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SIMPLE TYPE III SELF-INJECTIVE RINGS AND RINGS OF COLUMN-FINITE MATRICES

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1. Introduction. Relatively little is known about simple, Type III, right self-injective rings Q. This is despite their common occurrence, for example as $Q_{\max}(R)$ for any prime, nonsingular, countable-dimensional algebra R without uniform right ideals. (In particular Q can be constructed with a given field as its centre.) As with their directly finite, $SP(1)$, right self-injective counterparts, division rings, there are few obvious invariants apart from the centre.

One reason perhaps why little interest has been shown in their structure is that the usual construction of such Q, namely as a suitable $Q_{\max}(R)$, is not concrete enough; in general R sits far too loosely inside Q and not enough information transfers to Q from R. Thus, for example, taking R to be a non-right-Ore domain and $Q = Q_{\max}(R)$ tells us little about Q (although it has been conjectured that all Q arise this way).

The purpose of this paper is twofold. Firstly we wish to draw attention to the easily established fact that some of these Q can be constructed as a ring of fractions of the ring R_∞ of all (countably infinite) column-finite matrices over very concrete R. For example, suppose R is a countable-dimensional non-right-Ore domain or an $SP(1)$ regular ring, but not a division ring, with only countable direct sums of right ideals (such as certain direct limits of infinite-dimensional full linear rings). Then R_∞ has a ring of fractions, relative to its (unique) maximum Ore set of regular elements, which is a simple Type III right self-injective ring. Secondly, we raise the possibility of describing a general Q in terms of certain natural infinite-dimensional full linear subrings $T \cong F_\infty$ over the centre F of Q. These subrings, which we term Q-full-linear subrings, behave somewhat analogously to n-dimensional full linear subalgebras over F ($\cong F_n$ for fixed n), with centre F, of a given central simple finite-dimensional algebra over F: any two are conjugate and they satisfy the double centralizer condition $T = C_Q(C_Q(T))$. Also a large part of Q is covered by their union. For example, if $Q = Q_{\max}(R)$ where R is any "locally finite-dimensional semisimple" algebra over an algebraically closed field F, then the union of these Q-full-linear subrings contains R. This raises the question of just how "locally full linear" is a general Q with an algebraically closed centre.

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2. Preliminaries and background. Rings are assumed to be associative with identity. Our notation for the left annihilator of a set X in a ring R is $l_R(X)$. Similarly $r_R(X)$ denotes the right annihilator.

The injective hull of a module A is denoted by $E(A)$. For a cardinal α (finite or infinite), αA denotes the direct sum of α copies of A. We write $A \lesssim B$ to indicate a module A is subisomorphic to a module B, and $A \leq_e B$ to indicate the submodule A is essential in the module B.

For background on (right) nonsingular rings and the maximal right quotient ring of such rings, the reader is referred to [**2**]. We denote the maximal right quotient ring of a right nonsingular ring R by $Q_{\max}(R)$. As is well-known, $Q_{\max}(R)$ is a regular, right self-injective ring. For the general theory of regular, right self-injective rings, and the associated theory of types, see [**3**].

Generally our notation and terminology follow [**2**] and [**3**]. References [**4**] and [**6**] are the original sources of some important results for prime, regular, right self-injective rings (although these results are now in [**3**]).

We remind the reader that any regular, right self-injective ring Q is uniquely a direct product of rings of Types I, II, III. A good deal is known about regular, right self-injective rings of Type I or II, especially if Q is simple; in the Type I case, Q is then simple Artinian, while in the Type II case for example, Q is directly finite and possesses a rank function. In the simple Type III case, which is characterized by the property that Q is not a division ring but satisfies the $SP(1)$ condition (strongly prime with one insulator)

$$aQ \cong Q \quad \forall 0 \neq a \in Q,$$

not much is known beyond the properties which hold in any directly infinite, prime, regular, right self-injective ring. Yet this case occurs frequently. For as shown in [**7**], if R is any countable-dimensional, prime, nonsingular algebra without uniform right ideals, then $Q_{\max}(R)$ is always simple Type III.

By a *countable-dimensional full linear ring over a field F* we mean the ring $\operatorname{End}_F V$ of all linear transformations of a countably-infinite dimensional vector space V over F (with transformations written on the left of vectors). This ring is prime, regular, and right self-injective. Of course, $\operatorname{End}_F V$ is isomorphic to the ring F_∞ of all countably infinite, column-finite matrices over F.

Now let Q be a simple, Type III, right self-injective ring (which is necessarily regular). The directly finite counterpart of Q is of course a division ring. In the case where the division ring is finite-dimensional over its centre, the classical theory tells us a lot about its structure in terms of its subfields, particularly its maximal subfields. Is there an analogue of this (albeit much weaker) for Q, or at least for some Q? In place of the finite-dimensional restriction we could insist that Q have only countable direct sums of nonzero right ideals. In place of a subfield, we should perhaps pick out a directly infinite, prime, regular, right self-injective subring which is well-behaved and well-understood. A countable-dimensional full linear subring over a field immediately springs to mind. Just how these can arise as subrings of Q, is considered in Section 4. There it also becomes clear that their existence does not depend on the countability restriction. If Q does contain uncountable direct sums of right ideals, then probably the appropriate full linear subring to consider is one whose dimension matches the "dimension" of Q. We have not followed this line. Instead, although most of our results apply without the countability restriction, the particular case we have in mind is when Q does have only countable direct sums of right ideals (that is $\mu(Q) = \aleph_1$, where μ is the Goodearl-Boyle [**5**] infinite dimension function). This covers the principal motivating case $Q = Q_{\max}(R)$ where R is any countable-dimensional, prime, nonsingular algebra without uniform right ideals, such as

$$R = \lim_{\to} F_{2^n}$$

for a field F.

3. Self-injectivity and column-finiteness. Throughout Q denotes a regular, right self-injective ring. We term a set $\{e_i\}_1^\infty$ of nonzero orthogonal idempotents of Q *complete* if

$$l_Q\{e_i\}_1^\infty = 0,$$

equivalently

$$\bigoplus_1^\infty e_iQ$$

is a large right ideal of Q. One very powerful consequence of right self-injectivity when applied to a complete set is the (well-known) property of being able to string together arbitrary "columns" of elements

$$\alpha_1, \alpha_2, \dots, \alpha_n, \dots,$$

where each $\alpha_i \in Qe_i$, to obtain a (unique) element $x \in Q$.

