom}_{\mathrm{Sh}_c(E)}(\mathrm{int}(v)_* \mathbb{C}_{\mathrm{int}(U^\circ)}, \mathcal{F})$$

for all $\mathcal{F} \in \mathrm{Sh}_c(E/\mathbb{R}^+)$. (Also, when $u : U \hookrightarrow E^*$ equals all of E , the closed polar cone $v : U^\circ \hookrightarrow E$ is the zero section $X \hookrightarrow E$.) On both sides of the above identification, we have a pairing with the standard or costandard sheaf of a submanifold. It is this form of the calculation that we will use to characterize the Fourier transform.

2.3.5 Sections of limit. For future reference, we collect here the calculation of the sections of the limit $\Upsilon(\mathcal{F}) \in \mathrm{Sh}_c(\mathbb{R}^n/\mathbb{R}^+)$ of an object $\mathcal{F} \in \mathrm{Sh}_c(\mathbb{R}^n)$ with respect to the dilation \mathbb{R}^+ -action.

For $t \in \mathbb{R}^+$, let $\alpha^t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ denote the dilation \mathbb{R}^+ -action. For each $\varepsilon \in \mathbb{R}$, consider the open quadrant

$$q^\varepsilon : Q^\varepsilon = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i > \varepsilon\} \hookrightarrow \mathbb{R}^n.$$

We will write $q : Q \hookrightarrow \mathbb{R}^n$ for the open cone $q^0 : Q^0 \hookrightarrow \mathbb{R}^n$. Note that linear transformations of Q , together with \mathbb{R}^n itself, form a basis for the conic topology of \mathbb{R}^n .

LEMMA 2.3.1. *For any object $\mathcal{F} \in \mathrm{Sh}_c(\mathbb{R}^n)$ and any $\varepsilon \in \mathbb{R}^+$, there exists $\delta(\varepsilon) \in \mathbb{R}^+$ such that for any $\delta \in (0, \delta(\varepsilon))$, there are canonical identifications*

$$\begin{aligned} \mathrm{hom}_{\mathrm{Sh}_c(E)}(q_! \mathbb{C}_Q, \Upsilon(\mathcal{F})) &\simeq \mathrm{hom}_{\mathrm{Sh}_c(E)}(q_!^\varepsilon \mathbb{C}_{Q^\varepsilon}, \alpha_*^\delta(\mathcal{F})), \\ \mathrm{hom}_{\mathrm{Sh}_c(E)}(q_* \mathbb{C}_Q, \Upsilon(\mathcal{F})) &\simeq \mathrm{hom}_{\mathrm{Sh}_c(E)}(q_*^{-\varepsilon} \mathbb{C}_{Q^{-\varepsilon}}, \alpha_*^\delta(\mathcal{F})). \end{aligned}$$

Proof. Recall that $i : \mathbb{R}^n \hookrightarrow \overline{\mathbb{R}^n}$ denotes the spherical compactification and, by assumption, there is a stratification \mathcal{S} of $\overline{\mathbb{R}^n}$ such that $i_* \mathcal{F}$ is \mathcal{S} -constructible.

Consider the \mathbb{R}^+ -conic stratification $C(\mathcal{S})$ of $\mathbb{R}^n \times \mathbb{R}^+$ obtained by taking the cone over \mathcal{S} together with the origin. Observe that the object $i_{+*} \alpha^* \mathcal{F}$ appearing in the definition of $\Upsilon(\mathcal{F})$ is $C(\mathcal{S})$ -constructible.

Fix any $\varepsilon_0 \in \mathbb{R}^+$. Let $\mathcal{Q}^{\pm\varepsilon_0}$ be the stratification of \mathbb{R}^n given by the facets of $Q^{\pm\varepsilon_0}$ and let $\mathcal{Q}^{\pm\varepsilon_0} \times \{\mathbb{R}^+, \{0\}\}$ be the product stratification of $\mathbb{R}^n \times \mathbb{R}^{\geq 0}$.

Let $\mathcal{R}^{\pm\varepsilon_0}$ be a stratification of $\mathbb{R}^n \times \mathbb{R}^{\geq 0}$ that extends to $\overline{\mathbb{R}}^n \times \mathbb{R}^{\geq 0}$ and refines $C(\mathcal{S})$ and $\mathcal{Q}^{\pm\varepsilon_0} \times \{\mathbb{R}^+, \{0\}\}$. The restriction of the projection $\mathbb{R}^n \times \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$ to the strata of $\mathcal{R}^{\pm\varepsilon_0}$ will have discrete critical values. In particular, there exists $\delta(\varepsilon_0) \in \mathbb{R}^+$ such that there are no critical values in the interval $(0, \delta(\varepsilon_0))$. Thus, by the Thom isotopy lemma, there is a stratified homeomorphism of $\overline{\mathbb{R}}^n \times (0, \delta(\varepsilon_0))$ taking the restriction of the stratification $\mathcal{R}^{\pm\varepsilon_0}$ to a product stratification.

Observe that the above constructions behave well with respect to dilation. For $t \in \mathbb{R}^+$ and $\varepsilon = t\varepsilon_0$, we can take the dilation of the above stratifications and homeomorphism for ε_0 . In particular, for $\delta(\varepsilon) = t\delta(\varepsilon_0)$, there is a stratified homeomorphism of $\overline{\mathbb{R}}^n \times (0, \delta(\varepsilon))$ taking the restriction of the stratification $\mathcal{R}^{\pm\varepsilon}$ to a product stratification.

With the above system of open neighborhoods in hand, standard sheaf identities establish the identifications for any $\delta \in (0, \delta(\varepsilon))$. □

3. Microlocal branes

In this section, we review some basic aspects of the Fukaya category of a cotangent bundle T^*X of a compact real analytic manifold X following [Nad09, NZ09]. In particular, we recall the quasi-equivalence between constructible sheaves on X and the triangulated envelope of the Fukaya category of T^*X . For the foundations of the Fukaya category, we refer the reader to the fundamental work of Fukaya–Oh–Ohta–Ono [FOOO09] and Seidel [Sei08].

3.1 Preliminaries

In what follows, we work with a fixed compact real analytic manifold X with cotangent bundle $\pi : T^*X \rightarrow X$. We often denote points of T^*X by pairs (x, ξ) , where $x \in X$ and $\xi \in T_x^*X$. The material of this section is a condensed version of the discussion of [NZ09].

Let $\theta \in \Omega^1(T^*X)$ denote the canonical one-form $\theta(v) = \xi(\pi_*v)$ for $v \in T_{(x,\xi)}(T^*X)$ and let $\omega = d\theta \in \Omega^2(T^*X)$ denote the canonical symplectic structure. For a fixed Riemannian metric on X , let $|\xi| : T^*X \rightarrow \mathbb{R}$ denote the corresponding fiberwise linear length function.

3.1.1 Compactification. To better control non-compact Lagrangians in T^*X , it is useful to work with the cospherical compactification $\overline{\pi} : \overline{T^*X} \rightarrow X$ of the projection $\pi : T^*X \rightarrow X$ obtained by attaching the cosphere bundle at infinity $\pi^\infty : T^\infty X \rightarrow X$.

Concretely, we can realize the compactification $\overline{T^*X}$ as the quotient

$$\overline{T^*X} = ((T^*X \times \mathbb{R}_{\geq 0}) \setminus (X \times \{0\})) / \mathbb{R}_+,$$

where \mathbb{R}_+ acts by dilations on both factors. The canonical inclusion $T^*X \hookrightarrow \overline{T^*X}$ sends a covector ξ to the class of $[\xi, 1]$. The boundary at infinity $T^\infty X = \overline{T^*X} \setminus T^*X$ consists of classes of the form $[\xi, 0]$ with ξ a non-zero covector. Given a Riemannian metric on X , one can identify $\overline{T^*X}$ with the closed unit disk bundle D^*X and $T^\infty X$ with the unit cosphere bundle S^*X via the map

$$[\xi, r] \mapsto (\hat{\xi}, \hat{r}), \quad \text{where } |\hat{\xi}|^2 + \hat{r}^2 = 1.$$

The boundary at infinity $T^\infty X$ carries a canonical contact distribution $\kappa \subset T(T^\infty X)$ with a well-defined notion of positive normal direction. Given a Riemannian metric on X , under the

induced identification of $T^\infty X$ with the unit cosphere bundle S^*X , the distribution κ is the kernel of the restriction of θ .

3.1.2 *Conical almost complex structure.* To better control holomorphic disks in T^*X , it is useful to work with an almost complex structure $J_{\text{con}} \in \text{End}(T(T^*X))$, which near infinity is invariant under dilations.

A fixed Riemannian metric on X provides a canonical splitting $T(T^*X) \simeq T_b \oplus T_f$, where T_b denotes the horizontal base directions and T_f the vertical fiber directions, along with a canonical isomorphism $j_0 : T_b \rightarrow T_f$ of vector bundles over T^*X . We refer to the resulting almost complex structure

$$J_{\text{Sas}} = \begin{pmatrix} 0 & j_0^{-1} \\ -j_0 & 0 \end{pmatrix} \in \text{End}(T_b \oplus T_f)$$

as the Sasaki almost complex structure since, by construction, the Sasaki metric on T^*X is given by $g_{\text{Sas}}(v, v) = \omega(u, J_{\text{Sas}}v)$.

Fix positive constants $r_0, r_1 > 0$, a bump function $b : \mathbb{R} \rightarrow \mathbb{R}$ such that $b(r) = 0$ for $r < r_0$ and $b(r) = 1$ for $r > r_1$, and set $w(x, \xi) = |\xi|^{b(|\xi|)}$, where as usual $|\xi|$ denotes the length of a covector with respect to the original metric on X . We refer to the compatible almost complex structure

$$J_w = \begin{pmatrix} 0 & w^{-1}j_0^{-1} \\ -wj_0 & 0 \end{pmatrix} \in \text{End}(T_b \oplus T_f)$$

as a (asymptotically) conical almost complex structure since, near infinity, J_{con} is invariant under dilations. The corresponding metric $g_{\text{con}}(u, v) = \omega(v, J_{\text{con}}v)$ presents T^*X near infinity as a metric cone over the unit cosphere bundle S^*X equipped with the Sasaki metric.

One can view the conical metric g_{con} as being compatible with the compactification $\overline{T^*X}$ in the sense that near infinity it treats base and angular fiber directions on equal footing. Near infinity, the metrics on the level sets of $|\xi|$ are given by scaling the Sasaki metric on the unit cosphere bundle by the factor $|\xi|^{1/2}$.

3.2 Brane structures

By a Lagrangian $j : L \hookrightarrow T^*X$, we mean a closed (but not necessarily compact) half-dimensional submanifold such that TL is isotropic for the symplectic form ω . One says that L is exact if the pullback of the one-form $j^*\theta$ is cohomologous to zero.

By a brane structure on a Lagrangian $L \hookrightarrow T^*X$, we mean a three-tuple $(\mathcal{E}, \tilde{\alpha}, \mathfrak{b})$ consisting of a flat (finite-dimensional) vector bundle $\mathcal{E} \rightarrow L$ along with a grading $\tilde{\alpha} : L \rightarrow \mathbb{R}$ (with respect to the canonical bicanonical trivialization) and a relative pin structure \mathfrak{b} (with respect to the background class $\pi^*(w_2(X))$). To remind the interested reader, we include below a short summary of what the latter two structures entail.

3.2.1 *Gradings.* The almost complex structure $J_{\text{con}} \in \text{End}(T(T^*X))$ provides a holomorphic canonical bundle $\kappa = (\wedge^{\dim X} T^{\text{hol}}(T^*X))^{-1}$. According to [NZ09], there is a canonical trivialization η^2 of the bicanonical bundle $\kappa^{\otimes 2}$ (and a canonical trivialization of κ itself if X is assumed oriented). Consider the bundle of Lagrangian planes $\mathcal{L}ag_{T^*X} \rightarrow T^*X$ and the squared phase map

$$\alpha : \mathcal{L}ag_{T^*X} \rightarrow U(1),$$

$$\alpha(\mathcal{L}) = \eta(\wedge^{\dim X} \mathcal{L})^2 / |\eta(\wedge^{\dim X} \mathcal{L})|^2.$$

For a Lagrangian $L \hookrightarrow T^*X$ and a point $x \in L$, we obtain a map $\alpha : L \rightarrow U(1)$ by setting $\alpha(x) = \alpha(T_x L)$. The Maslov class $\mu(L) \in H^1(L, \mathbb{Z})$ is the obstruction class $\mu = \alpha^*(dt)$, where dt denotes the standard one-form on $U(1)$. Thus, α has a lift to a map $\tilde{\alpha} : L \rightarrow \mathbb{R}$ if and only if $\mu = 0$, and choices of a lift form a torsor over the group $H^0(L, \mathbb{Z})$. Such a lift $\tilde{\alpha} : L \rightarrow \mathbb{R}$ is called a grading of the Lagrangian $L \hookrightarrow T^*X$.

3.2.2 Relative pin structures. Recall that the group $\text{Pin}^+(n)$ is the double cover of $O(n)$ with center $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. A pin structure on a Riemannian manifold L is a lift of the structure group of TL to $\text{Pin}^+(n)$. The obstruction to a pin structure is the second Stiefel–Whitney class $w_2(L) \in H^2(L, \mathbb{Z}/2\mathbb{Z})$, and choices of pin structures form a torsor over the group $H^1(L, \mathbb{Z}/2\mathbb{Z})$.

A relative pin structure on a submanifold $L \hookrightarrow M$ with background class $[w] \in H^2(M, \mathbb{Z}/2\mathbb{Z})$ can be defined as follows. Fix a Čech cocycle w representing $[w]$ and let $w|_L$ be its restriction to L . Then a pin structure on L relative to $[w]$ can be defined to be a $w|_L$ -twisted pin structure on TL . Concretely, this can be represented by a $\text{Pin}^+(n)$ -valued Čech 1-cochain on L whose coboundary is $w|_L$. Such structures are canonically independent of the choice of Čech representatives.

For Lagrangians $L \hookrightarrow T^*X$, we will always consider relative pin structures \flat on L with respect to the fixed background class $\pi^*(w_2(X)) \in H^2(T^*X, \mathbb{Z}/2\mathbb{Z})$.

3.3 Fukaya category

We recall here the construction of the Fukaya A_∞ -category of the cotangent bundle T^*X of a compact real analytic manifold X . Our aim is not to review all of the details, but only those relevant to our later proofs. For more details, the reader could consult [NZ09] and the references therein.

3.3.1 Objects. An object of the Fukaya category of T^*X is a four-tuple $(L, \mathcal{E}, \tilde{\alpha}, \flat)$ consisting of an exact (not necessarily compact) closed Lagrangian submanifold $L \hookrightarrow T^*X$ equipped with a brane structure: this includes a flat vector bundle $\mathcal{E} \rightarrow L$ along with a grading $\tilde{\alpha} : L \rightarrow \mathbb{R}$ (with respect to the canonical bicanonical trivialization) and a relative pin structure \flat (with respect to the background class $\pi^*(w_2(X))$).

To ensure reasonable behavior near infinity, we place two assumptions on the Lagrangian L . First, consider the compactification \bar{T}^*X obtained by adding to T^*X the cosphere bundle at infinity $T^\infty X$. Then we fix an analytic–geometric category \mathcal{C} once and for all, and assume that the closure $\bar{L} \hookrightarrow \bar{T}^*X$ is a \mathcal{C} -subset. Along with other nice properties, this implies the following two key facts.

- (i) The boundary at infinity

$$L^\infty = \bar{L} \cap T^\infty X$$

is an isotropic subset of $T^\infty X$ with respect to the induced contact structure.

- (ii) There is a real number $r > 0$ such that the restriction of the length function

$$|\xi| : L \cap \{|\xi| > r\} \rightarrow \mathbb{R}$$

has no critical points.

As discussed below, the above properties guarantee that we can make sense of ‘intersections at infinity’.

Second, to have a manageable theory of pseudoholomorphic maps with boundary on such Lagrangians, we also assume the existence of a perturbation ψ that moves the initial Lagrangian L

to a nearby Lagrangian tame (in the sense of [Sik94]) with respect to the conical metric g_{con} . As explained in [Nad09, Appendix], all such perturbations lead to equivalent calculations, though not necessarily by the most direct comparison of equations. For a recollection of the basic idea, see §3.5 below. It is worth commenting that the above encompasses all restrictions placed on the objects of the Fukaya category studied herein. In particular, though it is often convenient to think of an object as ‘asymptotically conical’, the proofs do not appeal to such an independent notion, only the technical assumptions above.

We use the term Lagrangian brane to refer to objects of the Fukaya category. When there is no chance for confusion, we often write L alone to signify the Lagrangian brane.

3.3.2 Morphisms. To define the morphisms between two branes, we must perturb Lagrangians so that their intersections occur in some bounded domain. To organize the perturbations, we recall the inductive notion of a fringed set $R_{d+1} \subset \mathbb{R}_+^{d+1}$. A fringed set $R_1 \subset \mathbb{R}_+$ is any interval of the form $(0, r)$ for some $r > 0$. A fringed set $R_{d+1} \subset \mathbb{R}_+^{d+1}$ is a subset satisfying the following.

- (i) R_{d+1} is open in \mathbb{R}_+^{d+1} .
- (ii) Under the projection $\pi : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$ forgetting the last coordinate, the image $\pi(R_{d+1})$ is a fringed set.
- (iii) If $(r_1, \dots, r_d, r_{d+1}) \in R_{d+1}$, then $(r_1, \dots, r_d, r'_{d+1}) \in R_{d+1}$ for $0 < r'_{d+1} < r_{d+1}$.

A Hamiltonian function $H : T^*X \rightarrow \mathbb{R}$ is said to be controlled if there is a real number $r > 0$ such that in the region $|\xi| > r$ we have $H(x, \xi) = |\xi|$. The corresponding Hamiltonian isotopy $\varphi_{H,t} : T^*X \rightarrow T^*X$ equals the normalized geodesic flow γ_t in the region $|\xi| > r$.

As explained in [NZ09], given Lagrangian branes $L_0, \dots, L_d \subset T^*X$ and controlled Hamiltonian functions H_0, \dots, H_d , we may choose a fringed set $R \subset \mathbb{R}^{d+1}$ such that for $(\delta_0, \dots, \delta_d) \in R$, there is a real number $r > 0$ such that for any $i \neq j$, we have

$$\varphi_{H_i, \delta_i}(\bar{L}_i) \cap \varphi_{H_j, \delta_j}(\bar{L}_j) \text{ lies in the region } |\xi| < r.$$

By a further compactly supported Hamiltonian perturbation, we may also arrange so that the intersections are transverse.

We consider finite collections of Lagrangian branes $L_0, \dots, L_d \subset T^*X$ to come equipped with such perturbation data, with the brane structures $(\mathcal{E}_i, \tilde{\alpha}_i, b_i)$ and taming perturbations ψ_i transported via the perturbations. Note that the latter makes sense since the normalized geodesic flow γ_t is an isometry of the metric g_{con} . Then, for branes L_i, L_j with $i < j$, the graded vector space of morphisms between them is defined to be

$$\text{hom}_{F(T^*X)}(L_i, L_j) = \bigoplus_{p \in \psi_i(\varphi_{H_i, \delta_i}(L_i)) \cap \psi_j(\varphi_{H_j, \delta_j}(L_j))} \mathcal{H}om(\mathcal{E}_i|_p, \mathcal{E}_j|_p)[- \text{deg}(p)],$$

where the integer $\text{deg}(p)$ denotes the Maslov grading of the linear Lagrangian subspaces at the intersection.

It is worth emphasizing that near infinity the salient aspect of the above perturbation procedure is the relative position of the perturbed branes rather than their absolute position. The following informal viewpoint can be a useful mnemonic to keep the conventions straight. In general, we always think of morphisms as ‘propagating forward in time’. Thus, to calculate the morphisms $\text{hom}_{F(T^*X)}(L_0, L_1)$, we have required that L_0, L_1 are perturbed near infinity by normalized geodesic flow so that L_1 is further in the future than L_0 . But, what is important

is not that they are both perturbed forward in time, only that L_1 is further along the timeline than L_0 . So, for example, we could perturb L_0, L_1 near infinity by normalized anti-geodesic flow as long as L_0 is further in the past than L_1 .

3.3.3 Compositions. Signed counts of pseudoholomorphic polygons provide the differential and higher composition maps of the A_∞ -structure. We use the following approach of Sikorav [Sik94] (or, equivalently, Audin–Lalonde–Polterovich [ALP94]) to ensure that the relevant moduli spaces are compact and hence the corresponding counts are finite.

First, as explained in [NZ09], the cotangent bundle T^*X equipped with the canonical symplectic form ω , conical almost complex structure J_{con} , and conical metric g_{con} is tame in the sense of [Sik94]. To see this, one can verify that g_{con} is conical near infinity, and so it is easy to derive an upper bound on its curvature and a positive lower bound on its injectivity radius.

Next, given a finite collection of branes L_0, \dots, L_d , denote by L the union of their perturbations $\psi_i(\varphi_{H_i, \delta_i}(L_i))$ as described above. By construction, the intersection of L with the region $|\xi| > r$ is a tame submanifold (in the sense of [Sik94]) with respect to the structures ω, J_{con} , and g_{con} . Namely, there exists $\rho_L > 0$ such that for every $x \in L$, the set of points $y \in L$ of distance $d(x, y) \leq \rho_L$ is contractible, and there exists C_L giving a two-point distance condition $d_L(x, y) \leq C_L d(x, y)$ whenever $x, y \in L$ with $d(x, y) < \rho_L$.

Now, consider a fixed topological type of pseudoholomorphic map

$$u : (D, \partial D) \rightarrow (T^*X, L).$$

Assume that all $u(D)$ intersect a fixed compact region and there is an *a priori* area bound $\text{Area}(u(D)) < A$. Then, as proven in [Sik94], one has compactness of the moduli space of such maps u . In fact, one has a diameter bound (depending only on the given constants) constraining how far the image $u(D)$ can stretch from the compact set.

In the situation at hand, for a given A_∞ -structure constant, we must consider pseudoholomorphic maps u from polygons with labeled boundary edges. In particular, all such maps u have image intersecting the compact set given by a single intersection point. The area of the image $u(D)$ can be expressed as the contour integral

$$\text{Area}(u(D)) = \int_{u(\partial D)} \theta.$$

Since each of the individual Lagrangian branes making up L is exact, the contour integral only depends upon the integral of θ along minimal paths between intersection points. Thus, such maps u satisfy an *a priori* area bound. We conclude that for each A_∞ -structure constant, the moduli space defining the structure constant is compact, and its points are represented by maps u with image bounded by a fixed distance from any of the intersection points.

Finally, as usual, the composition map

$$m^d : \text{hom}_{F(T^*X)}(L_0, L_1) \otimes \dots \otimes \text{hom}_{F(T^*X)}(L_{d-1}, L_d) \rightarrow \text{hom}_{F(T^*X)}(L_0, L_d)[2-d]$$

is defined as follows. Consider elements $p_i \in \text{hom}(L_i, L_{i+1})$, for $i = 0, \dots, d-1$ and $p_d \in \text{hom}(L_0, L_d)$. Then the coefficient of p_d in $m^d(p_0, \dots, p_{d-1})$ is defined to be the signed sum over pseudoholomorphic maps from a disk with $d+1$ counterclockwise cyclically ordered marked points mapping to the p_i and corresponding boundary arcs mapping to the perturbations of L_{i+1} . Each map contributes according to the holonomy of its boundary, where adjacent perturbed components L_i and L_{i+1} are glued with p_i .

Continuation maps with respect to families of perturbed branes ensure the consistency of all of our definitions. For a recollection of the basic idea, see § 3.5 below.

One should also note that it is not immediately evident that we have a well-defined A_∞ -category. Bounding the behavior of each moduli space involves fixing the complexity of the input data of which branes, intersection points, and structure constants are in play. There are potentially (at least) two more or less equivalent ways to proceed. The first is very formal and somewhat standard: as explained by Stasheff [Sta63], there is a family of operads called A_n which parameterize ‘partial’ A_∞ -structures. Compatible A_n -categories, for all n , provide an A_∞ -category since the A_∞ -operad is the union of the A_n -operads. Second, and more geometrically, to make any given calculation, we may need to insist upon smaller and smaller perturbation data. This leads one to think of each object as the ‘limit’ of a brane under smaller and smaller perturbation data. Either formulation can be implemented with the invariance of Floer calculations established in [Nad09, NZ09] and reviewed in § 3.5 below. For an alternative geometric (rather than homotopical) approach, one could consult Oh’s paper [Oh] for a detailed analysis of the relevant pseudoholomorphic disk theory.

Consider the dg category of right modules over the Fukaya category of T^*X . Throughout this paper, we write $F(T^*X)$ for the full subcategory of twisted complexes of representable modules, and refer to it as the triangulated envelope of the Fukaya category. We use the term Lagrangian brane to refer to an object of the Fukaya category, and brane to refer to an object of its triangulated envelope $F(T^*X)$.

3.4 Microlocalization

We review here the microlocalization quasi-equivalence constructed in [NZ09]. Some useful notation: for a function $m : X \rightarrow \mathbb{R}$ and a number $r \in \mathbb{R}$, we write $X_{m=r}$ for the subset $\{x \in X \mid m(x) = r\}$ and similarly for inequalities.

Let $i : U \hookrightarrow X$ be an open submanifold that is a \mathcal{C} -subset of X . Since the complement $X \setminus U$ is a closed \mathcal{C} -subset of X , we can find a non-negative function $m : X \rightarrow \mathbb{R}_{\geq 0}$ such that $X \setminus U$ is precisely the zero-set of m . Since the complement of the critical values of m forms an open \mathcal{C} -subset of \mathbb{R} , the subset $X_{m>\eta}$ is an open submanifold with smooth hypersurface boundary $X_{m=\eta}$ for any sufficiently small $\eta > 0$.

Now let $i_\alpha : U_\alpha \hookrightarrow X$, for $\alpha = 0, \dots, d$, be a finite collection of open submanifolds that are \mathcal{C} -subsets of X . Fix a non-negative function $m_\alpha : X \rightarrow \mathbb{R}_{\geq 0}$, for $\alpha = 0, \dots, d$, such that $X \setminus U_\alpha$ is precisely the zero-set of m_α . There is a fringed set $R \subset \mathbb{R}_+^{d+1}$ such that for any $(\eta_d, \dots, \eta_0) \in R$, the following holds. First, the hypersurfaces $X_{m_\alpha=\eta_\alpha}$ are all transverse. Second, for $\alpha < \beta$, there is a quasi-isomorphism of complexes

$$\text{hom}_{\text{Sh}_c(X)}(i_{\alpha*}\mathbb{C}_{U_\alpha}, i_{\beta*}\mathbb{C}_{U_\beta}) \simeq (\Omega(X_{m_\alpha \geq \eta_\alpha} \cap X_{m_\beta > \eta_\beta}, X_{m_\alpha = \eta_\alpha} \cap X_{m_\beta > \eta_\beta}), d),$$

where (Ω, d) denotes the relative de Rham complex which calculates the cohomology of the pair. Furthermore, the composition of morphisms in $\text{Sh}_c(X)$ corresponds to the wedge product of forms.

Next, let $f_\alpha : X_{m_\alpha > \eta_\alpha} \rightarrow \mathbb{R}$, for $\alpha = 0, \dots, d$, be the logarithm $f_\alpha = \log m_\alpha$. While choosing the sequence of parameters (η_d, \dots, η_0) , we can also choose a sequence of small positive parameters $(\epsilon_d, \dots, \epsilon_0)$ such that the following holds. For any $\alpha < \beta$, consider the open submanifold $X_{m_\alpha > \eta_\alpha, m_\beta > \eta_\beta} = X_{m_\alpha > \eta_\alpha} \cap X_{m_\beta > \eta_\beta}$ with corners equipped with the function $f_{\alpha,\beta} = \epsilon_\beta f_\beta - \epsilon_\alpha f_\alpha$. Then there is an open set of Riemannian metrics on X such that for all $\alpha < \beta$, it makes sense to consider the Morse complex $\mathcal{M}(X_{m_\alpha > \eta_\alpha, m_\beta > \eta_\beta}, f_{\alpha,\beta})$, and there is a

quasi-isomorphism

$$(\Omega(X_{m_\alpha \geq \eta_\alpha} \cap X_{m_\beta > \eta_\beta}, X_{m_\alpha = \eta_\alpha} \cap X_{m_\beta > \eta_\beta}), d) \simeq \mathcal{M}(X_{m_\alpha > \eta_\alpha, m_\beta > \eta_\beta}, f_{\alpha, \beta}).$$

Furthermore, homological perturbation theory provides a quasi-isomorphism between the A_∞ -composition structure on the collection of Morse complexes and the dg structure given by the wedge product of forms.

Finally, we define the microlocalization functor

$$\mu_X : \text{Sh}_c(X) \longrightarrow F(T^*X)$$

as follows. The standard objects $i_*\mathbb{C}_U$ associated to open submanifolds $i : U \hookrightarrow X$ generate the constructible dg derived category $\text{Sh}_c(X)$. Thus, to construct μ_X , it suffices to find a parallel collection of standard objects of $F(T^*X)$.

Given an open submanifold $i : U \hookrightarrow X$ and a function $m : X \rightarrow \mathbb{R}_{\geq 0}$ with zero-set the complement $X \setminus U$, define the standard Lagrangian $L_{U, f_*} \hookrightarrow T^*X|_U$ to be the graph

$$L_{U, f_*} = \Gamma_{df},$$

where df denotes the differential of the logarithm $f = \log m$.

The standard Lagrangian L_{U, f_*} comes equipped with a canonical brane structure $(\mathcal{E}, \tilde{\alpha}, \flat)$ and taming perturbation ψ . Its flat vector bundle \mathcal{E} is trivial, and its grading $\tilde{\alpha}$ and relative pin structure \flat are the canonical structures on a graph. Its taming perturbation ψ is given by the family of standard Lagrangians

$$L_{X_{m=\eta}, f_{\eta^*}} = \Gamma_{df_\eta} \quad \text{for sufficiently small } \eta > 0,$$

where $f_\eta = \log m_\eta$ is the logarithm of the shifted function $m_\eta = m - \eta$.

Now one can extend the fundamental result of Fukaya–Oh [FO97] identifying Morse moduli spaces and Fukaya moduli spaces to the current setting. Namely, one can show that for any finite ordered collection of open submanifolds $i_\alpha : U_\alpha \hookrightarrow X$, for $\alpha = 0, \dots, d$, and any finite collection of A_∞ -compositions respecting the order, there is a fringed set $R \subset \mathbb{R}^{d+1}$ such that for any parameters $(\eta_d, \dots, \eta_0) \in R$, the Morse moduli spaces of the ordered collection of functions f_{η_α} are isomorphic to the Fukaya moduli spaces of the ordered collection of standard branes $L_{X_{m=\eta}, f_{\eta^*}}$ (after further variable dilations of the functions and branes).

Once and for all, for each $U \subset X$, let us choose a non-negative function $m : X \rightarrow \mathbb{R}_{\geq 0}$ such that the complement $X \setminus U$ is the zero-set of m . We denote the resulting standard brane L_{U, f_*} , where $f = \log m$, by the abbreviated notation L_{U^*} . We define the functor μ_X so that on objects we have

$$\mu_X(i_*\mathbb{C}_U) = L_{U^*}.$$

The arguments of [NZ09] outlined above show that calculations among standard branes are equivalent to calculations among corresponding standard sheaves. In particular, given $U \subset X$, there is no preference as to which function $m : X \rightarrow \mathbb{R}_{\geq 0}$ is used, and hence no preference as to which standard brane is used. It follows that any standard brane for $U \subset X$ will have the same structure with respect to other standard branes. For example, a standard brane and its dilations will have the same structure with respect to other standard branes.