PROPOSITION 1. *Let $\{e_i\}_1^\infty$ be a complete set of orthogonal idempotents of Q. Given $\alpha_i \in Qe_i$ for $i = 1, 2, \dots$, there exists a unique $x \in Q$ with $xe_i = \alpha_i \quad \forall i$. Consequently*

$$_QQ \cong \prod_1^\infty Qe_i.$$

Proof. Right self-injectivity of Q ensures that the map

$$\psi:\oplus e_iQ \to Q, \quad \sum a_i \mapsto \sum \alpha_i a_i$$

is given by left multiplication by a suitable $x \in Q$. Then

$$xe_i = \psi(e_i) = \alpha_i e_i = \alpha_i \quad \forall i.$$

Uniqueness of x follows from $l_Q\{e_i\} = 0$.

For the class of regular rings R in which each large right ideal is an essential extension of some countably generated right ideal, the property in Proposition 1 actually characterizes injectivity of R_R. For suppose $L \leq_e R_R$ and $\psi:L \to R$ is an R-map. Choose a complete set $\{e_i\}_1^\infty$ of orthogonal idempotents of R with

$$\bigoplus_1^\infty e_iR \leq_e L.$$

By the "column-stringing" property of R, there exists $x \in R$ with

$$xe_i = \psi(e_i) \in Re_i \quad \forall i,$$

and this x clearly induces ψ.

In view of Proposition 1, a natural question to ask is: when can we string together "rows" of elements $\beta_1, \beta_2, \ldots, \beta_n, \ldots$ with each $\beta_i \in e_iQ$, that is when does there exist $x \in Q$ with $e_ix = \beta_i \quad \forall i$? The answer is that the matrix (β_ie_j) must be "almost column-finite":

PROPOSITION 2. *Let $\{e_i\}_1^\infty$ be a complete set of orthogonal idempotents of Q. Given $\beta_i \in e_iQ$ for $i = 1, 2, \ldots, n, \ldots$ there exists $x \in Q$ with $e_ix = \beta_i \quad \forall i$ if and only if for each $j = 1, 2, \ldots$ the right ideal*

$$A_j = \{a \in Q:\beta_ie_ja = 0 \text{ for almost all } i\}$$

is large in Q.

Proof. Although this proposition can be deduced from Propositions 1 and 3, the following argument is more enlightening.

($\Rightarrow$) Fix j. Let $0 \neq y \in Q$. If $xe_jy = 0$ then $y \in A_j$ because

$$\beta_ie_jy = e_ixe_jy = 0 \quad \forall i.$$

Suppose $xe_jy \neq 0$. Since

$$\sum_1^\infty e_iQ \leq_e Q,$$

we have

$$xe_jyQ \cap \sum_1^\infty e_iQ \neq 0,$$

say

$$0 \neq xe_j yz \in \sum_1^n e_i Q.$$

Then for all $k > n$,

$$(\beta_k e_j)yz = e_k(xe_j yz) = 0,$$

whence $0 \neq yz \in A_j$. Hence $A_j \leq_e Q$.

($\Leftarrow$) Fix j and assume $L = A_j$ is large. Set

$$L_n = \{a \in Q:\beta_i e_j a = 0 \quad \forall i > n\}$$

so that

$$L = \bigcup_1^\infty L_n \quad \text{and} \quad L_1 \subseteq L_2 \subseteq \dots .$$

Define a map $\psi:L \to Q$ by

$$\psi(a) = (\beta_1 e_j + \beta_2 e_j + \dots + \beta_n e_j)a \quad \text{if } a \in L_n.$$

This ψ is a well-defined Q-homomorphism, so by the right injectivity of Q there exists $w \in Q$ with

$$\psi(a) = wa \quad \forall a \in L.$$

Let $\alpha_j = we_j$. Let $a \in L$. Fix i and choose $n > i$ such that $a \in L_n$. Then

$$\begin{aligned}(e_i\alpha_j - \beta_i e_j)a &= e_i\alpha_j a - \beta_i e_j a \\ &= e_i(w(e_j a)) - \beta_i e_j a \\ &= e_i(\beta_1 + \dots + \beta_n)e_j a - \beta_i e_j a\end{aligned}$$

(since $e_j a \in L_n$)

$$\begin{aligned}&= \beta_i e_j a - \beta_i e_j a \\ &= 0.\end{aligned}$$

Hence

$$(e_i\alpha_j - \beta_i e_j)L = 0,$$

whence

$$e_i\alpha_j - \beta_i e_j = 0$$

because $L \leq_e Q$ and $Z(Q_Q) = 0$. Thus for each j, there exists $\alpha_j \in Qe_j$ with

$$e_i\alpha_j = \beta_i e_j \quad \forall i.$$

By Proposition 1, there exists $x \in Q$ with $xe_j = \alpha_j \quad \forall j$. Now $\forall j$

$$(e_ix - \beta_i)e_j = e_ixe_j - \beta_ie_j = e_i\alpha_j - \beta_ie_j = 0.$$

By completeness of $\{e_j\}_1^\infty$,

$$e_ix - \beta_i = 0$$

and so $e_ix = \beta_i$.

An easy corollary of Propositions 1 and 2 is the well-known fact that a regular, two-sided injective ring Q cannot contain an infinite independent family of nonzero pairwise isomorphic right ideals (and in particular Q is directly finite). Otherwise there exists a complete set $\{e_i\}_1^\infty$ with $e_1Q \lesssim e_iQ$ for infinitely many i, and thus elements $\beta_i \in e_iQe_1$ with

$$r_Q(\beta_i) \cap e_1Q = 0 \quad \text{for infinitely many } i.$$

According to the left-sided version of Proposition 1 (note $r_Q\{e_i\} = 0$ because $r_Q\{e_i\} \cap \sum e_iQ = 0$), the left injectivity of Q implies there exists $x \in Q$ with

$$e_ix = \beta_i \quad \forall i,$$

whereas the right injectivity of Q and Proposition 2 say x cannot exist because $A_1 \cap e_1Q = 0$.

Using a two-sided Peirce decomposition we can give a truly column-finite interpretation of Proposition 2.

PROPOSITION 3. *Let $\{e_i\}_1^\infty$ be a complete set of orthogonal idempotents of a regular, right self-injective ring Q, and let $\beta_i \in e_iQ$ be given for $i = 1, 2, \ldots$. Then there exists $x \in Q$ with $e_ix = \beta_i \quad \forall i$ if and only if there is a complete set $\{f_i\}_1^\infty$ of orthogonal idempotents such that the matrix*

$$(\beta_if_j)$$

is column-finite.