THEOREM 3.4.1 [Nad09, NZ09]. *Microlocalization is a quasi-equivalence*

$$\mu_X : \text{Sh}_c(X) \xrightarrow{\sim} F(T^*X).$$

One can rephrase the equivalence of the theorem to say that every brane can be expressed in terms of standard branes. Thus, to understand properties of branes it suffices to study collections of standard branes. It follows from the discussion immediately preceding the theorem that all of the standard branes for a given subset are equivalent to each other (since the arguments of [NZ09] show that they lead to the same calculations as the unambiguous standard sheaves), and all branes are equivalent to themselves under dilation (since the dilation of a standard brane is equally well a standard brane and thus leads to the same calculations).

The theorem admits the following refinement. Given a conical Lagrangian $\Lambda \subset T^*X$, let $F_\Lambda(T^*X) \subset F(T^*X)$ denote the full subcategory generated by branes $L \subset T^*X$ whose boundary $L^\infty = \bar{L} \cap T^\infty X$ lies in the boundary $\Lambda^\infty = \bar{\Lambda} \cap T^\infty X$.

For any stratification $\mathcal{S} = \{S_\alpha\}$ of X , let $\Lambda_{\mathcal{S}} \subset T^*X$ denote the union of conormal bundles $\Lambda_{\mathcal{S}} = \bigcup_\alpha T^*_{S_\alpha} X$. By construction, the microlocalization μ_X takes the full subcategory $\text{Sh}_{\mathcal{S}}(X) \subset \text{Sh}_c(X)$ to the full subcategory $F(T^*X)_{\Lambda_{\mathcal{S}}} \subset F(T^*X)$.

Conversely, given an object L of $F(T^*X)$, let $\Lambda \subset T^*X$ be a conical Lagrangian such that the boundary L^∞ lies in the boundary Λ^∞ (for instance, one can minimally take Λ to be the cone over L^∞). Then, for any object \mathcal{F} of $\text{Sh}_c(X)$ such that $\mu_X(\mathcal{F}) \simeq L$, and for any stratification $\mathcal{S} = \{S_\alpha\}$ of X such that $\Lambda \subset \Lambda_{\mathcal{S}} = \bigcup_\alpha T^*_{S_\alpha} X$, the object \mathcal{F} belongs to $\text{Sh}_{\mathcal{S}}(X)$.

One proves the theorem and the above refinement by studying non-characteristic families of branes. By a one-parameter family of closed (but not necessarily compact) submanifolds (without boundary) in T^*X , we mean a closed submanifold

$$\mathcal{L} \hookrightarrow \mathbb{R} \times T^*X$$

satisfying the following.

- (i) The restriction of the projection $p_{\mathbb{R}} : \mathbb{R} \times T^*X \rightarrow \mathbb{R}$ to the submanifold \mathcal{L} is non-singular.
- (ii) There is a real number $r > 0$ such that the restriction of the product $p_{\mathbb{R}} \times |\xi| : \mathbb{R} \times T^*X \rightarrow \mathbb{R} \times [0, \infty)$ to the subset $\{|\xi| > r\} \cap \mathcal{L}$ is proper and non-singular.
- (iii) There is a compact interval $[a, b] \hookrightarrow \mathbb{R}$ such that the restriction of the projection $p_X : \mathbb{R} \times T^*X \rightarrow T^*X$ to the submanifold $p_{\mathbb{R}}^{-1}([\mathbb{R} \setminus [a, b]] \cap \mathcal{L}$ is locally constant.

Note that conditions (1) and (2) will be satisfied if the restriction of the projection $\bar{p}_{\mathbb{R}} : \mathbb{R} \times \bar{T}^*X \rightarrow \mathbb{R}$ to the closure $\bar{\mathcal{L}} \hookrightarrow \bar{T}^*X$ is non-singular as a stratified map, but the weaker condition stated is a useful generalization. It implies in particular that the fibers $\mathcal{L}_s = p_{\mathbb{R}}^{-1}(s) \cap \mathcal{L} \hookrightarrow T^*X$ are all diffeomorphic, but imposes no requirement that their boundaries at infinity should all be homeomorphic as well.

By a one-parameter family of tame Lagrangian branes in T^*X , we mean a one-parameter family of closed submanifolds $\mathcal{L} \hookrightarrow \mathbb{R} \times T^*X$ in the above sense such that the fibers $\mathcal{L}_s = p_{\mathbb{R}}^{-1}(s) \cap \mathcal{L} \hookrightarrow T^*X$ also satisfy:

- (i) the fibers \mathcal{L}_s are exact tame Lagrangians with respect to the usual symplectic structure and any almost complex structure conical near infinity;
- (ii) the fibers \mathcal{L}_s are equipped with a locally constant brane structure $(\mathcal{E}_s, \tilde{\alpha}_s, b_s)$ with respect to the usual background classes.

Note that if we assume that \mathcal{L}_0 is an exact Lagrangian, then \mathcal{L}_s being an exact Lagrangian is equivalent to the family \mathcal{L} being given by the flow φ_{H_s} of the vector field of a time-dependent Hamiltonian $H_s : T^*X \rightarrow \mathbb{R}$. Note as well that a brane structure consists of topological data, so it can be transported unambiguously along the fibers of such a family.

Fix a conical Lagrangian $\Lambda \subset T^*X$ with boundary $\Lambda^\infty = \overline{\Lambda} \cap T^\infty X$. As above, let $F_\Lambda(T^*X)$ be the full subcategory of $F(T^*X)$ generated by Lagrangian branes L whose boundary $L^\infty = \overline{L} \cap T^\infty X$ lies in Λ^∞ .

Suppose that $\mathfrak{L} \hookrightarrow \mathbb{R} \times T^*X$ is a one-parameter family of tame Lagrangian branes. We will say that \mathfrak{L} is Λ -non-characteristic if

$$\overline{\mathfrak{L}}_s \cap \Lambda^\infty = \emptyset \quad \text{for all } s \in \mathbb{R}.$$

PROPOSITION 3.4.2 [Nad09]. *Suppose that $\mathfrak{L} \hookrightarrow \mathbb{R} \times T^*X$ is a Λ -non-characteristic one-parameter family of tame Lagrangian branes. For any test object P of $F_\Lambda(T^*X)$, there are functorial quasi-isomorphisms among the Floer complexes*

$$\text{hom}_{F(T^*X)}(P, \mathfrak{L}_s) \quad \text{for all } s \in \mathbb{R}.$$

The proof of the proposition is very general and does not use that X is compact in any serious way. For example, it holds when X is complete, or in fact for tame Lagrangian branes in more general exact symplectic targets. We will use it in later sections for the cotangent bundle of a vector space.

3.5 Floer invariance

In the definition of the Fukaya category of T^*X recalled in § 3.3, as well as in the microlocalization quasi-equivalence recalled in § 3.4, we have appealed to results of [Nad09, NZ09] on the invariance of Floer calculations under suitable motions of non-compact branes. To make the current paper as self contained as possible, we include here a brief section reviewing the (somewhat *ad hoc*) arguments which establish the following basic example of this invariance. One could also consult Oh’s paper [Oh], which contains a detailed analysis of the canonical structures provided by pseudoholomorphic disk theory.

PROPOSITION 3.5.1. *Suppose that \mathfrak{L}_s is a family of objects of $F(T^*X)$. Suppose that L' is a fixed test object which is disjoint from \mathfrak{L}_s near infinity for all s . Suppose that \mathfrak{L}_s is transverse to L' except for finitely many points.*

Then, for any a, b with \mathfrak{L}_a and \mathfrak{L}_b transverse to L' , the Floer chain complexes $CF(\mathfrak{L}_a, L')$ and $CF(\mathfrak{L}_b, L')$ are quasi-isomorphic.

Before proving the proposition in full, it is convenient to first prove the following special case.

LEMMA 3.5.2. *Suppose that \mathfrak{L}_s is a family of objects of $F(T^*X)$. Suppose that L' is a fixed test object which is disjoint from \mathfrak{L}_s near infinity for all s .*

Fix s_0 and assume that \mathfrak{L}_{s_0} is transverse to L' . Then there is an $\epsilon > 0$ so that for all $s_1 \in (s_0 - \epsilon, s_0 + \epsilon)$, the Floer chain complexes $CF(\mathfrak{L}_{s_0}, L')$ and $CF(\mathfrak{L}_{s_1}, L')$ are quasi-isomorphic.

Proof. By our assumptions on the tame behavior (in the sense of [Sik94]) of \mathfrak{L}_{s_0} and L' near infinity, the moduli spaces giving the differential of $CF(\mathfrak{L}_{s_0}, L')$ are compact. This follows from the *a priori* C^0 -bound: there is some $r_0 \gg 0$ such that no disk in the moduli space leaves the region $|\xi| < r_0$, where (x, ξ) are local coordinates on T^*X and $|\xi|$ is the Riemannian metric.

Choose some $r_1 > r_0$. Then, for very small $\epsilon > 0$ and any $s_1 \in (s_0 - \epsilon, s_0 + \epsilon)$, we may decompose the motion $\mathfrak{L}_{s_0} \rightsquigarrow \mathfrak{L}_{s_1}$ into two parts: first, a motion $\mathfrak{L}_{s_0} \rightsquigarrow L$ supported in the region $|\xi| > r_0$; and then second, a compactly supported motion $L \rightsquigarrow \mathfrak{L}_{s_1}$. We must show that each of the above two motions leads to a quasi-isomorphism.

First, for the motion $\mathfrak{L}_{s_0} \rightsquigarrow L$, since we have not changed \mathfrak{L}_{s_0} or L' in the region $|\xi| < r_0$, the same *a priori* C^0 -bounds of [Sik94] hold (they only depend on the Lagrangians in the region $|\xi| < r_0$), and the pseudoholomorphic strips for the pair (\mathfrak{L}_{s_0}, L') and for the pair (L, L') are in fact exactly the same (we could perversely attach ‘wild’ non-intersecting ends to either and it would not make a difference). Thus, we can take the ‘continuation map’ to be the identity.

(One should probably not use the term ‘continuation map’ for such a construction. Rather, it is an example of the more general setup of parameterized moduli spaces. In the above setting, one can obtain a uniform C^0 -bound over the family, so the parameterized moduli space is compact, and hence one can apply standard cobordism arguments to prove that the matrix coefficients at the initial and final times are the same. We thank an anonymous referee for this perspective on the argument.)

Second, the motion $L \rightsquigarrow \mathfrak{L}_{s_1}$ is compactly supported, so standard partial differential equation (PDE) techniques provide a continuation map. □

Proof of Proposition 3.5.1. By the previous lemma, it suffices to show that for any s_0 with \mathfrak{L}_{s_0} not (necessarily) transverse to L' , there is a small $\epsilon > 0$ such that the Floer chain complexes $CF(\mathfrak{L}_{s_0-\epsilon}, L')$ and $CF(\mathfrak{L}_{s_0+\epsilon}, L')$ are quasi-isomorphic.

To see this, let $H_s(x, \xi)$ be a (time-dependent) Hamiltonian giving the motion \mathfrak{L}_s . Choose a bump function $b(|\xi|)$ which is 0 near infinity and 1 on a compact set containing all of the (possibly non-transverse) intersection points $\mathfrak{L}_{s_0} \cap L'$.

The product Hamiltonian $\tilde{H}(x, \xi) = b(|\xi|)H_s(x, \xi)$ gives a family $\tilde{\mathfrak{L}}_s$ through the base object \mathfrak{L}_{s_0} satisfying: (1) $\tilde{\mathfrak{L}}_s$ is transverse to L' whenever $|s - s_0|$ is small and non-zero and (2) $\tilde{\mathfrak{L}}_s$ is equal to \mathfrak{L}_{s_0} near infinity. Therefore, since the motion of $\tilde{\mathfrak{L}}_s$ is compactly supported, standard PDE techniques provide a continuation map giving a quasi-isomorphism between $CF(\tilde{\mathfrak{L}}_{s_0-\epsilon}, L')$ and $CF(\tilde{\mathfrak{L}}_{s_0+\epsilon}, L')$ for small enough $\epsilon > 0$.

Finally, returning to the bump function $b(|\xi|)$, one can construct motions $\mathfrak{L}_{s_0-\epsilon} \rightsquigarrow \tilde{\mathfrak{L}}_{s_0-\epsilon}$ and $\tilde{\mathfrak{L}}_{s_0+\epsilon} \rightsquigarrow \mathfrak{L}_{s_0+\epsilon}$ which are supported near infinity and thus in particular always transverse to L' . Thus, we may apply the previous lemma to obtain quasi-isomorphisms between $CF(\mathfrak{L}_{s_0-\epsilon}, L')$ and $CF(\tilde{\mathfrak{L}}_{s_0-\epsilon}, L')$ and, similarly, between $CF(\tilde{\mathfrak{L}}_{s_0+\epsilon}, L')$ and $CF(\mathfrak{L}_{s_0+\epsilon}, L')$. Putting together the above, we obtain a quasi-isomorphism between $CF(\mathfrak{L}_{s_0-\epsilon}, L')$ and $CF(\mathfrak{L}_{s_0+\epsilon}, L')$. □

Remark 3.5.3. The above proposition (which is a condensed form of arguments of [Nad09, NZ09]) is closely related to Question 1.3 of Oh’s paper [Oh], which asks whether a homology-level continuation map constructed by a careful limiting argument with PDE techniques is induced by a chain-level morphism. While we have not investigated this, it is not hard to believe that the quasi-isomorphism of the above proposition provides the desired lift.

4. Fourier transform for branes

In this section, we study the symplectic topology of the cotangent bundle of a real finite-dimensional vector space V . Our aim is to describe a Fukaya theory of branes in $T^*V \simeq V \times V^*$ that treats the horizontal and vertical directions as symmetrically as possible. Wherever possible, we will appeal to arguments of the preceding section and restrict the discussion here to the new aspects which arise.

4.1 Preliminaries

Fix a real finite-dimensional vector space V .

We will write V_1 in place of V , and V_2 for its dual V^* . Let \mathbb{R}_x^n denote standard Euclidean space with coordinate $x = (x_i)$, and let \mathbb{R}_ξ^n denote the dual Euclidean space with coordinate $\xi = (\xi_i)$, so that $\langle x, \xi \rangle = \sum_{i=1}^n x_i \xi_i$. By choosing an isomorphism $V_1 \simeq \mathbb{R}_x^n$, we obtain a dual isomorphism $V_2 \simeq \mathbb{R}_\xi^n$. For concreteness, we will often assume such identifications have been fixed (though our constructions will not depend on the specific identifications).

Let ω_1, ω_2 denote the respective canonical exact symplectic forms on T^*V_1, T^*V_2 . Under the canonical identifications

$$T^*V_1 \simeq V_1 \times V_2 \simeq T^*V_2,$$

the canonical exact symplectic forms are related by $\omega_1 = -\omega_2$ since, in local coordinates, we have

$$\omega_1 = \sum_{i=1}^n d\xi_i dx_i, \quad \omega_2 = \sum_{i=1}^n dx_i d\xi_i.$$

In what follows, unless otherwise stated, we will break symmetry and work with the symplectic structure ω_1 . Thus, to identify $V_1 \times V_2$ and T^*V_2 as symplectic manifolds, we will compose the above canonical identification with the negation map on the first factor: $x \mapsto -x, \xi \mapsto \xi$. When it is not clear from context, we will write

$$\iota : T^*V_2 \xrightarrow{\sim} V_1 \times V_2$$

for the symplectic identification.

Given a positive-definite quadratic form on V_1 , we obtain an identification $V_1 \simeq V_2$. For vectors $v_1 \in V_1, v_2 \in V_2$, we write $|v_1|, |v_2|$ for the respective lengths of v_1, v_2 .