Proof. ($\Leftarrow$) This follows by applying Proposition 1 to

$$\alpha_i = \sum_j \beta_jf_i \in Qf_i \quad \text{for } i = 1, 2, \ldots.$$

($\Rightarrow$) The map

$$\psi:Q \to \prod_1^\infty e_iQ, \quad x \mapsto (e_ix)$$

is a Q-monomorphism. (Note $r_Q\{e_i\} = 0$ because

$$r_Q\{e_i\} \cap (\oplus e_iQ) = 0 \quad \text{and} \quad \oplus e_iQ \leq_e Q.)$$

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$$Q = E\left(\sum_1^\infty e_iQ\right)$$

we must have

$$\begin{aligned}\psi(Q) &= \text{the injective hull of (external) } \oplus e_iQ \\ &\qquad\qquad \text{within the nonsingular injective } \prod e_iQ \\ &= \mathscr{S}\text{-closure of } \oplus e_iQ \text{ in } \prod e_iQ \\ &= \{\gamma \in \prod e_iQ \mid \gamma L \subseteq \oplus e_iQ \text{ for some } L \leq_e Q\} \\ &= \{(\gamma_i) \in \prod e_iQ \mid \{y \in Q \mid \gamma_i y = 0 \quad \forall' i\} \leq_e Q\}.\end{aligned}$$

Thus for the given β_i,

$$L = \{y \in Q \mid \beta_i y = 0 \quad \forall' i\}$$

is a countably generated large right ideal of Q and we can choose a complete set $\{f_i\}_1^\infty \subseteq L$ of orthogonal idempotents of Q. Since for each j, $\beta_i f_j = 0$ for almost all i, the matrix $(\beta_i f_j)$ is column-finite.

This (admittedly tenuous) link between right self-injectivity and column-finiteness of Peirce decompositions makes the following (known) corollary a little unexpected.

COROLLARY 4. *For a right nonsingular ring R, the ring R_∞ of all (countably infinite) column-finite matrices over R is right self-injective if and only if R is semisimple Artinian.* (*Note*:

$$R_\infty \cong \operatorname{End}_R\left(\bigoplus_1^\infty R\right).)$$

Proof. Although this follows easily from Proposition 3, there is in fact a stronger result in the literature. In [**8**], Shanny showed that R_∞ is regular if and only if R is semisimple Artinian. See also [**9**].

Remark. An unpublished related result due to D. Handelman (Ph.D. thesis, 1975) asserts that if R is a right $SP(1)$ ring but not an Ore domain, then there exists an infinite cardinal $\aleph$ such that $\operatorname{End}_R(\aleph R)$ is right but not left $SP(1)$. If there are no uncountable direct sums of nonzero right ideals in R, then $\aleph = \aleph_0$ will work.

Suppose $x_1, \ldots, x_n, \ldots$ is a sequence of elements in a regular, right self-injective ring R, and suppose the sequence is "almost finite" in the sense that

$$L = \{y \in R \mid x_i y = 0 \quad \forall' i\}$$

is essential in R_R (equivalently $(x_1, x_2, \ldots, x_n, \ldots)$ is in the $\mathscr{S}$-closure of $\oplus_1^\infty R$ within $\prod_1^\infty R$). Then it makes sense to talk about the "sum"

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$$\sum_1^\infty x_i$$

as the unique element $x \in R$ which induces the R-map

$$\psi:L \rightarrow R, \quad y \mapsto (x_1 + \ldots + x_n)y \quad \text{if } x_iy = 0 \quad \forall i > n.$$

Inspired by Proposition 2 we could consider the set $\bar{R}_\infty$ of those countably-infinite matrices over R whose columns are almost finite. This $\bar{R}_\infty$ becomes a ring under the obvious addition, and an extended matrix product AB based on the fact that the inner product of a row $(a_1, a_2, \ldots)$ of $A \in \bar{R}_\infty$ and a column $(b_1, b_2, \ldots)$ of $B \in \bar{R}_\infty$ makes sense as $\sum_1^\infty a_ib_i$ (because $a_1b_1, a_2b_2, \ldots$ is almost finite). The ring $\bar{R}_\infty$ is regular, right self-injective, but it is not new. It is a matrix representation of

$$\mathrm{End}_R\left(E\left(\bigoplus_1^\infty R\right)\right)$$

and is also $Q_{\max}(R_\infty)$. (Recall that the endomorphism ring of a non-singular injective module is regular, right self-injective.)

Recall that a ring S is a *ring of fractions* of a ring R if elements of S are expressible as ac^{-1} for suitable $a, c \in R$. A *regular* (*right*) *Ore set* for R is any (non-empty) multiplicatively closed set C of regular elements of R with the common right multiple property: for each $c \in C, r \in R, \exists c_1 \in C, r_1 \in R$ with $cr_1 = rc_1$. With each ring of fractions S we have a regular Ore set $C = S^* \cap R$ (here S^* is the group of units of S). Conversely, given a regular Ore set C there exists a unique ring $S = RC^{-1}$ of fractions of R such that $C \subseteq S^* \cap R$ and each $x \in S$ is expressible as

$$x = ac^{-1} \quad a \in R, c \in C.$$

Since then S is also a right quotient ring of R, if R is right nonsingular we have

$$R \subseteq S \subseteq Q_{\max}(R) \quad \text{and}$$

$$C \subseteq Q^*_{\max}(R) \cap R = \{y \in R: r_R(y) = 0, yR \leq_e R_R\}.$$

All this is very well-known. Less well-known is the following:

PROPOSITION 5. *Each ring R (with identity) has a unique largest regular Ore set M (which contains all other regular Ore sets), and the corresponding ring of fractions RM^{-1} is the (unique) largest ring of fractions of R.*

Proof. R possesses regular Ore sets, for example $C = \{1\}$ or $C = R^*$. Let M be the multiplicative semigroup generated by all the regular Ore subsets of R. Let $c \in M, r \in R$. We can suppose $c = c_1c_2 \ldots c_n$ where $c_i \in C_i$ for some regular Ore set C_i. Choose $d_1 \in C_1, r_1 \in R$ such that

$$c_1r_1 = rd_1.$$

Choose $d_2 \in C_2$, $r_2 \in R$ such that

$$c_2 r_2 = r_1 d_2$$

and so on up to $d_n \in C_n$, $r_n \in R$ such that

$$c_n r_n = r_{n-1} d_n.$$

Then

$$\begin{aligned} c r_n &= c_1 c_2 \dots c_n r_n \\ &= (c_1 c_2 \dots c_{n-1})(r_{n-1} d_n) \\ &\vdots \\ &= c_1 r_1 (d_2 \dots d_n) \\ &= r(d_1 d_2 \dots d_n) \end{aligned}$$

which establishes the common right multiple property.

We denote RM^{-1} by $Q_{\text{max-cl}}(R)$ and refer to it as the *maximal* (*right*) *classical ring of fractions of R*.

Remark. A ring R without identity need not have any regular Ore sets let alone maximal (non-empty) ones, even if R has regular elements. For example consider $R = F[x]x$, where $F[x]$ is the skew polynomial ring consisting of all polynomials $a_0 + a_1 x + \dots + a_n x^n$ with multiplication determined from

$$xa = \sigma(a)x$$

for a fixed monomorphism $\sigma: F \rightarrow F$ which is not a surjection.

In view of Corollary 4, one expects to lose injective-related properties in going from R to R_∞. But viewed from their maximal quotient rings, sometimes R_∞ is more tightly embedded than R (even if $Q_{\max}(R) \cong Q_{\max}(R_\infty)$). One such case is:

THEOREM 6. *Suppose R is a right nonsingular ring containing only countable direct sums of nonzero right ideals* (*for example, a countable-dimensional algebra*), *but not finite-dimensional* (*on the right*), *and satisfies $R \lesssim aR$ for all nonzero $a \in R$. Then R_∞ has a ring of fractions, relative to its maximum regular Ore set, which is a simple, Type* III, *right self-injective ring.*

Remark. The restriction on countable direct sums could be weakened to the existence of a right ideal

$$J = \bigoplus_1^\infty a_i R \leq_e R$$

such that for each $x \in Q_{\max}(R)$, there exists

$$\bigoplus_1^\infty b_iR \leq_e \{r \in J \mid xr \in J\}.$$

Proof. Because of our countable direct sum restriction, each right ideal K of R is an essential extension of some $\oplus_1^\infty c_iR$ with each $c_iR \cong R$. Fix a large right ideal of R of the form

$$J = \bigoplus_1^\infty a_iR$$

with each $a_iR \cong R$. Let $Q = Q_{\max}(R)$ and let

$$Y = \{y \in Q \mid yJ \subseteq J\}.$$

Since Q_R is injective and nonsingular, and each $a_iR \cong R$, we have

$$\begin{aligned} Y &\cong \operatorname{End} J_R \\ &= \operatorname{End} \bigoplus_1^\infty a_iR \\ &\cong \operatorname{End} \bigoplus_1^\infty R \\ &\cong R_\infty. \end{aligned}$$

Thus it suffices to establish the claims for the ring Y.

Let $x \in Q$. Choose a large right ideal

$$L = \bigoplus_1^\infty b_iR \subseteq J,$$

with each $b_iR \cong R$, such that $xL \subseteq J$ (recall inverse images of large submodules are large). Since $a_iR \cong b_iR\ \forall i$, we can construct an R-isomorphism

$$\psi: J \to L.$$

This ψ is induced by some $c \in Q$ because Q_R is injective. Now

$$cJ \subseteq L \subseteq J \quad \text{and}$$

$$(xc)J = x(cJ) \subseteq xL \subseteq J,$$

which shows both c and xc are in Y. Also $r_J(c) = 0$ and $J \leq_e Q_R$ imply $r_Q(c) = 0$, while $LQ \subseteq cQ$ and $LQ \leq_e Q$ imply

$$cQ = Q \quad \text{and} \quad l_Q(c) = 0.$$

Thus c is regular in Q and hence a unit in Q. Since $x = (xc)c^{-1}$ and $xc, c \in Y$, this shows Q is a ring of fractions of Y relative to the regular Ore set $C = Y \cap Q^*$. Now

$$C = \{c \in Y \mid r_Y(c) = 0 \text{ and } cY \leq_e Y\}$$

so necessarily C is the maximum regular Ore set of Y.

Given $0 \neq x \in Q$, choose $0 \neq a \in xQ \cap R$. Then

$$R \lesssim aR \leqq xQ \cap R$$

implies $Q \lesssim xQ$. Hence Q is an $SP(1)$, right self-injective ring, but not a division ring (otherwise R is finite-dimensional), whence Q is a simple, Type III, right self-injective ring.

The following two corollaries are immediate.

COROLLARY 7. *Let R be an integral domain which is countable or countable-dimensional (on the right) over a subdivision ring, but not right Ore. Then R_∞ has a ring of fractions, relative to its maximum regular Ore set, which is a simple Type* III *right self-injective ring. (Note: here*

$$Q_{\text{max-cl}}(R_\infty) \cong Q_{\text{max}}(R).)$$

Remark. For the classical example of such R, namely the skew polynomial ring $(F[x], \sigma)$ over a field F with

$$1 < [F:\sigma(F)] \leqq \aleph_0,$$

the maximum regular Ore set of R itself is just F^*, so that

$$Q_{\text{max-cl}}(R) = R.$$

This contrasts sharply with $Q_{\text{max-cl}}(R_\infty)$.

COROLLARY 8. *The conclusions of Corollary 7 also hold for any $SP(1)$ regular ring R which has only countable direct sums of right ideals, but is not a division ring.*

The maximal ring of fractions provides us with a much more concrete and satisfying construction of a simple, Type III, regular right self-injective ring Q with a given field F as centre than does the maximal quotient ring of, say, even $\lim_{\rightarrow} F_{2^n}$ (see Introduction). This is because a ring of fractions (of some known ring) gives a much closer fit than does a general quotient ring. The construction is in three steps. First we form a direct limit

$$U = \lim_{\rightarrow} T_i$$

of a countable sequence of full linear rings $T_i \cong F_\infty$ where each embedding $T_i \rightarrow T_{i+1}$ induces an isomorphism of the centres and splits primitive idempotents of T_i into infinitely many primitives of T_{i+1}, that is

$$\operatorname{soc}(T_{i+1}) \cap T_i = 0.$$

(If T_{i+1} is also nonsingular over T_i, then the embedding $T_i \rightarrow T_{i+1}$ is actually equivalent to the diagonal embedding

$$a \mapsto \begin{bmatrix} a & & & & & \\ & a & & & & \\ & & a & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \end{bmatrix}$$

of F_∞ into $(F_\infty)_\infty \subseteq F_\infty$.) Next form the ring U_∞ of column-finite matrices over U. Since U is a regular $SP(1)$ ring with only countable direct sums of right ideals, by Corollary 8 if we now let

$$Q = Q_{\text{max-cl}}(U_\infty)$$

be the maximal ring of fractions of U_∞, then Q is a simple, Type III right self-injective ring. Also

$$Q \cong Q_{\max}(U)$$

and U is simple with centre F, so the centre of Q is F. (One can show, however, that U_∞ is not a right order in Q, so that some regular elements of U_∞ are not invertible in Q. In other words, the maximum regular Ore set of U_∞ is not the complete set of its regular elements.)