4.1.1 Symmetric compactification. To control non-compact Lagrangians in $V_1 \times V_2$, we will work with a symmetric product compactification.

Given a vector space V , consider the spherical compactification

$$\bar{V} = (V \times \mathbb{R}_{\geq 0} \setminus \{(0, 0)\}) / \mathbb{R}^+,$$

where \mathbb{R}_+ acts by dilations on both factors. The canonical inclusion $V \hookrightarrow \bar{V}$ sends a vector v to the class of $[v, 1]$. The boundary sphere at infinity $V^\infty = \bar{V} \setminus V$ consists of classes of the form $[v, 0]$ with v a non-zero vector.

Now let \bar{V}_1, \bar{V}_2 be the spherical compactifications of V_1, V_2 with spheres at infinity V_1^∞, V_2^∞ . We will work with the symmetric product compactification $\bar{V}_1 \times \bar{V}_2$. Its boundary at infinity is the disjoint union of a codimension one boundary

$$B = (V_1 \times V_2^\infty) \amalg (V_1^\infty \times V_2),$$

along with a codimension two corner

$$C = V_1^\infty \times V_2^\infty.$$

4.1.2 Symmetric almost complex structure. To control holomorphic disks in $V_1 \times V_2$, we will work with a symmetric almost complex structure.

Fix a positive-definite quadratic form on V_1 and let $j_0 : V_1 \xrightarrow{\sim} V_2$ be the corresponding identification.

Fix $0 < r_1 < r_2$ and a smooth increasing function $b : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $b(r) = 0$ for $r < r_1$ and $b(r) = 1$ for $r_2 < r$. Consider the functions

$$w(v_1, v_2) = \frac{1 + |v_2|}{1 + |v_1|}, \quad \rho(v_1, v_2) = |v_1|^2 + |v_2|^2$$

and define the ω_1 -compatible almost complex structure

$$J_{\text{sym}} = \begin{pmatrix} 0 & w^{-b(\rho)}j_0^{-1} \\ -w^{b(\rho)}j_0 & 0 \end{pmatrix} \in \text{End}(T(V_1 \oplus V_2)).$$

We refer to J_{sym} as a symmetric (asymptotically) conical almost complex structure. The corresponding metric $g_{\text{sym}}(v, v) = \omega_1(v, J_{\text{sym}}v)$ is complete and tame.

4.2 From branes to sheaves

In this section, we explain how to associate constructible sheaves to branes in $V_1 \times V_2$.

4.2.1 *Branes in $V_1 \times V_2$.* To define Lagrangian branes in $V_1 \times V_2$, we must first fix background structures. Of course, the background class for relative pin structures is trivial since it lies in $H^2(V_1 \times V_2, \mathbb{Z}/2\mathbb{Z}) \simeq 0$. Thus, the only background structures of note are the bicanonical bundle and its trivialization. We will break symmetry and work with the bicanonical bundle $\kappa_1^{\otimes 2}$ and trivialization η_1^2 coming from the canonical identification $T^*V_1 \simeq V_1 \times V_2$.

By a Lagrangian brane $L \hookrightarrow V_1 \times V_2$, we mean a four-tuple $(L, \mathcal{E}, \tilde{\alpha}, \flat)$ consisting of an exact (not necessarily compact) closed Lagrangian submanifold $L \hookrightarrow V_1 \times V_2$ equipped with a brane structure: this includes a flat vector bundle $\mathcal{E} \rightarrow L$ along with a grading $\tilde{\alpha} : L \rightarrow \mathbb{R}$ (with respect to the bicanonical trivialization η_1^2 of the bicanonical bundle $\kappa_1^{\otimes 2}$) and a pin structure \flat .

Furthermore, we place two assumptions on the Lagrangian L . Recall the symmetric product compactification $\bar{V}_1 \times \bar{V}_2$, and the symmetric conical almost complex structure J_{sym} and corresponding metric g_{sym} . First, we assume that the closure $\bar{L} \hookrightarrow \bar{V}_1 \times \bar{V}_2$ is a \mathcal{C} -subset. Second, we assume the existence of a perturbation ψ that moves the initial Lagrangian L to a nearby Lagrangian tame (in the sense of [Sik94]) with respect to the symmetric conical metric g_{sym} .

Let us take a moment to comment on the asymmetry of the above definition. Recall that to identify T^*V_2 and $V_1 \times V_2$ as symplectic manifolds, we compose the canonical identification with the negation map on the first factor: $x \mapsto -x, \xi \mapsto \xi$. Thus, the bicanonical bundle $\kappa_2^{\otimes 2}$ coming from the resulting identification $\iota : T^*V_2 \xrightarrow{\sim} V_1 \times V_2$ is canonically identified with $\kappa_1^{\otimes 2}$. The bicanonical trivialization η_2^2 arising via ι satisfies $\eta_2^2 = -\eta_1^2$. We will identify the two bicanonical trivializations via the path $\eta_1^2 \rightsquigarrow \eta_2^2$ induced by the positively oriented path $1 \rightsquigarrow -1$ inside \mathbb{C}^\times .

Example 4.2.1. Suppose that $V_1 = \mathbb{R}_x, V_2 = \mathbb{R}_\xi$, and let $L \hookrightarrow \mathbb{R}_x \times \mathbb{R}_\xi$ be the Lagrangian $L = \{x\xi = 1\}$. Then the canonical grading of L as a graph in $T^*\mathbb{R}_x$ coincides with its canonical grading as a graph in $T^*\mathbb{R}_\xi$. Note that if we write η for the dual coordinate of ξ , then, as a graph in $T^*\mathbb{R}_\xi \simeq \mathbb{R}_\xi \times \mathbb{R}_\eta$, we have $L = \{\xi\eta = -1\}$.

4.2.2 *Fukaya A_∞ -structures.* Suppose that we have a collection of Lagrangian branes in $V_1 \times V_2$ that are pairwise transverse and whose boundaries at infinity are disjoint.

We define a Fukaya pre- A_∞ -category structure (or partially define an A_∞ -category structure) on the collection as follows. The graded vector space of morphisms between distinct branes is

defined to be

$$\text{hom}(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \mathcal{H}om(\mathcal{E}_0|_p, \mathcal{E}_1|_p)[- \deg(p)],$$

where the integer $\deg(p)$ denotes the Maslov grading of the linear Lagrangian subspaces at the intersection.

Signed counts of pseudoholomorphic polygons provide the differential and higher composition maps of the pre- A_∞ -structure. To ensure that the relevant moduli spaces are compact, we appeal to the same arguments used for cotangent bundles in §3.3. First, the target $V_1 \times V_2$ with the symplectic form ω_1 , almost complex structure J_{sym} , and corresponding metric g_{sym} is tame. Next, let $L \hookrightarrow V_1 \times V_2$ be the union of any finite number of branes from the collection. By assumption, the intersection of L with the region $\rho(x, \xi) = |x|^2 + |\xi|^2 > r$, for large $r > 0$, is a tame submanifold. Since our branes are exact, for a given A_∞ -structure constant, the relevant pseudoholomorphic maps

$$u : (D, \partial D) \rightarrow (V_1 \times V_2, L)$$

satisfy an *a priori* area bound. Thus, we have a diameter bound on their images $u(D)$, and hence the moduli space of all such maps is compact.

Finally, for distinct branes, the composition map

$$m^d : \text{hom}_{F(T^*X)}(L_0, L_1) \otimes \cdots \otimes \text{hom}_{F(T^*X)}(L_{d-1}, L_d) \rightarrow \text{hom}_{F(T^*X)}(L_0, L_d)[2 - d]$$

is defined as follows. Consider elements $p_i \in \text{hom}(L_i, L_{i+1})$, for $i = 0, \dots, d - 1$, and $p_d \in \text{hom}(L_0, L_d)$. Then the coefficient of p_d in $m^d(p_0, \dots, p_{d-1})$ is defined to be the signed sum over pseudoholomorphic maps from a disk with $d + 1$ counterclockwise cyclically ordered marked points mapping to p_i and corresponding boundary arcs mapping to the perturbations of L_{i+1} . Each map contributes according to the holonomy of its boundary, where adjacent perturbed components L_i and L_{i+1} are glued with p_i .

Our key technical tool for understanding calculations in $F(V_1 \times V_2)_{\text{pre}}$ is their invariance under certain motions of branes. The following is a direct generalization of Proposition 3.4.2 and the discussion preceding it.

By a one-parameter family of closed (but not necessarily compact) submanifolds (without boundary) in $V_1 \times V_2$, we mean a closed submanifold

$$\mathcal{L} \hookrightarrow \mathbb{R} \times V_1 \times V_2$$

satisfying the following.

(i) The restriction of the projection $p_{\mathbb{R}} : \mathbb{R} \times V_1 \times V_2 \rightarrow \mathbb{R}$ to the submanifold \mathcal{L} is non-singular.

(ii) There is a real number $r > 0$ such that the restriction of the product $p_{\mathbb{R}} \times \rho : V_1 \times V_2 \rightarrow \mathbb{R} \times [0, \infty)$ to the subset $\{\rho > r\} \cap \mathcal{L}$ is proper and non-singular.

(iii) There is a compact interval $[a, b] \hookrightarrow \mathbb{R}$ such that the restriction of the projection $p_X : \mathbb{R} \times T^*X \rightarrow T^*X$ to the submanifold $p_{\mathbb{R}}^{-1}([\mathbb{R} \setminus [a, b]]) \cap \mathcal{L}$ is locally constant.

By a one-parameter family of tame Lagrangian branes in $V_1 \times V_2$, we mean a one-parameter family of closed submanifolds $\mathcal{L} \hookrightarrow \mathbb{R} \times V_1 \times V_2$ in the above sense such that the fibers $\mathcal{L}_s = p_{\mathbb{R}}^{-1}(s) \cap \mathcal{L} \hookrightarrow V_1 \times V_2$ are exact tame Lagrangians equipped with a locally constant brane structure $(\mathcal{E}_s, \tilde{\alpha}_s, \flat_s)$.

Fix a biconical Lagrangian $\Lambda \subset V_1 \times V_2$ with boundary Λ^∞ . Let $F_\Lambda(V_1 \times V_2)_{\text{pre}}$ be the pre- A_∞ -category of Lagrangian branes L whose boundary L^∞ lies in Λ^∞ . Suppose that $\mathfrak{L} \hookrightarrow \mathbb{R} \times V_1 \times V_2$ is a one-parameter family of tame Lagrangian branes. We will say that \mathfrak{L} is Λ -non-characteristic if

$$\overline{\mathfrak{L}}_s \cap \Lambda^\infty = \emptyset \quad \text{for all } s \in \mathbb{R}.$$

As with Proposition 3.4.2, one can repeat the proof from [Nad09] to establish the following assertion. In fact, the same argument gives a further generalization for tame Lagrangian branes in other exact symplectic targets.

PROPOSITION 4.2.2. *Suppose that $\mathfrak{L} \hookrightarrow \mathbb{R} \times V_1 \times V_2$ is a Λ -non-characteristic one-parameter family of tame Lagrangian branes. For any test object P of $F_\Lambda(V_1 \times V_2)_{\text{pre}}$, there are functorial quasi-isomorphisms among the Floer complexes*

$$\text{hom}_{F(V_1 \times V_2)_{\text{pre}}}(P, \mathfrak{L}_s) \quad \text{for all } s \in \mathbb{R}.$$

4.2.3 *Functionals on sheaves.* Let us single out a special class of branes in $V_1 \times V_2$. Consider the situation from the perspective of V_1 , so that we have $T^*V_1 \simeq V_1 \times V_2$. We say that a brane L is compact along the first factor if its projection to V_1 is compact or, equivalently, the closure \overline{L} lies in $\overline{T^*V_1} \simeq V_1 \times \overline{V_2}$.

Consider the collection of branes in $V_1 \times V_2$ that are compact along the first factor. Then we can regard them as branes in T^*V_1 , and accordingly define an honest Fukaya A_∞ -category structure on them repeating our constructions for cotangent bundles in §3.3. In particular, we can find a fringed set parameterizing controlled Hamiltonian perturbations that move the branes so that they do not intersect at infinity. We write $F(T^*V_1)_\kappa$ for the triangulated envelope of the Fukaya category of such branes.

Consider the full subcategory $\text{Sh}_c(V_1)_\kappa \subset \text{Sh}_c(V_1)$ of compactly supported objects. Since V_1 is complete (though non-compact), we can repeat the construction of microlocalization to obtain a quasi-embedding

$$\mu_{V_1} : \text{Sh}_c(V_1)_\kappa \hookrightarrow F(V_1)_\kappa.$$

Now fix a brane $L \hookrightarrow V_1 \times V_2$, without any assumption on whether it is compact in either direction. Let us measure the structure of L using the quasi-embedding μ_{V_1} . Namely, we can consider the right module

$$\tilde{\pi}_{V_1}(L) : \text{Sh}_c(V_1)_\kappa^{\text{op}} \rightarrow Ch, \quad \tilde{\pi}_{V_1}(L)(\mathcal{F}) = \text{hom}_{F(V_1 \times V_2)_{\text{pre}}}(\mu_{V_1}(\mathcal{F}), L).$$

By definition, if the boundary of $\mu_{V_1}(\mathcal{F})$ intersects the boundary of L , then we simply perturb the former according to our usual conventions for cotangent bundles. Thus, we can always unambiguously make the necessary calculations to define an honest module.

By Proposition 3.4.2 and the results on quasi-representability from [Nad09], the module $\tilde{\pi}_{V_1}(L)$ is quasi-represented by an object of $\text{Sh}_c(V_1)$, which we denote by $\pi_{V_1}(L)$. Given a relatively compact open submanifold $i : U \hookrightarrow V_1$, we have quasi-isomorphisms of complexes

$$\pi_{V_1}(L)(U) \simeq \text{hom}_{\text{Sh}_c(V_1)}(i^! \mathcal{C}U, \mathcal{F}) \simeq \text{hom}_{F(V_1 \times V_2)_{\text{pre}}}(LU^! \otimes \text{or}_{V_1}[-\dim V_1], L).$$

Fix a conical (with respect to the second factor) Lagrangian $\Lambda \hookrightarrow T^*V_1$ such that the part of the boundary L^∞ that lies in $V_1 \times V_2^\infty$ in fact lies in the boundary Λ^∞ . Then, for any stratification $\mathcal{S} = \{S_\alpha\}$ of V_1 such that $\Lambda \subset \Lambda_{\mathcal{S}} = \bigcup_\alpha T_{S_\alpha}^* V_1$, the object $\pi_{V_1}(L)$ lies in $\text{Sh}_{\mathcal{S}}(V_1)$.

Similarly, we say that a brane L is compact along the second factor if its projection to V_2 is compact or, equivalently, the closure \bar{L} lies in $\bar{T}^*V_2 \simeq \bar{V}_1 \times V_2$. We can define a Fukaya A_∞ -structure on the collection of branes in $V_1 \times V_2$ that are compact along the second factor. We write $F(T^*V_2)_\kappa$ for the triangulated envelope of the Fukaya category of such branes. Consider the full subcategory $\text{Sh}_c(V_2)_\kappa \subset \text{Sh}_c(V_2)$ of compactly supported objects. In parallel with the above discussion, we have a quasi-embedding

$$\mu_{V_2} : \text{Sh}_c(V_2)_\kappa \hookrightarrow F(T^*V_2)_\kappa$$

that leads to a right module

$$\tilde{\pi}_{V_2}(L) : \text{Sh}_c(V_2)_\kappa^{\text{op}} \rightarrow Ch, \quad \tilde{\pi}_{V_2}(L)(\mathcal{F}) = \text{hom}_{F(V_1 \times V_2)_{\text{pre}}}(\mu_{V_2}(\mathcal{F}), L)$$

that is quasi-represented by an object of $\text{Sh}_c(V_2)$, which we denote by $\pi_{V_2}(L)$. Fix a conical (with respect to the first factor) Lagrangian $\Lambda \hookrightarrow T^*V_2$ such that the part of the boundary L^∞ that lies in $V_1^\infty \times V_2$ in fact lies in the boundary Λ^∞ . Then, for any stratification $\mathcal{S} = \{S_\alpha\}$ of V_2 such that $\Lambda \subset \Lambda_\mathcal{S} = \bigcup_\alpha T_{S_\alpha}^*V_2$, the object $\pi_{V_2}(L)$ lies in $\text{Sh}_\mathcal{S}(V_2)$.