Remarks. (1) Note that for R as in Theorem 6 (or Corollaries 7, 8), the unique maximum right Ore set of regular elements of the ring $A = R_\infty$ is

$$\{a \in A \mid r_A(a) = 0 \text{ and } aA \leq_e A\}.$$

(2) By Camillo's recent theorem [**1**], if R and S are not Morita equivalent rings then $R_\infty \not\cong S_\infty$. Thus we can produce many non-isomorphic R_∞ in Corollaries 7 and 8. Nevertheless we are still faced with the question of when their corresponding maximal rings of fractions are isomorphic (which is analogous to the difficult question of when right Ore domains have isomorphic division rings of fractions).

(3) Observe that if R and S are countable-dimensional algebras over the same field F, then as rings

$$R_\infty \lesssim S_\infty \quad \text{and} \quad S_\infty \lesssim R_\infty.$$

This is because both R and S embed in $T = F_\infty$ and both R_∞ and S_∞ contain a copy of T, whence $R_\infty \lesssim T_\infty \lesssim T \lesssim S_\infty$ implies $R_\infty \lesssim S_\infty$. Similarly $S_\infty \lesssim R_\infty$. (By a similar argument $R_\infty \lesssim U_\infty$ and $U_\infty \lesssim R_\infty$ for U the countable direct limit of full linear rings above.) Thus we can expect R_∞ and S_∞ (and perhaps their rings of fractions) to have similar local properties.

(4) The particular embedding $R_\infty \lesssim S_\infty$ described above need not extend to an embedding of $Q_{\text{max-cl}}(R_\infty)$ into $Q_{\text{max-cl}}(S_\infty)$ because regular, non-unit, elements of R_∞ can become zero-divisors in T and hence in S_∞. In general a ring homomorphism

$$\psi:R_\infty \to S_\infty,$$

for R and S satisfying the conditions of Theorem 6, will extend to a ring homomorphism

$$\bar{\psi}:Q_{\text{max-cl}}(R_\infty) \to Q_{\text{max-cl}}(S_\infty)$$

if and only if ψ maps the maximum regular Ore set of R_∞ into the maximum regular Ore set of S_∞.

4. Full linear subrings of simple type III self-injective rings. Let Q be a prime, regular, right self-injective ring with centre F (a field). We make the following:

Definition 9. A subring T of Q is a *Q-full-linear subring* if

centre (T) = centre $(Q) = F$,

$T \cong F_\infty$, and

Q_T is nonsingular.

If Q is simple Type III, then Q-full-linear subrings arise quite naturally from complete sets of orthogonal idempotents and Q-isomorphisms between the associated principal right ideals, as a consequence of the injectivity of Q_Q and the fact that

$$eQ \cong fQ \quad \forall 0 \neq e, f \in Q.$$

For let $\{e_i\}_1^\infty$ be a complete set of (nonzero) orthogonal idempotents. Choose Q-isomorphisms

$$e_1Q \xrightarrow{m_{21}} e_2Q \xrightarrow{m_{32}} e_3Q \to \dots$$

induced by $m_{i+1,i} \in e_{i+1}Qe_i$, and for $i > j$ derive the isomorphisms

$$m_{ij}:e_jQ \to e_iQ, \quad m_{ij} \in e_iQe_j.$$

Set $m_{ii} = e_i$ and for $i < j$ let

$$m_{ij} = m_{ji}^{-1} \in e_iQe_j.$$

Then $M = \{m_{ij}\}_1^\infty$ is a complete set of matrix units of Q. Set

$$C_i = \sum_{j=1}^{\infty} Fm_{ji} \quad \text{(the } i\text{th "column space")}$$

for $i = 1, 2, \dots$. Since Q is right self-injective, by Proposition 1 we can string the C_i together to get a subring

$$\begin{aligned} T &= \{x \in Q : xe_i \in C_i \quad \forall i\} \\ &= \prod_1^{\infty} C_i \quad \text{(as an additive group)} \\ &\cong F_{\infty} \quad \text{(as a ring).} \end{aligned}$$

Also

$$\text{centre } (T) = F \quad \text{and} \quad \text{soc}(T) = \sum_1^{\infty} e_i T.$$

If $x \in Z(Q_T)$, then $x\,(\text{soc}(T)) = 0$ whence

$$xe_i = 0 \quad \forall i$$

and so $x = 0$ because $\{e_i\}$ is complete. Thus Q_T is nonsingular and T is therefore a Q-full-linear subring.

Conversely, every Q-full-linear subring T arises this way: choose a complete set of matrix units $\{m_{ij}\}_1^{\infty}$ for T with each m_{ii} a primitive idempotent of T, and let $e_i = m_{ii}$ for $i = 1, 2, \ldots$. The above construction then recovers T.

The following lemma is useful for determining when two Q-full-linear subrings are equal or conjugate.

LEMMA 10. *Let Q be a prime, regular, right self-injective ring. Then*:

(1) *A Q-full-linear subring T is completely determined by any complete set $\{e_i\}_1^{\infty}$ of orthogonal primitive idempotents of T and any system of Q-isomorphisms*

$$e_1Q \xrightarrow{m_{21}} e_2Q \xrightarrow{m_{32}} e_3Q \rightarrow \ldots$$

induced by $m_{i+1,i} \in e_{i+1}Te_i$ for $i = 1, 2, \ldots$.

(2) *Suppose S is any other Q-full-linear subring with $\{f_i\}_1^{\infty}$ a complete set of orthogonal primitive idempotents of S and*

$$f_1Q \xrightarrow{n_{21}} f_2Q \xrightarrow{n_{32}} f_3Q \rightarrow \ldots$$

a system of Q-isomorphisms induced by $n_{i+1,i} \in f_{i+1}Sf_i$. Then any $c \in Q$ which (by its left multiplication) makes the diagram

$$\begin{array}{ccccccc} e_1Q & \xrightarrow{m_{21}} & e_2Q & \xrightarrow{m_{32}} & e_3Q & \longrightarrow & \cdots \\ \downarrow c & & \downarrow c & & \downarrow c & & \\ f_1Q & \xrightarrow{n_{21}} & f_2Q & \xrightarrow{n_{32}} & f_3Q & \longrightarrow & \cdots \end{array}$$

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commutative and all vertical maps isomorphisms, is a unit and satisfies $S = T^c$ $(= cTc^{-1})$.

(3) *When* $S = T$ *in* (2) *with* $f_i = e_i$ *and* $n_{i+1,i} = m_{i+1,i}$ *for all* i, *then any such* c *must centralize* S.

Proof. (1) Since $T \cong F_\infty$ and centre $(T) = F$, we have that e_iTe_j is 1-dimensional over F for all i, j, and so

$$e_iTe_j = Fa_{ij} \quad \text{for any } 0 \neq a_{ij} \in e_iTe_j.$$

From the given $m_{i+1,i}$ we can produce $0 \neq m_{ij} \in e_iTe_j$ in the obvious way:

for $j = i$, set $m_{ii} = e_i$

for $j < i$, set $m_{ij} = m_{i,i-1}m_{i-1,i-2} \dots m_{j+1,j}$

for $j > i$, set $m_{ij} = m_{ji}^{-1}$ (= the unique element in e_iQe_j which induces the inverse of the map $e_iQ \xrightarrow{m_{ji}} e_jQ$).

Then

$$e_iTe_j = Fm_{ij} \quad \forall i, j.$$

Since $\{e_i\}_1^\infty$ is a complete set of orthogonal primitive idempotents of T,

$$\text{soc}(T) = \sum_1^\infty e_iT.$$

Hence for each j,

$$Te_j = \text{soc}(T)e_j = \sum_{i=1}^\infty e_iTe_j = \sum_{i=1}^\infty Fm_{ij}.$$

By Proposition 1 applied to T,

$$T = \prod_{j=1}^\infty Te_j = \prod_{j=1}^\infty \left(\sum_{i=1}^\infty Fm_{ij} \right)$$

which, by Proposition 1 applied to Q, shows T is completely determined by $\{e_i\}_1^\infty$ and $\{m_{i+1,i}\}_1^\infty$.

(2) Clearly c is a unit because

$$\bigoplus_1^\infty e_iQ \leq_e Q \quad \text{and} \quad \bigoplus_1^\infty f_iQ \leq_e Q.$$

Also $ce_i \in f_iQ \quad \forall i$, whence

$$(f_ic - f_ice_i)e_j = 0 \quad \forall j$$

and so $f_i c - f_i c e_i = 0$. Hence $f_i c = f_i c e_i = c e_i$ and

$$c e_i c^{-1} = f_i \in S.$$

From commutativity of the diagram,

$$c m_{i+1,i} c^{-1} = n_{i+1,i} \in S.$$

Hence S and T^c are Q-full-linear subrings sharing a common complete set $\{f_i\}_1^\infty$ of primitive orthogonal idempotents and isomorphisms

$$f_i Q \xrightarrow{n_{i+1,i}} f_{i+1} Q,$$

so $S = T^c$ by (1).

(3) From the proof of (2), it is clear that

$$m_{ij}^c = m_{ij} \quad \forall i, j,$$

where the m_{ij} are the matrix units of T derived as in the proof of (1). Then c centralizes each

$$T e_j = \sum_{i=1}^{\infty} F m_{ij}.$$

For a general $x \in T$, we have

$$(cx - xc)e_j = c(xe_j) - x(ce_j) = xe_j c - xe_j c = 0 \quad \forall j$$

whence $cx - xc = 0$ because $\{e_j\}$ is complete. Thus $cx = xc$, showing c centralizes S.

The next proposition shows there are three classes of regular, right self-injective rings which contain many countable dimensional full linear subrings. We recall that for a prime, regular, right self-injective ring Q, the Goodearl-Boyle [**5**] infinite dimension function μ, defined on the class of nonsingular, injective, right Q-modules, is given by $\mu(A) = 0$ if $A = 0$, while if $A \neq 0$ then

$\mu(A) =$ smallest infinite cardinal α such that αA is not subisomorphic to A

(see [**3**, Chapter 12]). In terms of μ, Goodearl's description of the (two-sided) ideals of Q is that they take the form

$$H(\alpha) = \{x \in Q : \mu(xQ) \leqq \alpha\}$$

for infinite cardinals α [**3**, Proposition 12.19]. In particular $H(\aleph_0)$ is the ideal consisting of all x for which xQ is directly finite, and when it is nonzero, $H(\aleph_0)$ is the unique minimum ideal of Q.

PROPOSITION 11. *For a regular, right self-injective ring Q, the following are equivalent*:

(1) *Each idempotent* $e \in Q$ *belongs to some countably-infinite dimensional full linear subring* T *with* Q_T *nonsingular.*

(2) $Q \cong E(\aleph_0(eQ)) \quad \forall 0 \neq e \in Q$.

(3) Q *is prime and* $Q/H(\aleph_0)$ *is simple.*

(4) *Either*

Q *is a countably-infinite dimensional full linear ring* (*over a division ring*)

or

Q *is simple*, *Type* III

or

Q *is prime*, *Type* II_∞ *with* $Q/H(\aleph_0)$ *simple.*

Proof. (1) $\Rightarrow$ (2). Let $0 \neq e = e^2 \in Q$ and suppose $e \in T$ for some countably-infinite dimensional full linear subring T with Q_T nonsingular. Since $T \cong E(\aleph_0(eT))$, there is a complete set $\{e_i\}_1^\infty$ of orthogonal idempotents of T with

$$\bigoplus_1^\infty e_iT \leq_e T \quad \text{and} \quad e_iT \cong eT \quad \forall i.$$

Now Q_T nonsingular implies $l_Q\{e_i\}_1^\infty = 0$, whence

$$\bigoplus_1^\infty e_iQ \leq_e Q.$$

Also $e_iT \cong eT$ implies $e_iQ \cong eQ \quad \forall i$, so

$$Q = E(\oplus e_iQ) \cong E(\aleph_0(eQ)).$$

(2) $\Rightarrow$ (3). First note that $H(\aleph_0) \neq Q$ because Q is directly infinite from (2). Let $e \neq 0, 1$ be an idempotent of Q. Then $E(\aleph_0(eQ)) \cong Q$ implies

$$\mathrm{Hom}_Q(n(eQ), (1-e)Q) \neq 0 \quad \text{for some } n$$

and hence

$$\mathrm{Hom}_Q(eQ, (1-e)Q) \neq 0.$$

Thus $(1-e)Qe \neq 0$ which shows Q is prime. Now suppose $e \notin H(\aleph_0)$. Then $\mu(eQ) > \aleph_0$, whence by [**3**, Theorem 12.16]

$$\mu(Q) = \mu(E(\aleph_0(eQ))) = \max(\aleph_1, \mu(eQ)) = \mu(eQ).$$

Hence $Q \cong eQ$ (since Q is prime and directly infinite; see [**3**, Corollary 12.11]) and thus $Q = QeQ$. This shows $Q/H(\aleph_0)$ is simple.