4.2.4 Dilation invariance. Recall that for X a compact manifold, the natural dilation \mathbb{R}^+ -action on T^*X is not a symplectomorphism, but, as explained immediately after Theorem 3.4.1, any brane $L \hookrightarrow T^*X$ is quasi-isomorphic to its dilations.

The target $V_1 \times V_2$ admits two commuting \mathbb{R}^+ -dilation actions, which we denote by α_1^t, α_2^t , for any $t \in \mathbb{R}^+$. In general, given a brane $L \hookrightarrow V_1 \times V_2$, its dilations along either factor will not define quasi-isomorphic sheaves on the corresponding factor.

Example 4.2.3. Suppose that $V_1 = \mathbb{R}_x, V_2 = \mathbb{R}_\xi$, and let $L \hookrightarrow \mathbb{R}_x \times \mathbb{R}_\xi$ be a brane with underlying Lagrangian $L = \{\xi = 1\}$. Then $\pi_{V_1}(L)$ is quasi-isomorphic to its dilations, but $\pi_{V_2}(L)$ is not.

PROPOSITION 4.2.4. *There are quasi-isomorphisms*

$$\begin{aligned} \pi_{V_1}(\alpha_2^t(L)) &\simeq \pi_{V_1}(L), & \pi_{V_2}(\alpha_1^t(L)) &\simeq \pi_{V_2}(L), \\ \pi_{V_1}(\alpha_1^t(L)) &\simeq \alpha_{1*}^t(\pi_{V_1}(L)), & \pi_{V_2}(\alpha_2^t(L)) &\simeq \alpha_{2*}^t(\pi_{V_2}(L)). \end{aligned}$$

Proof. We prove the left-hand column of assertions; the right-hand column is the same.

For the first assertion, observe that the microlocalization μ_{V_1} is independent of the dilation α_2^t . To be precise, rather than scale the module L , we can scale the image of μ_{V_1} by the inverse. But, such scalings lead to quasi-isomorphic calculations among standard branes.

For the second assertion, observe that the linear diffeomorphism α_1^t of the base V_1 induces the scaling $\alpha_1^t \circ \alpha_2^{t^{-1}}$ of its cotangent $T^*V_1 \simeq V_1 \times V_2$. Thus, we have a quasi-isomorphism

$$\pi_{V_1}(\alpha_1^t \alpha_2^{t^{-1}}(L)) \simeq \alpha_{1*}^t(\pi_{V_1}(L))$$

since all of our constructions are invariant under linear diffeomorphisms. On the other hand, by the first assertion applied to $\alpha_1^t(L)$, we have a quasi-isomorphism

$$\pi_{V_1}(\alpha_2^{t^{-1}} \alpha_1^t(L)) \simeq \pi_{V_1}(\alpha_1^t(L)).$$

Combining the above quasi-isomorphisms proves the second assertion. □

4.3 Balanced branes

Our main theorem will not apply to all branes $L \hookrightarrow V_1 \times V_2$, but rather those satisfying a technical assumption which we explain here. (The main theorem fails without it.)

Recall that the compactification $\overline{V}_1 \times \overline{V}_2$ has a codimension one boundary $B = (V_1 \times V_2^\infty) \amalg (V_1^\infty \times V_2)$ and a codimension two corner $C = V_1^\infty \times V_2^\infty$.

Given a brane $L \hookrightarrow V_1 \times V_2$, let $L_C = \overline{L} \cap C \hookrightarrow L^\infty$ be the intersection of the closure \overline{L} with the corner C . Consider the cone

$$\text{Cone}(L_C) = \{(v_1, v_2) \in (\overline{V}_1 \setminus \{0\}) \times (\overline{V}_2 \setminus \{0\}) \mid ([v_1], [v_2]) \in L_C\} \hookrightarrow (\overline{V}_1 \setminus \{0\}) \times (\overline{V}_2 \setminus \{0\})$$

of non-zero elements whose projectivizations lie in L_C . By definition, it is invariant under the two commuting dilations α_1^t, α_2^t .

For $\delta_1, \delta_2 \in \mathbb{R}^+$, consider the dilated brane $L(\delta_1, \delta_2) = \alpha_1^{\delta_1} \alpha_2^{\delta_2} L$. We would expect that as $\delta_1, \delta_2 \rightarrow 0$, the dilated brane $L(\delta_1, \delta_2)$ would consist of two parts: a limit that collects along the closure of the axes $\overline{V}_1 \times \{0\}, \{0\} \times V_2$, and the conical trace $\text{Cone}(L_C)$. In general, the situation is much more complicated.

By a balanced Lagrangian brane $L \hookrightarrow V_1 \times V_2$, we mean a Lagrangian brane satisfying the following additional hypothesis: the intersection L_C of its closure \overline{L} with the corner C is of the expected dimension:

$$\dim L_C = \dim L - 2.$$

This implies that for every neighborhood $\mathcal{N}_{\text{axes}}$ of the closure of the axes $\overline{V}_1 \times \{0\}, \{0\} \times V_2$, and every neighborhood $\mathcal{N}_{\text{cone}}$ of the cone $\text{Cone}(L_C)$, there exist $\delta_1, \delta_2 \in \mathbb{R}^+$ such that the dilated brane $L(\delta_1, \delta_2)$ lies in the union of the neighborhoods:

$$L(\delta_1, \delta_2) \subset \mathcal{N}_{\text{axes}} \cup \mathcal{N}_{\text{cone}}.$$

Example 4.3.1. Suppose that $V_1 = \mathbb{R}_x, V_2 = \mathbb{R}_\xi$, and let $L \hookrightarrow \mathbb{R}_x \times \mathbb{R}_\xi$ be a Lagrangian brane. Then L is symmetric if and only if the closure $\overline{L} \hookrightarrow \overline{\mathbb{R}}_x \times \overline{\mathbb{R}}_\xi$ is disjoint from the four corners $\{(\pm\infty, \pm\infty)\} \subset \overline{\mathbb{R}}_x \times \overline{\mathbb{R}}_\xi$. For instance, the Lagrangian $\{\xi = 1/x\}$ can underlie a balanced brane, but the Lagrangian $\{x = \xi\}$ cannot underlie a balanced brane.

The above example explains our use of the term balanced. The intersection of the closure \overline{L} with the corners C is unstable, and threatens to teeter over in the direction of either factor.

4.4 Main theorem

This section contains the main technical result of this paper. In order to help the reader follow our arguments, we have isolated the case when $\dim V_1 = 1$. Furthermore, the arguments of the general case are best understood as a product of copies of the dimension one case.

4.4.1 Case of dimension one. The fearless reader could skip this section, and continue in the next section with the general case. But, many of the intricacies of the general case already appear here. Furthermore, the constructions of the general case can be understood as a product of constructions described here. With this in mind, our aim here is not to give the most concrete proof possible, but rather to argue in parallel with what will be required for the general case.

We will write \mathbb{R}_x to denote V_1 with coordinate x , and \mathbb{R}_ξ to denote $V_2 = V_1^*$ with dual coordinate ξ . Recall that a brane $L \hookrightarrow \mathbb{R}_x \times \mathbb{R}_\xi$ is balanced if its closure $\overline{L} \hookrightarrow \overline{\mathbb{R}}_x \times \overline{\mathbb{R}}_\xi$ is disjoint from the four corners $\{(\pm\infty, \pm\infty)\} \subset \overline{\mathbb{R}}_x \times \overline{\mathbb{R}}_\xi$.

THEOREM 4.4.1. *Let $L \hookrightarrow \mathbb{R}_x \times \mathbb{R}_\xi$ be a balanced brane. Then there are quasi-isomorphisms*

$$(\Upsilon(\pi_{\mathbb{R}_x}(L)))^\wedge \simeq \Upsilon(\pi_{\mathbb{R}_\xi}(L)), \quad (\Upsilon(\pi_{\mathbb{R}_\xi}(L)))^\vee \simeq \Upsilon(\pi_{\mathbb{R}_x}(L)).$$

Proof. We prove the first identity; the second follows immediately by applying the inverse Fourier transform to the first.

Consider an object $\mathcal{F} \in \text{Sh}_c(\mathbb{R}_x/\mathbb{R}^+)$ and its Fourier transform $\mathcal{F}^\wedge \in \text{Sh}_c(\mathbb{R}_\xi/\mathbb{R}^+)$. Recall that for any open convex cone $u : U \hookrightarrow \mathbb{R}_\xi$, with closed polar cone $v : U^\circ \hookrightarrow \mathbb{R}_x$ with interior $\text{int}(v) : \text{int}(U^\circ) \hookrightarrow \mathbb{R}_x$, we have a quasi-isomorphism

$$\text{hom}_{\text{Sh}_c(\mathbb{R}_\xi)}(u_! \mathbb{C}_U, \mathcal{F}^\wedge) \simeq \text{hom}_{\text{Sh}_c(\mathbb{R}_x)}(\text{int}(v)_* \mathbb{C}_{\text{int}(U^\circ)}, \mathcal{F}).$$

Furthermore, for the inclusion of open convex cones $U_0 \hookrightarrow U_1 \hookrightarrow \mathbb{R}_\xi$, the above quasi-isomorphisms fit into a commutative (at the level of cohomology) square. The resulting compatible collection of quasi-isomorphisms characterizes \mathcal{F}^\wedge .

Thus, to prove the first assertion, it suffices to establish the formula

$$\text{hom}_{\text{Sh}_c(\mathbb{R}_\xi)}(u_! \mathbb{C}_U, \Upsilon(\pi_{\mathbb{R}_\xi}(L))) \simeq \text{hom}_{\text{Sh}_c(\mathbb{R}_x)}(\text{int}(v)_* \mathbb{C}_{\text{int}(U^\circ)}, \Upsilon(\pi_{\mathbb{R}_x}(L))) \tag{†}$$

compatibly for all open convex cones.

In dimension one, it is possible to list all of the open convex cones

$$\begin{aligned} U &= \mathbb{R}_\xi^+, & U &= \mathbb{R}_\xi^-, & U &= \mathbb{R}_\xi, \\ U^\circ &= \overline{\mathbb{R}_x^+}, & U^\circ &= \overline{\mathbb{R}_x^-}, & U^\circ &= \{0\}. \end{aligned}$$

We will establish formula (†) for $U = \mathbb{R}_\xi^+$, $U^\circ = \overline{\mathbb{R}_x^+}$. We leave it to the reader to modify the arguments for the other cases, and to check that the constructions are compatible with inclusions.

Thus, our aim is to show that there is a quasi-isomorphism

$$\text{hom}_{\text{Sh}_c(\mathbb{R}_\xi)}(u_! \mathbb{C}_{\mathbb{R}_\xi^+}, \Upsilon(\pi_{\mathbb{R}_\xi}(L))) \simeq \text{hom}_{\text{Sh}_c(\mathbb{R}_x)}(u_* \mathbb{C}_{\overline{\mathbb{R}_x^+}}, \Upsilon(\pi_{\mathbb{R}_x}(L))). \tag{‡}$$

Our strategy will be to construct a brane in $\mathbb{R}_x \times \mathbb{R}_\xi$ such that both sides of formula (‡) are quasi-isomorphic to its Floer pairing with a dilation of L .

Fix a pair $\varepsilon_x, \varepsilon_\xi \in \mathbb{R}$ (soon to be specialized to the case $\varepsilon_x < 0, \varepsilon_\xi > 0$), and consider the open subsets

$$q(\varepsilon_x) : Q(\varepsilon_x) = \{x \in \mathbb{R}_x \mid x > \varepsilon_x\} \hookrightarrow \mathbb{R}_x, \quad q(\varepsilon_\xi) : Q(\varepsilon_\xi) = \{\xi \in \mathbb{R}_\xi \mid \xi > \varepsilon_\xi\} \hookrightarrow \mathbb{R}_\xi$$

and the Lagrangian

$$P(\varepsilon_x, \varepsilon_\xi) = \{(x, \xi) \in \mathbb{R}_x \times \mathbb{R}_\xi \mid x > \varepsilon_x, (x - \varepsilon_x)(\xi - \varepsilon_\xi) = 1\} \hookrightarrow \mathbb{R}_x \times \mathbb{R}_\xi.$$

We equip $P(\varepsilon_x, \varepsilon_\xi)$ with the brane structure coming from its identification with the standard brane $L_{Q(\varepsilon_x)*} \hookrightarrow T^*\mathbb{R}_x$ or, equivalently, its identification with the costandard brane $L_{Q(\varepsilon_\xi)!} \hookrightarrow T^*\mathbb{R}_\xi$. Note that the boundary of $P(\varepsilon_x, \varepsilon_\xi)$ inside $\overline{\mathbb{R}_x} \times \overline{\mathbb{R}_\xi}$ consists of the two points $(\varepsilon_x, +\infty)$ and $(+\infty, \varepsilon_\xi)$.

For any $\delta_x, \delta_\xi \in \mathbb{R}^+$, consider the dilated brane

$$L(\delta_x, \delta_\xi) = \alpha_x(\delta_x)\alpha_\xi(\delta_\xi)L \hookrightarrow \mathbb{R}_x \times \mathbb{R}_\xi.$$

By assumption, L is balanced, so the boundary of L inside $\overline{\mathbb{R}_x} \times \overline{\mathbb{R}_\xi}$ is disjoint from the corners $\{(\pm\infty, \pm\infty)\}$. Thus, for small $\delta_x, \delta_\xi \in \mathbb{R}^+$, the boundary of $L(\delta_x, \delta_\xi)$ is arbitrarily close to the points $\{(\pm\infty, 0), (0, \pm\infty)\}$. In particular, the boundary of $L(\delta_1, \delta_2)$ does not intersect the boundary of $P(\varepsilon_x, \varepsilon_\xi)$. Thus, for sufficiently small $\delta_x, \delta_\xi \in \mathbb{R}^+$, it makes sense to consider the Floer complex

$$\text{hom}_{F(\mathbb{R}_x \times \mathbb{R}_\xi)}(P(\varepsilon_x, \varepsilon_\xi), L(\delta_x, \delta_\xi)). \tag{Fl}$$

We claim that for fixed $\varepsilon_x < 0$, $\varepsilon_\xi > 0$, and sufficiently small $\delta_x, \delta_\xi \in \mathbb{R}^+$, both sides of formula (‡) are quasi-isomorphic to the Floer complex (F1). We will first explain why the right-hand side of (‡) is quasi-isomorphic to (F1), and then give the parallel arguments for the left-hand side.