(3) $\Rightarrow$ (2). Let $e \in Q$. If $e \notin H(\aleph_0)$ then, since $Q/H(\aleph_0)$ is simple, $\mu(eQ) = \mu(Q)$. Hence

$$\mu(E(\aleph_0(eQ))) = \max\{\aleph_1, \mu(eQ)\} = \mu(eQ) = \mu(Q)$$

and so $E(\aleph_0(eQ)) \cong Q$ because Q is prime and directly infinite. On the other hand, if $e \in H(\aleph_0)$ and $e \neq 0$, then

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$$E(\aleph_0(eQ)) \cong fQ \quad \text{for some } f \in Q,\ f \notin H(\aleph_0),$$

whence

$$E(\aleph_0(eQ)) \cong fQ \cong E(\aleph_0(fQ)) \cong Q.$$

(2) $\Rightarrow$ (1). Let $e \in Q$. If $1 - e \notin H(\aleph_0)$, then by the argument in (2) $\Rightarrow$ (3),

$$(1 - e)Q \cong Q \cong E(\aleph_0(eQ))$$

and so there is a complete set $\{e_i\}_1^\infty$ of orthogonal idempotents of Q with

$$e = e_1 \quad \text{and} \quad e_iQ \cong e_jQ \quad \forall i, j.$$

By the construction outlined at the beginning of this section, $e \in T$ for some Q-full-linear subring T. On the other hand, if $1 - e \in H(\aleph_0)$ then $e \notin H(\aleph_0)$, and the above argument produces T with $1 - e \in T$ and hence $e \in T$.

(3) $\Leftrightarrow$ (4). When Q is prime and directly infinite, Q is of Type I_∞, II_∞, or III. In the Type I_∞ case, $Q/H(\aleph_0)$ is simple if and only if Q is a countably-infinite dimensional full linear ring over a division ring.

Of the three classes in (4), the class of simple Type III self-injective rings can be distinguished as in either of the following two corollaries.

COROLLARY 12. *Let Q be a regular, right self-injective ring but not a division ring. Then Q is simple Type* III *if and only if each idempotent $e \in Q$, with $e \neq 0, 1$, is a primitive idempotent of some countably-infinite dimensional full linear subring T with Q_T nonsingular.*

Proof. ($\Rightarrow$) This is shown in the proof of (2) $\Rightarrow$ (1) of Proposition 11.

($\Leftarrow$) Suppose $H(\aleph_0) \neq 0$. Observe that $H(\aleph_0) \neq Q$ by Proposition 11(3). Choose an idempotent $e \in H(\aleph_0)$, $e \neq 0$. Let T be a full linear subring containing $1 - e$ as a primitive idempotent. Then

$$(1 - e)T \lesssim eT.$$

But now this implies $(1 - e)Q \lesssim eQ$, which is a contradiction because $(1 - e)Q$ is directly infinite whereas eQ is directly finite. We conclude $H(\aleph_0) = 0$. By Proposition 11, Q is simple Type III.

COROLLARY 13. *Let Q satisfy the conditions of Proposition* 11. *Then Q is simple Type* III *if and only if all Q-full-linear subrings are conjugate in Q.*

Proof. Assume all Q-full-linear subrings are conjugate. Suppose $H(\aleph_0) \neq 0$. Then we can find nonzero orthogonal idempotents $f, g \in H(\aleph_0)$. Let $e = f + g$. As shown in the proof of (2) $\Rightarrow$ (1) of Proposition 11, since

$$1 - e \notin H(\aleph_0) \quad \text{and} \quad 1 - f \notin H(\aleph_0),$$

there exist Q-full-linear subrings T_1, T_2 in which e is a primitive idempotent of T_1 and f is a primitive idempotent of T_2. By assumption $T_2 = T_1^c$ for some unit $c \in Q$. Since cec^{-1} must be primitive in T_2 we have

$$(cec^{-1})T_2 \cong fT_2$$

and hence

$$eQ \cong (cec^{-1})Q \cong fQ,$$

which implies eQ is directly infinite, a contradiction. Hence $H(\aleph_0) = 0$ and this makes Q simple Type III.

Conversely, assume Q is simple Type III. Let T_1, T_2 be Q-full-linear subrings, and let $\{m_{ij}\}_1^\infty$ and $\{n_{ij}\}_1^\infty$ be complete sets of matrix units for T_1, T_2 respectively such that the m_{ii} and n_{ii} are primitive idempotents of T_1, T_2 respectively. Let $e_i = m_{ii}$, $f_i = n_{ii}$ for $i = 1, 2, \ldots$. Since Q is right self-injective and $eQ \cong fQ$ for all nonzero $e, f \in Q$, we can choose $c \in Q$ such that, under left multiplication by c, the following diagram is commutative and the vertical maps are isomorphisms:

$$\begin{array}{ccccccc}
e_1Q & \xrightarrow{m_{21}} & e_2Q & \xrightarrow{m_{32}} & e_3Q & \longrightarrow & \cdots \\
\downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c} & & \\
f_1Q & \xrightarrow{n_{21}} & f_2Q & \xrightarrow{n_{32}} & f_3Q & \longrightarrow & \cdots
\end{array}$$

By Lemma 10(2), c is a unit and $T_2 = T_1^c$.

Of course we could have derived similar (but less natural) characterizations of a simple Type III Q if, instead of taking an F_∞ subring in our definition of a Q-full-linear subring, we had considered F_n for a fixed integer $n \geqq 2$. However not all elements of Q lie in a copy of F_n, and not all finitely generated subrings can be embedded in F_n, whereas at least every countable subring of Q is isomorphic to a subring of F_∞; so one expects the Q-full-linear subrings to cover much of Q. For example:

PROPOSITION 14. *Suppose Q is a simple, Type* III, *right self-injective ring with an algebraically closed centre F. Then any semisimple, finite dimensional F-subalgebra A of Q is contained in some Q-full-linear subring.*