Thus, our immediate aim is to show that for fixed $\varepsilon_x < 0$, $\varepsilon_\xi > 0$, and sufficiently small $\delta_x, \delta_\xi \in \mathbb{R}^+$, there is a quasi-isomorphism

$$\text{hom}_{F(\mathbb{R}_x \times \mathbb{R}_\xi)}(P(\varepsilon_x, \varepsilon_\xi), L(\delta_x, \delta_\xi)) \simeq \text{hom}_{\text{Sh}_c(\mathbb{R}_x)}(u_*\mathbb{C}_{\mathbb{R}_x^+}, \Upsilon(\pi_{\mathbb{R}_x}(L))). \tag{rhs}$$

Let us unpack the right-hand side of the sought-after identity (rhs). To that end, for large $r_x \in \mathbb{R}^+$, consider the intervals

$$[\varepsilon_x, r_x] \xrightarrow{c} [\varepsilon_x, r_x] \xrightarrow{d} \mathbb{R}_x$$

and define the pushforward

$$\mathcal{R}(\varepsilon_x, r_x) = d_*c!\mathbb{C}_{[\varepsilon_x, r_x]} \in \text{Sh}_c(\mathbb{R}_x).$$

Consider the corresponding brane $R(\varepsilon_x, r_x) = \mu_{\mathbb{R}_x}(\mathcal{R}(\varepsilon_x, r_x))$ with underlying Lagrangian

$$R(\varepsilon_x, r_x) = \{(x, \xi) \in \mathbb{R}_x \times \mathbb{R}_\xi \mid x \in (\varepsilon_x, r_x), \xi(x - \varepsilon_x)(r_x - x) = 1\} \hookrightarrow \mathbb{R}_x \times \mathbb{R}_\xi.$$

As long as $\delta_\xi \in \mathbb{R}^+$ is sufficiently small, by Lemma 2.3.1, Proposition 4.2.4, standard adjunctions, and the definition of $\pi_{\mathbb{R}_x}$, we have quasi-isomorphisms

$$\begin{aligned} \text{hom}_{\text{Sh}_c(\mathbb{R}_x)}(u_*\mathbb{C}_{\mathbb{R}_x^+}, \Upsilon(\pi_{\mathbb{R}_x}(L))) &\simeq \text{hom}_{\text{Sh}_c(E)}(\mathcal{R}(\varepsilon_x, r_x), \pi_{\mathbb{R}_x}(L(\delta_x, \delta_\xi))) \\ &\simeq \text{hom}_{F(\mathbb{R}_x \times \mathbb{R}_\xi)}(R(\varepsilon_x, r_x), L(\delta_x, \delta_\xi)). \end{aligned}$$

Thus, to establish the sought-after identity (rhs), it suffices to establish a quasi-isomorphism

$$\text{hom}_{F(\mathbb{R}_x \times \mathbb{R}_\xi)}(P(\varepsilon_x, \varepsilon_\xi), L(\delta_x, \delta_\xi)) \simeq \text{hom}_{F(\mathbb{R}_x \times \mathbb{R}_\xi)}(R(\varepsilon_x, r_x), L(\delta_x, \delta_\xi)). \tag{★_x}$$

Again, since L is balanced, for small $\delta_x, \delta_\xi \in \mathbb{R}^+$, the boundary of $L(\delta_x, \delta_\xi)$ inside $\overline{\mathbb{R}_x} \times \overline{\mathbb{R}_\xi}$ is arbitrarily close to the points $\{(\pm\infty, 0), (0, \pm\infty)\}$. Thus, we can find a $[0, 1]$ -family of branes $\mathfrak{P}_t \hookrightarrow \mathbb{R}_x \times \mathbb{R}_\xi$ such that

$$\mathfrak{P}_0(\varepsilon_x, \varepsilon_\xi) = P(\varepsilon_x, \varepsilon_\xi), \quad \mathfrak{P}_1(\varepsilon_x, \varepsilon_\xi) = R(\varepsilon_x, r_x),$$

and \mathfrak{P}_t is non-characteristic with respect to $L(\delta_x, \delta_\xi)$. In other words, the boundary of \mathfrak{P}_t is disjoint from the boundary of $L(\delta_x, \delta_\xi)$, for all times $t \in [0, 1]$.

Thus, by Proposition 4.2.2, we have the sought-after identity (★_x), and in turn the identity (rhs).

Our next aim is to give parallel arguments explaining why the left-hand side of (‡) is quasi-isomorphic to (F1). So, we need to verify that for fixed $\varepsilon_x < 0$, $\varepsilon_\xi > 0$, and sufficiently small $\delta_x, \delta_\xi \in \mathbb{R}^+$, there is a quasi-isomorphism

$$\text{hom}_{F(\mathbb{R}_x \times \mathbb{R}_\xi)}(P(\varepsilon_x, \varepsilon_\xi), L(\delta_x, \delta_\xi)) \simeq \text{hom}_{\text{Sh}_c(\mathbb{R}_\xi)}(u!\mathbb{C}_{\mathbb{R}_\xi^+}, \Upsilon(\pi_{\mathbb{R}_\xi}(L))). \tag{lhs}$$

Let us unpack the right-hand side of the sought-after identity (lhs). To that end, for large $r_\xi \in \mathbb{R}^+$, consider the interval $c : (\varepsilon_\xi, r_\xi) \hookrightarrow \mathbb{R}_\xi$ and define the pushforward

$$\mathcal{T}(\varepsilon_\xi, r_\xi) = c!\mathbb{C}_{(\varepsilon_\xi, r_\xi)} \in \text{Sh}_c(\mathbb{R}_\xi).$$

Consider the corresponding costandard brane $T(\varepsilon_\xi, r_\xi) = \mu_{\mathbb{R}_\xi}(\mathcal{T}(\varepsilon_\xi, r_\xi))$ with underlying Lagrangian

$$T(\varepsilon_\xi, r_\xi) = \{(x, \xi) \in \mathbb{R}_x \times \mathbb{R}_\xi \mid \xi \in (\varepsilon_\xi, r_\xi), x(\xi - \varepsilon_\xi)(r_\xi - \xi) = 2\xi - r_\xi - \varepsilon_\xi\} \hookrightarrow \mathbb{R}_x \times \mathbb{R}_\xi.$$

As long as $\delta_x \in \mathbb{R}^+$ is sufficiently small, by Lemma 2.3.1, Proposition 4.2.4, standard adjunctions, and the definition of $\pi_{\mathbb{R}_\xi}$, we have quasi-isomorphisms

$$\begin{aligned} \text{hom}_{\text{Sh}_c(\mathbb{R}_\xi)}(u_! \mathbb{C}_{\mathbb{R}_\xi^+}, \Upsilon(\pi_{\mathbb{R}_\xi}(L))) &\simeq \text{hom}_{\text{Sh}_c(E)}(\mathcal{T}(\varepsilon_\xi, r_\xi), \pi_{\mathbb{R}_\xi}(L(\delta_x, \delta_\xi))) \\ &\simeq \text{hom}_{F(\mathbb{R}_x \times \mathbb{R}_\xi)}(T(\varepsilon_\xi, r_\xi), L(\delta_x, \delta_\xi)). \end{aligned}$$

Thus, to establish the sought-after identity (lhs), it suffices to establish a quasi-isomorphism

$$\text{hom}_{F(\mathbb{R}_x \times \mathbb{R}_\xi)}(P(\varepsilon_x, \varepsilon_\xi), L(\delta_x, \delta_\xi)) \simeq \text{hom}_{F(\mathbb{R}_x \times \mathbb{R}_\xi)}(T(\varepsilon_\xi, r_\xi), L(\delta_x, \delta_\xi)). \quad (\star_\xi)$$

Again, since L is balanced, for small $\delta_x, \delta_\xi \in \mathbb{R}^+$, the boundary of $L(\delta_x, \delta_\xi)$ inside $\overline{\mathbb{R}}_x \times \overline{\mathbb{R}}_\xi$ is arbitrarily close to the points $\{(\pm\infty, 0), (0, \pm\infty)\}$. Thus, we can find a $[0, 1]$ -family of branes $\mathfrak{P}_t \hookrightarrow \mathbb{R}_x \times \mathbb{R}_\xi$ such that

$$\mathfrak{P}_0(\varepsilon_x, \varepsilon_\xi) = P(\varepsilon_x, \varepsilon_\xi), \quad \mathfrak{P}_1(\varepsilon_x, \varepsilon_\xi) = T(\varepsilon_\xi, r_\xi),$$

and \mathfrak{P}_t is non-characteristic with respect to $L(\delta_x, \delta_\xi)$. In other words, the boundary of \mathfrak{P}_t is disjoint from the boundary of $L(\delta_x, \delta_\xi)$, for all times $t \in [0, 1]$.

Thus, by Proposition 4.2.2, we have the sought-after identity (\star_ξ) , and in turn the identity (lhs).

Putting the preceding together, we have identified the left (lhs) and right-hand (rhs) sides of formula (‡) with the Floer complex (F1). As mentioned above, we leave the reader to modify the arguments to establish formula (†) in the other cases, and to check its compatibility with inclusions. This establishes the first assertion of the theorem. \square

4.4.2 *General case.* Now we arrive at the main technical result of this paper in Theorem 4.4.2. We hope that the reader has followed the arguments of the preceding section in the case of dimension one. Here we return to the general setting of an arbitrary real finite-dimensional vector space V_1 with dual V_2 .

THEOREM 4.4.2. *Let $L \hookrightarrow V_1 \times V_2$ be a balanced brane. Then there are quasi-isomorphisms*

$$(\Upsilon(\pi_{V_1}(L)))^\wedge \simeq \Upsilon(\pi_{V_2}(L)), \quad (\Upsilon(\pi_{V_2}(L)))^\vee \simeq \Upsilon(\pi_{V_1}(L)).$$

Before giving the proof, let us record the following special case, which we will apply in the context of Springer theory.

COROLLARY 4.4.3. *Suppose that L is conical along the second factor in the sense that $\alpha_t^! L = L$ for all $t \in \mathbb{R}^+$. Then we have $(\Upsilon(\pi_{V_1}(L)))^\wedge \simeq \pi_{V_2}(L)$.*

Proof. This follows from the theorem, Proposition 4.2.4, and the fact that Υ is the identity functor on conic objects. \square

Proof of Theorem 4.4.2. We prove the first identity; the second follows immediately by applying the inverse Fourier transform to the first.

Consider an object $\mathcal{F} \in \text{Sh}_c(V_1/\mathbb{R}^+)$ and its Fourier transform $\mathcal{F}^\wedge \in \text{Sh}_c(V_2/\mathbb{R}^+)$. Recall that for any convex open cone $u: U \hookrightarrow V_2$, with closed polar cone $v: U^\circ \hookrightarrow V_1$ with interior

$\text{int}(v) : \text{int}(U^\circ) \hookrightarrow V_1$, we have a quasi-isomorphism

$$\text{hom}_{\text{Sh}_c(V_2)}(u! \mathbb{C}_U, \mathcal{F}^\wedge) \simeq \text{hom}_{\text{Sh}_c(V_1)}(\text{int}(v)_* \mathbb{C}_{\text{int}(U^\circ)}, \mathcal{F}).$$

Furthermore, for the inclusion of open convex cones $U_0 \hookrightarrow U_1 \hookrightarrow V_2$, the above quasi-isomorphisms fit into a commutative (at the level of cohomology) square. The resulting compatible collection of quasi-isomorphisms characterizes \mathcal{F}^\wedge .

Thus, to prove the first assertion, it suffices to establish the formula

$$\text{hom}_{\text{Sh}_c(V_2)}(u! \mathbb{C}_U, \Upsilon(\pi_{V_2}(L))) \simeq \text{hom}_{\text{Sh}_c(V_1)}(\text{int}(v)_* \mathbb{C}_{\text{int}(U^\circ)}, \Upsilon(\pi_{V_1}(L))) \tag{†}$$

compatibly for all open convex cones.

In fact, it suffices to establish formula (†) for any collection of open cones as long as they generate the conic topology. For technical convenience, we will focus on open cones $u : U \hookrightarrow V_2$ which become identified $U \simeq Q_\xi^n$ with the standard open quadrant

$$q : Q_\xi^n = \{(\xi_1, \dots, \xi_n) \in \mathbb{R}_\xi^n \mid \xi_i > 0\} \hookrightarrow \mathbb{R}_\xi^n$$

under some linear isomorphism $V_2 \simeq \mathbb{R}_\xi^n$. We refer to such open cones as open quadrant cones. Note that the interior of the closed polar cone of the standard open quadrant $q : Q_\xi^n \hookrightarrow \mathbb{R}_\xi^n$ is nothing other than the standard open quadrant $q : Q_x^n \hookrightarrow \mathbb{R}_x^n$. The collection of open quadrant cones, together with V_2 itself, generate the conic topology.

It will be useful to specialize further to a particular collection of open quadrant cones. For any $\delta_x, \delta_\xi \in \mathbb{R}^+$, consider the dilated brane

$$L(\delta_x, \delta_\xi) = \alpha_x(\delta_x) \alpha_\xi(\delta_\xi) L \hookrightarrow V_1 \times V_2.$$

Recall that since L is balanced, there exists a biconic Lagrangian $\Lambda \hookrightarrow V_1 \times V_2$ such that for every neighborhood $\mathcal{N}_{\text{axes}}$ of the closure of the axes $\overline{V_1} \times \{0\}$, $\{0\} \times V_2$, and every neighborhood $\mathcal{N}_{\text{cone}}$ of Λ , there exist $\delta_1, \delta_2 \in \mathbb{R}^+$ such that the dilated brane $L(\delta_1, \delta_2)$ lies in the union of the neighborhoods

$$L(\delta_1, \delta_2) \subset \mathcal{N}_{\text{axes}} \cup \mathcal{N}_{\text{cone}}.$$

Given an open quadrant cone $u : U \hookrightarrow V_2$, consider the biconic Lagrangian

$$\Lambda_U \hookrightarrow T^* \mathbb{R}_\xi^n \simeq V_1 \times V_2$$

obtained by taking the union of the conormals to the facets of the boundary $\partial U \hookrightarrow \mathbb{R}_\xi^n$. We will focus on open quadrant cones $u : U \hookrightarrow V_2$ such that Λ_U is disjoint from Λ away from $\mathcal{N}_{\text{axes}}$. By a dimension count, one can check that this is a generic condition. Hence, the collection of all such open quadrant cones, together with V_2 itself, generate the conic topology.

Now, without loss of generality, to establish formula (†) for a given generic quadrant cone $u : U \hookrightarrow V_2$, it suffices to choose an identification $V_2 \simeq \mathbb{R}_\xi^n$, and to establish (†) for the standard open quadrant $q : Q_\xi^n \hookrightarrow \mathbb{R}_\xi^n$. This case of formula (†) will occupy the remainder of our arguments. We leave it to the reader to handle the case $U = \mathbb{R}_\xi^n$ itself, and to check that our constructions are compatible with inclusions.

Thus, our aim is to show that there is a quasi-isomorphism

$$\text{hom}_{\text{Sh}_c(\mathbb{R}_\xi^n)}(q! \mathbb{C}_{Q_\xi^n}, \Upsilon(\pi_{V_2}(L))) \simeq \text{hom}_{\text{Sh}_c(\mathbb{R}_x^n)}(q_* \mathbb{C}_{Q_x^n}, \Upsilon(\pi_{V_1}(L))). \tag{†}$$

Our strategy will be to construct a brane in $\mathbb{R}_x^n \times \mathbb{R}_\xi^n$ such that both sides of formula (†) are quasi-isomorphic to its Floer pairing with a dilation of L .

Fix a pair $\varepsilon_x, \varepsilon_\xi \in \mathbb{R}$ (to be specialized to the case $\varepsilon_x < 0, \varepsilon_\xi > 0$ momentarily) and consider the open subsets

$$\begin{aligned} q(\varepsilon_x) : Q_x^n(\varepsilon_x) &= \{x \in \mathbb{R}_x^n \mid x_i > \varepsilon_x, \text{ for } i = 1, \dots, n\} \hookrightarrow \mathbb{R}_x^n, \\ q(\varepsilon_\xi) : Q_\xi^n(\varepsilon_\xi) &= \{\xi \in \mathbb{R}_\xi^n \mid \xi_i > \varepsilon_\xi, \text{ for } i = 1, \dots, n\} \hookrightarrow \mathbb{R}_\xi^n. \end{aligned}$$

By definition, we have $Q_x^n(0) = Q_x^n$ and $Q_\xi^n(0) = Q_\xi^n$.

Consider inside of $\mathbb{R}_x^n \times \mathbb{R}_\xi^n$ the Lagrangian

$$P(\varepsilon_x, \varepsilon_\xi) = \{(x, \xi) \in \mathbb{R}_x^n \times \mathbb{R}_\xi^n \mid x_i > \varepsilon_x, (x_i - \varepsilon_x)(\xi_i - \varepsilon_\xi) = 1, \text{ for } i = 1, \dots, n\}$$

equipped with the brane structure coming from its identification with the standard brane

$$L_{Q_x^n(\varepsilon_x)^*} \hookrightarrow T^*\mathbb{R}_x^n$$

or, equivalently, its identification with the costandard brane

$$L_{Q_\xi^n(\varepsilon_\xi)!} \hookrightarrow T^*\mathbb{R}_\xi^n.$$

By construction, for fixed $\varepsilon_x < 0, \varepsilon_\xi > 0$, and sufficiently small $\delta_x, \delta_\xi \in \mathbb{R}^+$, the boundary of the dilated brane $L(\delta_1, \delta_2)$ does not intersect the boundary of $P(\varepsilon_x, \varepsilon_\xi)$. Thus, it makes sense to consider the Floer complex

$$\text{hom}_{F(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)}(P(\varepsilon_x, \varepsilon_\xi)L(\delta_x, \delta_\xi)) \quad \text{for sufficiently small } \delta_x, \delta_\xi \in \mathbb{R}^+. \tag{F1}$$

We claim that both sides of formula (‡) are quasi-isomorphic to the Floer complex (F1). We will first explain why the right-hand side of (‡) is quasi-isomorphic to (F1), and then give the parallel arguments for the left-hand side.

Thus, our immediate aim is to show that for fixed $\varepsilon_x < 0, \varepsilon_\xi > 0$, and sufficiently small $\delta_x, \delta_\xi \in \mathbb{R}^+$, there is a quasi-isomorphism

$$\text{hom}_{F(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)}(P(\varepsilon_x, \varepsilon_\xi), L(\delta_x, \delta_\xi)) \simeq \text{hom}_{\text{Sh}_c(\mathbb{R}_x^n)}(q_*\mathbb{C}_{Q_x^n}, \Upsilon(\pi_{V_1}(L))). \tag{rhs}$$

Let us unpack the right-hand side of the sought-after identity (rhs). To that end, let us introduce the translated variables $\hat{x}_i = x_i - \varepsilon_x$ for $i = 1, \dots, n$. For large $r_x \in \mathbb{R}^+$, consider the truncations

$$Q_x^n(\varepsilon_x) \cap \{|\hat{x}|^2 < r_x\} \xrightarrow{c} Q_x^n(\varepsilon_x) \cap \{|\hat{x}|^2 \leq r_x\} \xrightarrow{d} \mathbb{R}_x^n$$

and define the pushforward

$$\mathcal{R}(\varepsilon_x, r_x) = d_*c_!\mathbb{C}_{Q_x^n(\varepsilon_x) \cap \{|\hat{x}|^2 < r_x\}} \in \text{Sh}_c(\mathbb{R}_x^n).$$

Consider the corresponding brane $R(\varepsilon_x, r_x) = \mu_{\mathbb{R}_x^n}(\mathcal{R}(\varepsilon_x, r_x))$ with underlying Lagrangian

$$R(\varepsilon_x, r_x) = \{(x, \xi) \in \mathbb{R}_x^n \times \mathbb{R}_\xi^n \mid \hat{x}_i > 0, \xi \hat{x}_i(r_x - |\hat{x}|^2) = 1, |\hat{x}|^2 < r_x\} \hookrightarrow \mathbb{R}_x^n \times \mathbb{R}_\xi^n.$$

As long as $\delta_\xi \in \mathbb{R}^+$ is sufficiently small, by Lemma 2.3.1, Proposition 4.2.4, standard adjunctions, and the definition of π_{V_1} , we have quasi-isomorphisms

$$\begin{aligned} \text{hom}_{\text{Sh}_c(\mathbb{R}_x^n)}(q_*\mathbb{C}_{Q_x^n}, \Upsilon(\pi_{\mathbb{R}_x^n}(L))) &\simeq \text{hom}_{\text{Sh}_c(E)}(\mathcal{R}(\varepsilon_x, r_x), \pi_{V_1}(L(\delta_x, \delta_\xi))) \\ &\simeq \text{hom}_{F(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)}(R(\varepsilon_x, r_x), L(\delta_x, \delta_\xi)). \end{aligned}$$

Thus, to establish the sought-after identity (rhs), it suffices to establish a quasi-isomorphism

$$\text{hom}_{F(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)}(P(\varepsilon_x, \varepsilon_\xi), L(\delta_x, \delta_\xi)) \simeq \text{hom}_{F(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)}(R(\varepsilon_x, r_x), L(\delta_x, \delta_\xi)). \tag{\star_x}$$

By construction, we can find a $[0, 1]$ -family of branes $\mathfrak{P}_t \hookrightarrow \mathbb{R}_x^n \times \mathbb{R}_\xi^n$ such that

$$\mathfrak{P}_0(\varepsilon_x, \varepsilon_\xi) = P(\varepsilon_x, \varepsilon_\xi), \quad \mathfrak{P}_1(\varepsilon_x, \varepsilon_\xi) = R(\varepsilon_x, r_x),$$

and \mathfrak{P}_t is non-characteristic with respect to $L(\delta_x, \delta_\xi)$. In other words, the boundary of \mathfrak{P}_t is disjoint from the boundary of $L(\delta_x, \delta_\xi)$ for all times $t \in [0, 1]$. To explicitly define \mathfrak{P}_t , one can exploit that an open quadrant cone is a product of one-dimensional cones.

Thus, by Proposition 4.2.2, we have the sought-after identity (\star_x) , and in turn the identity (rhs).

Next, we give parallel arguments explaining why the left-hand side of (\ddagger) is quasi-isomorphic to (Fl). So, we need to verify that for fixed $\varepsilon_x < 0$, $\varepsilon_\xi > 0$, and sufficiently small $\delta_x, \delta_\xi \in \mathbb{R}^+$, there is a quasi-isomorphism

$$\text{hom}_{F(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)}(P(\varepsilon_x, \varepsilon_\xi), L(\delta_x, \delta_\xi)) \simeq \text{hom}_{\text{Sh}_c(\mathbb{R}_\xi^n)}(u! \mathbb{C}_{\mathbb{R}_\xi^+}, \Upsilon(\pi_2(L))). \tag{lhs}$$

Let us unpack the right-hand side of the sought-after identity (lhs). To that end, let us introduce the translated variables $\hat{\xi}_i = \xi_i - \varepsilon_\xi$ for $i = 1, \dots, n$. For large $r_\xi \in \mathbb{R}^+$, consider the truncation

$$c : Q_\xi^n(\varepsilon_\xi) \cap \{|\hat{\xi}|^2 < r_\xi\} \hookrightarrow \mathbb{R}_\xi^n$$

and define the pushforward

$$\mathcal{T}(\varepsilon_\xi, r_\xi) = c! \mathbb{C}_{Q_\xi^n(\varepsilon_\xi) \cap \{|\hat{\xi}|^2 < r_\xi\}} \in \text{Sh}_c(\mathbb{R}_\xi^n).$$

Consider the corresponding costandard brane $T(\varepsilon_\xi, r_\xi) = \mu_{\mathbb{R}_\xi^n}(\mathcal{T}(\varepsilon_\xi, r_\xi))$ with underlying Lagrangian

$$T(\varepsilon_\xi, r_\xi) = \{(x, \xi) \in \mathbb{R}_x^n \times \mathbb{R}_\xi^n \mid \hat{\xi}_i > 0, \hat{\xi}_i x_i (r_\xi - |\hat{\xi}|^2) = r_\xi - |\hat{\xi}|^2 + 2\hat{\xi}_i^2, |\hat{\xi}|^2 < r_\xi\}.$$

As long as $\delta_x \in \mathbb{R}^+$ is sufficiently small, by Lemma 2.3.1, Proposition 4.2.4, standard adjunctions, and the definition of π_{V_2} , we have quasi-isomorphisms

$$\begin{aligned} \text{hom}_{\text{Sh}_c(\mathbb{R}_\xi^n)}(u! \mathbb{C}_{\mathbb{R}_\xi^+}, \Upsilon(\pi_2(L))) &\simeq \text{hom}_{\text{Sh}_c(E)}(\mathcal{T}(\varepsilon_\xi, r_\xi), \pi_{V_2}(L(\delta_x, \delta_\xi))) \\ &\simeq \text{hom}_{F(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)}(T(\varepsilon_\xi, r_\xi), L(\delta_x, \delta_\xi)). \end{aligned}$$

Thus, to establish the sought-after identity (lhs), it suffices to establish a quasi-isomorphism

$$\text{hom}_{F(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)}(P(\varepsilon_x, \varepsilon_\xi), L(\delta_x, \delta_\xi)) \simeq \text{hom}_{F(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)}(T(\varepsilon_\xi, r_\xi), L(\delta_x, \delta_\xi)). \tag{\star_\xi}$$

By construction, we can find a $[0, 1]$ -family of branes $\mathfrak{P}_t \hookrightarrow \mathbb{R}_x^n \times \mathbb{R}_\xi^n$ such that

$$\mathfrak{P}_0(\varepsilon_x, \varepsilon_\xi) = P(\varepsilon_x, \varepsilon_\xi), \quad \mathfrak{P}_1(\varepsilon_x, \varepsilon_\xi) = T(\varepsilon_\xi, r_\xi),$$

and \mathfrak{P}_t is non-characteristic with respect to $L(\delta_x, \delta_\xi)$. In other words, the boundary of \mathfrak{P}_t is disjoint from the boundary of $L(\delta_x, \delta_\xi)$ for all times $t \in [0, 1]$. To explicitly define \mathfrak{P}_t , one can exploit that an open quadrant cone is a product of one-dimensional cones.

Thus, by Proposition 4.2.2, we have the sought-after identity (\star_ξ) , and in turn the identity (lhs).

Putting the preceding together, we have identified the left (lhs) and right-hand (rhs) sides of formula (\ddagger) with the Floer complex (Fl). As mentioned above, we leave the reader to modify the arguments to establish formula (\ddagger) in the other cases, and to check its compatibility with inclusions. This establishes the first assertion of the theorem, and consequently the second. \square

5. Springer theory

In this section, we give an application of the preceding theory to branes in the cotangent bundle of a Lie algebra.

5.1 Springer theory via sheaves

We begin with a brief review of the Springer theory of Weyl group representations following Lusztig [Lus81], Borho–MacPherson [BM81], Ginzburg [Gin83], and Hotta–Kashiwara [HK84]. For an expanded summary, the reader could consult the discussion in [Gri98].

Let G be a reductive complex algebraic group with Lie algebra \mathfrak{g} and Weyl group W . Let \mathcal{B} be the flag variety of Borel subalgebras $\mathfrak{b} \subset \mathfrak{g}$. For a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, let $\mathfrak{u} \subset \mathfrak{b}$ be its unipotent radical and let $\mathfrak{h} = \mathfrak{b}/\mathfrak{u}$ be the universal Cartan algebra.

Let $\tilde{\mathfrak{g}} = \{(X, \mathfrak{b}) \mid x \in \mathfrak{b} \in \mathcal{B}\}$ be the Grothendieck–Springer space of pairs. Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone and $\tilde{\mathcal{N}} = \{(x, \mathfrak{b}) \mid x \in \mathfrak{b} \cap \mathcal{N}, \mathfrak{b} \in \mathcal{B}\}$ the Springer resolution. The following diagram summarizes some of the well-known relations among these spaces.

$$\begin{array}{ccccc}
 \mathcal{B} & \xleftarrow{p} & T^*\mathcal{B} \simeq \tilde{\mathcal{N}} & \xrightarrow{\tilde{i}} & \tilde{\mathfrak{g}} & \xrightarrow{\tilde{q}} & \mathfrak{h} \\
 & & \mu_{\mathcal{N}} \downarrow & & \mu_{\mathfrak{g}} \downarrow & & s \downarrow \\
 & & \mathcal{N} & \xrightarrow{i} & \mathfrak{g} & \xrightarrow{q} & \mathfrak{h} // W
 \end{array}$$

Here $p, \mu_{\mathcal{N}}, \mu_{\mathfrak{g}}$, and s are the obvious projections, \tilde{i} and i are the obvious inclusions, \tilde{q} assigns to (x, \mathfrak{b}) the class of x in $\mathfrak{b}/\mathfrak{u}$, and q is the affine adjoint quotient map.

Let $\mathfrak{g}_{rs} \subset \mathfrak{g}$ denote the regular semisimple locus and let $\mathfrak{h}_r \subset \mathfrak{h}$ denote the W -regular locus. The right-hand portion of the above diagram restricts to the following Cartesian diagram whose vertical arrows are W -torsors.

$$\begin{array}{ccc}
 \tilde{\mathfrak{g}}_{rs} & \xrightarrow{\tilde{q}} & \mathfrak{h}_r \\
 \mu_{\mathfrak{g}} \downarrow & & s \downarrow \\
 \mathfrak{g}_{rs} & \xrightarrow{q} & \mathfrak{h}_r // W
 \end{array}$$

Consider the constant perverse sheaves

$$\mathbb{C}_{\tilde{\mathfrak{g}}}[\dim_{\mathbb{C}} \mathfrak{g}] \in \text{Perv}(\tilde{\mathfrak{g}}), \quad \mathbb{C}_{\tilde{\mathcal{N}}}[\dim_{\mathbb{C}} \mathcal{N}] \in \text{Perv}(\tilde{\mathcal{N}}),$$

and their pushforwards

$$S_{\mathfrak{g}} = \mu_{\mathfrak{g}}! \mathbb{C}_{\tilde{\mathfrak{g}}}[\dim_{\mathbb{C}} \mathfrak{g}], \quad S_{\mathcal{N}} = \mu_{\mathcal{N}}! \mathbb{C}_{\tilde{\mathcal{N}}}[\dim_{\mathbb{C}} \mathcal{N}].$$

We refer to $S_{\mathfrak{g}}$ as the global Springer sheaf and $S_{\mathcal{N}}$ as the nilpotent Springer sheaf.

The map $\mu_{\mathfrak{g}}$ is small and the map $\mu_{\mathcal{N}}$ is semismall. Thus, the complexes $S_{\mathfrak{g}}, S_{\mathcal{N}}$ are in fact perverse, and $S_{\mathfrak{g}}$ is the middle extension of the local system

$$\mathcal{L}_{rs} = \mu_{\mathfrak{g}}! \mathbb{C}_{\tilde{\mathfrak{g}}_{rs}}[\dim_{\mathbb{C}} \mathfrak{g}] \in \text{Perv}(\mathfrak{g}_{rs}).$$

The Weyl group W acts on \mathcal{L}_{rs} by deck transformations, and hence on $S_{\mathfrak{g}}$ by the functoriality of the middle extension. The action identifies the group algebra $\mathbb{C}[W]$ with the (degree zero) endomorphisms of $S_{\mathfrak{g}}$.

There are two immediate ways in which $S_{\mathfrak{g}}$ and $S_{\mathcal{N}}$ are related. First, by proper base change, restriction along the inclusion $i : \mathcal{N} \hookrightarrow \mathfrak{g}$ induces an identification

$$i^* S_{\mathfrak{g}}[\dim_{\mathbb{C}} \mathfrak{h}] \simeq S_{\mathcal{N}}.$$

Second, if we identify \mathfrak{g} with its dual \mathfrak{g}^* via the Killing form, then we can regard the shifted Fourier transforms as endofunctors on perverse sheaves on \mathfrak{g} . With this understanding, the shifted Fourier transforms exchange the two perverse sheaves:

$$(S_{\mathcal{N}})^{\wedge}[\dim_{\mathbb{C}} \mathfrak{g}] \simeq S_{\mathfrak{g}}, \quad (S_{\mathfrak{g}})^{\vee}[-\dim_{\mathbb{C}} \mathfrak{g}] \simeq S_{\mathcal{N}}.$$

It is also possible to construct $S_{\mathcal{N}}$ as the nearby cycles of the constant perverse sheaf $\mathbb{C}_{\mathfrak{g}_{rs}}[\dim_{\mathbb{C}} \mathfrak{g}]$ in the (multi-dimensional) family defined by the adjoint quotient map $q : \mathfrak{g} \rightarrow \mathfrak{h} // W$. Namely, we have an identification

$$R\psi(\mathbb{C}_{\mathfrak{g}_{rs}}[\dim_{\mathbb{C}} \mathfrak{g}]) \simeq S_{\mathcal{N}},$$

where $R\psi$ can be taken to be the nearby cycles in the direction of any line $\mathbb{A}^1 \hookrightarrow \mathfrak{h} // W$ such that $\mathbb{A}^1 \setminus \{0\} \hookrightarrow \mathfrak{h}_r // W$. In this realization, the braid group $B_W \simeq \pi_1(\mathfrak{h}_r // W, pt)$ acts on the nearby cycles by monodromy transformations. This action factors through the projection $B_W \rightarrow W$ giving another realization of the W -action on $S_{\mathcal{N}}$.

5.2 Cotangent bundle of adjoint quotient

Our aim here is to explain how we will think about the cotangent bundle of the adjoint quotient \mathfrak{g}/G . The discussion is included to elucidate what follows and is not needed in any technical sense.

5.2.1 *General formalism.* Suppose that the group G acts on a smooth manifold X . Then the induced action of G on the cotangent bundle T^*X preserves the canonical one-form θ , and hence the symplectic form $\omega = d\theta$ as well.

Consider the moment map $m : T^*X \rightarrow \mathfrak{g}^*$ for the action of G . It is characterized by two properties: (1) m is G -equivariant with respect to the coadjoint action on \mathfrak{g}^* and (2) its differential dm satisfies the contraction formula

$$\langle dm, x \rangle = \iota_{\tilde{x}}\omega \quad \text{for } x \in \mathfrak{g},$$

where \tilde{x} is the vector field on T^*X corresponding to $x \in \mathfrak{g}$. Note that $\mathcal{L}_{\tilde{x}}\theta = 0$ and thus $\iota_{\tilde{x}}\omega = d\langle \theta, \tilde{x} \rangle$. Therefore, one can say that $m = \theta$ in the sense that $\langle m, x \rangle = \iota_{\tilde{x}}\theta$.

Consider the quotient stack X/G . By the cotangent stack $T^*(X/G)$, we mean the result of performing Hamiltonian reduction at the zero moment map value. Namely, we take the zero-fiber of the moment map $m^{-1}(0) \subset T^*X$ and then pass to the quotient stack $T^*(X/G) = m^{-1}(0)/G$.

Two immediate comments are in order.

First, there is no reason that the zero-fiber $m^{-1}(0)$ or that $T^*(X/G)$ should be smooth, and in our case of interest this is not so. This should not cause any consternation since we will always be working in the ambient target T^*X . To be precise, consider the correspondence

$$T^*(X/G) \xleftarrow{\ell} T^*(X/G) \times_{X/G} X \xrightarrow{r} T^*X,$$

where ℓ is the obvious smooth projection and r is the obvious inclusion. This is nothing more than the Lagrangian correspondence associated to the smooth projection $X \rightarrow X/G$. Note that applying the correspondence to $T^*(X/G)$ itself recovers the zero-fiber of the moment

map $m^{-1}(0) = r(\ell^{-1}(T^*(X/G)))$. In short, all of our concrete geometric arguments will take place in T^*X , and we only keep track of $T^*(X/G)$ to help us understand what is going on. For example, by a smooth Lagrangian $\mathcal{L} \hookrightarrow T^*(X/G)$, we will mean a substack such that

$$L = r(\ell^{-1}(\mathcal{L})) \hookrightarrow T^*X$$

is a smooth Lagrangian.

Second, just as one uses the sophisticated technology of stacks to deal with quotients, one should set the moment map m equal to zero in the appropriate homotopical sense. But, since our aims are purely topological, we can safely ignore this issue and work naively with $T^*(X/G)$ as a stack rather than as a derived stack. In other words, the reader unaccustomed to this kind of enhancement can safely ignore the issue, and in particular this comment itself.

5.2.2 Case of adjoint quotient. Now let us apply the preceding to the case $X = \mathfrak{g}$ with the adjoint action of G .

Under the identification of \mathfrak{g} with its dual \mathfrak{g}^* via the Killing form, the moment map becomes the Lie bracket

$$m : \mathfrak{g} \times \mathfrak{g} \simeq T^*\mathfrak{g} \rightarrow \mathfrak{g}^* \simeq \mathfrak{g}, \quad m(x, \xi) = [x, \xi].$$

Thus, the zero-fiber is the space of commuting pairs

$$m^{-1}(0) = \{(x, \xi) \in \mathfrak{g} \times \mathfrak{g} \mid [x, \xi] = 0\} \subset \mathfrak{g} \times \mathfrak{g}$$

and the cotangent bundle is the diagonal adjoint quotient

$$T^*(\mathfrak{g}/G) = \{(x, \xi) \in \mathfrak{g} \times \mathfrak{g} \mid [x, \xi] = 0\}/G.$$

5.3 Quantization of regular Hitchin fibers

Choose an embedding $\mathfrak{h} \hookrightarrow \mathfrak{g}$ of the universal Cartan algebra and let $H \hookrightarrow G$ be the corresponding maximal torus.

Recall that $q : \mathfrak{g} \rightarrow \mathfrak{h} // W$ denotes the affine adjoint quotient arising from Chevalley’s identification $\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{h}]^W$.

We use the phrase Hitchin fibration to refer to the map

$$\mathcal{H} : T^*(\mathfrak{g}/G) \rightarrow \mathfrak{h} // W, \quad \mathcal{H}(x, \xi) = q(\xi).$$

The definition and nomenclature come from the observation that the stack \mathfrak{g}/G is isomorphic to the stratum of semistable G -bundles on a cuspidal elliptic curve (or, equivalently, bundles whose pullbacks to the normalization \mathbb{P}^1 are trivializable).

Recall that $\mathfrak{h}_r \subset \mathfrak{h}$ denotes the W -regular locus. For $\lambda \in \mathfrak{h}_r // W$, we refer to the inverse image $\mathcal{L}_\lambda = \mathcal{H}^{-1}(\lambda) \subset T^*(\mathfrak{g}/G)$ as a regular Hitchin fiber. In terms of our usual identifications, we have the explicit description

$$\mathcal{L}_\lambda = \{(x, \xi) \in \mathfrak{g} \times \mathfrak{g} \mid [x, \xi] = 0, q(\xi) = \lambda\}/G.$$

Our immediate goal is to show that \mathcal{L}_λ is an exact Lagrangian and carries a canonical brane structure. (General formalism shows that it is Lagrangian, but we will give an explicit reason in a moment.)

To clarify the discussion, let us denote the Lie algebra by \mathfrak{g}_x and its dual by \mathfrak{g}_ξ . (We will continue to identify them via the Killing form.) Now let us switch our perspective, and consider $T^*(\mathfrak{g}_x/G)$ as the cotangent to the coadjoint quotient \mathfrak{g}_ξ/G . Then, by construction, the Hitchin fiber $\mathcal{L}_\lambda \hookrightarrow T^*(\mathfrak{g}_x/G)$ is nothing more than the conormal to the coadjoint orbit $\mathcal{O}_\lambda/G \hookrightarrow \mathfrak{g}_\xi/G$.

Thus, it is an exact Lagrangian and carries a canonical brane structure. By this, we mean that the base change

$$L_\lambda = r(\ell^{-1}(\mathcal{L}_\lambda)) \subset \mathfrak{g}_x \times \mathfrak{g}_\xi$$

carries a canonical G -equivariant brane structure. Namely, as the conormal to a closed submanifold, it comes equipped with a standard brane structure.

To keep with usual conventions, we will shift the standard brane structure of L_λ by the amount $\dim_{\mathbb{C}} \mathfrak{h} = \dim_{\mathbb{C}} \mathfrak{g} - \dim_{\mathbb{C}} \mathcal{O}_\lambda$. We use the phrase Hitchin brane to refer to $L_\lambda \hookrightarrow \mathfrak{g}_x \times \mathfrak{g}_\xi$ with this brane structure. Thus, by construction, the constructible complex $\pi_{\mathfrak{g}_\xi}(L_\lambda) \in \text{Sh}_c(\mathfrak{g}_\xi)$ is simply the shifted constant sheaf $\mathbb{C}_{\mathcal{O}_\lambda}[-\dim_{\mathbb{C}} \mathfrak{h}]$.

Now we will apply Theorem 4.4.2 to the Hitchin brane L_λ . Our aim is to identify the constructible complex $\mathcal{F}_\lambda = \pi_{\mathfrak{g}_x}(L_\lambda) \in \text{Sh}_c(\mathfrak{g})$. Observe that L_λ is conic along the factor \mathfrak{g}_x , and hence \mathcal{F}_λ is conic as well. We refer to \mathcal{F}_λ as the Hitchin sheaf.

THEOREM 5.3.1. *The Hitchin sheaf \mathcal{F}_λ is isomorphic to the global Springer sheaf $S_{\mathfrak{g}}$.*

Proof. By construction, the constructible complex shifted constant sheaf is $\mathbb{C}_{\mathcal{O}_\lambda}[-\dim_{\mathbb{C}} \mathfrak{h}]$. Note that L_λ is a balanced brane since its corner consists of projectivized pairs of commuting nilpotent elements. Thus, Theorem 4.4.2 provides an identification

$$\mathcal{F}_\lambda \simeq (\Upsilon(\mathbb{C}_{\mathcal{O}_\lambda}[-\dim_{\mathbb{C}} \mathfrak{h}]))^\wedge.$$

To complete the proof, we use the well-known identities of Springer theory

$$\Upsilon(\mathbb{C}_{\mathcal{O}_\lambda}[\dim_{\mathbb{C}} \mathcal{O}_\lambda]) \simeq R\psi(\mathbb{C}_{\mathcal{O}_\lambda}[\dim_{\mathbb{C}} \mathcal{O}_\lambda]) \simeq S_{\mathcal{N}}, \quad (S_{\mathcal{N}})^\wedge[\dim_{\mathbb{C}} \mathfrak{g}] \simeq S_{\mathfrak{g}}. \quad \square$$

The identification $\mathcal{F}_\lambda \simeq S_{\mathfrak{g}}$ is compatible with motions of the parameter $\lambda \in \mathfrak{h}_r // W$ as follows. By construction, the identification $\pi_{\mathfrak{g}_\xi}(L_\lambda) \simeq \mathbb{C}_{\mathcal{O}_\lambda}[-\dim_{\mathbb{C}} \mathfrak{h}]$ is compatible with parallel transport with respect to $\lambda \in \mathfrak{h}_r // W$. For this, recall that from the perspective of \mathfrak{g}_ξ , the brane $L_\lambda \hookrightarrow T^*\mathfrak{g}_\xi$ is nothing more than (a shift of) the standard brane with support the conormal to $\mathcal{O}_\lambda \hookrightarrow \mathfrak{g}_\xi$. Now, under the Fourier transform, motions of $\mathbb{C}_{\mathcal{O}_\lambda}[-\dim_{\mathbb{C}} \mathfrak{h}]$ induce the usual Weyl group action on $S_{\mathfrak{g}}$ of Springer theory. This is equivalent to the fact that the monodromy braid group action on the nearby cycles $R\psi(\mathbb{C}_{\mathcal{O}_\lambda}[-\dim_{\mathbb{C}} \mathfrak{h}])$ descends to the usual Weyl group action. Thus, in conclusion, the proof of the theorem shows that the braid group action on \mathcal{F}_λ descends to the usual Weyl group action as well.

ACKNOWLEDGEMENTS

It is a pleasure to thank Eric Zaslow for his generous interest and encouragement. I am grateful to Pierre Schapira for helpful remarks on the exposition. I am also grateful to an anonymous referee for careful, detailed comments on the proofs and the foundations they rely upon. Finally, I would like to acknowledge the support of NSF grant DMS-0600909 and a Sloan Research Fellowship.

REFERENCES

ALP94 M. Audin, F. Lalonde and L. Polterovich, *Symplectic rigidity: Lagrangian submanifolds*, in *Holomorphic curves in symplectic geometry*, Progress in Mathematics, vol. 117 (Birkhäuser, Basel, 1994), 271–321.

BD A. Beilinson and V. Drinfeld, *Quantization of Hitchin Hamiltonians and Hecke eigensheaves*, Preprint.

BM88 E. Bierstone and P. Milman, *Semianalytic and subanalytic sets*, Publ. Math. Inst. Hautes Études Sci. **67** (1988), 5–42.

- BM81 W. Borho and R. MacPherson, *Représentations des groupes de Weyl et homologie d'intersection pour les variétés nilpotentes*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), 707–710.
- Dri04 V. Drinfeld, *DG quotients of DG categories*, J. Algebra **272** (2004), 643–691.
- FO97 K. Fukaya and Y.-G. Oh, *Zero-loop open strings in the cotangent bundle and Morse homotopy*, Asian J. Math. **1** (1997), 96–180.
- FOOO09 K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian intersection Floer theory: anomaly and obstruction, Part I*, AMS/IP Studies in Advanced Mathematics, vol. 46 (American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009).
- Gin83 V. Ginzburg, *Intégrales sur les orbites nilpotentes et représentations des groupes de Weyl*, C. R. Acad. Sci. Paris Sér. I Math. **296** (1983), 249–252.
- Gri98 M. Grinberg, *A generalization of Springer theory using nearby cycles*, Represent. Theory **2** (1998), 410–431 (electronic).
- HK84 R. Hotta and M. Kashiwara, *The invariant holonomic system on a semisimple Lie algebra*, Invent. Math. **75** (1984), 327–358.
- KW07 A. Kapustin and E. Witten, *Electric–magnetic duality and the geometric Langlands program*, Commun. Number Theory Phys. **1** (2007), 1–236.
- KS94 M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften, vol. 292 (Springer, Berlin, 1994).
- KL80 D. Kazhdan and G. Lusztig, *A topological approach to Springer's representations*, Adv. Math. **38** (1980), 222–228.
- Kel06 B. Keller, *On differential graded categories*, International Congress of Mathematicians, vol. II (European Mathematical Society, Zürich, 2006), 151–190.
- Lus81 G. Lusztig, *Green polynomials and singularities of unipotent classes*, Adv. Math. **42** (1981), 169–178.
- Nad09 D. Nadler, *Microlocal branes are constructible sheaves*, Selecta Math. **15** (2009), 563–619.
- NZ09 D. Nadler and E. Zaslow, *Constructible sheaves and the Fukaya category*, J. Amer. Math. Soc. **22** (2009), 233–286.
- Oh Y.-G. Oh, *Unwrapped continuation invariance in Lagrangian Floer theory: energy and C^0 estimates*, arXiv:0910.1131.
- Sei08 P. Seidel, *Fukaya categories and Picard–Lefschetz theory*, Zurich Lectures in Advanced Mathematics (European Mathematical Society (EMS), Zürich, 2008).
- Sik94 J.-C. Sikorav, *Some properties of holomorphic curves in almost complex manifolds*, in *Holomorphic curves in symplectic geometry* (Birkhäuser, Basel, 1994), 165–189.
- Spr76 T. A. Springer, *Trigonometric sums, Green functions of finite groups and representations of Weyl groups*, Invent. Math. **36** (1976), 173–207.
- Spr78 T. A. Springer, *A construction of representations of Weyl groups*, Invent. Math. **44** (1978), 279–293.
- Spr82 T. A. Springer, *Quelques applications de la cohomologie d'intersection*, in *Bourbaki seminar, vol. 1981/1982*, Asterisque, vols. 92–93 (Soc. Math. France, Paris, 1982), 249–273.
- Sta63 J. D. Stasheff, *Homotopy associativity of H-spaces. I, II*, Trans. Amer. Math. Soc. **108** (1963), 275–312.
- vM96 L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), 497–539.

David Nadler nadler@math.northwestern.edu

Department of Mathematics, Northwestern University, 2033 Sheridan Road,
Evanston, IL 60208, USA