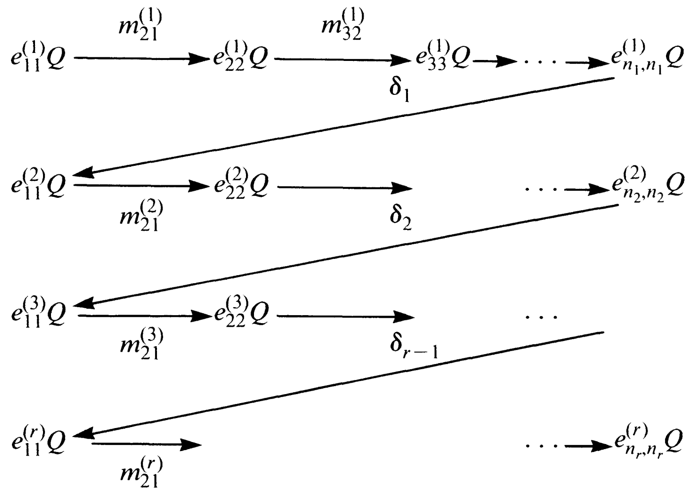
Proof. We have

$$A = A_1 \oplus A_2 \oplus \ldots \oplus A_r$$

for some ideals A_i of A, with $A_i \cong F_{n_i}$ for some positive integers n_i. For $k = 1, \ldots, r$, let

$$\{m_{ij}^{(k)}\}_{i,j=1,\ldots,n_k}$$

be a fixed set of matrix units for A_k. Since $xQ \cong yQ$ for any nonzero $x, y \in Q$, we can insert "links" $\delta_1, \dots, \delta_{r-1}$ to obtain the following system of Q-isomorphisms:



(where $e_{ii}^{(k)} = m_{ii}^{(k)}$).

From this we can derive in the usual way a set of matrix units

$$\{n_{ij}\}_{i,j=1,\dots,l}$$

where $l = n_1 + \dots + n_r$, such that the subalgebra

$$B = \sum Fn_{ij} \cong F_l$$

contains all the $m_{ij}^{(k)}$. Then $B \supseteq A$. Further infinite splittings of each $n_{ii}Q$, using

$$n_{11}Q \cong E(\aleph_0(n_{11}Q))$$

and the maps

$$n_{i1}:n_{11}Q \to n_{ii}Q,$$

then lead to a Q-full-linear subring T containing B and hence A. (Alternatively, for this second stage, we can start with any Q-full-linear subring T_1 and choose a subalgebra $B_1 \subseteq T_1$ with centre $(B_1) = F$ and $B_1 \cong F_l$. Then, as in Corollary 13, $B = B_1^c$ for some unit $c \in Q$, whence $T = T_1^c$ is a Q-full-linear subring containing B.)

Remarks. (1) If $Q = Q_{\max}(A)$ for

$$A = \varinjlim F_{2^n},$$

then there cannot exist a Q-full-linear subring T containing A, otherwise

$$Q = Q_{\max}(T) = T.$$

Thus Proposition 14 cannot be extended to F-subalgebras which are countable direct limits of semisimple, finite dimensional algebras.

(2) One important difference between a simple, Type III, right self-injective ring Q and its directly finite analogue, namely a division ring, is that Q has an abundance of idempotents; enough to generate Q as a ring [**3**, Theorem 13.16]. This property ensures that, at the very least, Q is always generated by the family of Q-full-linear subrings (see Proposition 11). By Corollary 13, Q is in fact generated by

$$\bigcup_{c \in Q^*} cTc^{-1}$$

for any fixed Q-full-linear subring T. In contrast, the analogous property of a division ring D being generated by its centre (which is the only candidate for a full linear subring, over a field, having the same centre as D) obviously fails when the ring is not commutative. (If, however, D is finite-dimensional over its centre, then by the Cartan-Brauer-Hua Theorem, D is generated by the conjugates of any subfield which properly contains the centre.)

Example 15. Let $Q = Q_{\max}(R)$ where R is a countable-dimensional, prime, nonsingular algebra over an algebraically closed field F. Assume further that $\operatorname{soc}(R) = 0$ but that R is "locally finite-dimensional semisimple", that is every finite subset of R is contained in some finite-dimensional semisimple subalgebra. (For instance this is true of the group algebra $R = F[G]$ where G is a countable, prime, locally finite group, and $\operatorname{char}(F) = 0$.) Then Q is a simple, Type III, right self-injective ring whose centre contains F, whence by the same argument used in the proof of Proposition 14, the union of the Q-full-linear subrings contains R. (In the group algebra example, the centre of Q is F so Proposition 14 applies directly.)

For a subset X of Q, let

$$C_Q(X) = \{a \in Q : ax = xa \quad \forall x \in X\}$$

be the centralizer of X in Q.

PROPOSITION 16. *Let Q be a simple, Type* III, *right self-injective ring, and let T be a Q-full-linear subring. Then T has the double centralizer property*:

$$C_Q(C_Q(T)) = T.$$

Proof. Let F be the centre of Q. Let $M = \{m_{ij}\}_1^{\infty}$ be a complete set of matrix units for T with centralizer F (so that

$$T = \prod_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} Fm_{ij} \right) \bigg),$$

and let $e_i = m_{ii}$ for $i = 1, 2, \ldots$. Then

$$C_Q(T) = C_Q(M) = D_Q(M)$$

where $D_Q(M)$ is the "diagonal of Q relative to M", that is

$$D_Q(M) = \{y \in Q{:}ye_i = \psi_i(e_1ye_1) \quad \forall i\}$$

where $\psi_i{:}e_1Qe_1 \to e_iQe_i$ is the isomorphism

$$a \mapsto m_{i1}am_{1i}.$$

(Note, by Proposition 1, $D_Q(M)$ is naturally isomorphic to e_1Qe_1.) Let

$$w \in C_Q(C_Q(T)),$$

so that $w \in C_Q(D_Q(M))$. Let

$$w_{ij} = e_iwe_j \quad \text{for } i, j = 1, 2, \ldots.$$

Then for all $a \in e_1Qe_1$

$$\begin{aligned} e_iw\psi_j(a) &= \psi_i(a)we_j \\ &\Rightarrow w_{ij}\psi_j(a) = \psi_i(a)w_{ij} \\ &\Rightarrow (m_{1i}w_{ij}m_{j1})a = a(m_{1i}w_{ij}m_{j1}), \end{aligned}$$

whence $m_{1i}w_{ij}m_{j1} \in$ centre $(e_1Qe_1) = Fe_1$ and thus $w_{ij} \in Fm_{ij}$.

Now for $0 \neq c \in F$,

$$r_Q(cm_{ij}) \cap e_jQ = 0$$

because $m_{ji}m_{ij} = e_j$. Hence for a given j, if infinitely many $w_{ij} \neq 0$, then the right ideal

$$A_j = \{a \in Q{:}w_{ij}a = 0 \quad \forall' i\}$$

is not large in Q, contrary to Proposition 2. Hence $w_{ij} \neq 0$ for only finitely many i, whence

$$we_j \in \sum_{i=1}^{\infty} Fm_{ij} = Te_j \subseteq T.$$

Hence (by Proposition 1) $w \in T$, showing $C_Q(C_Q(T)) \subseteq T$.

The properties of Q-full-linear subrings in Corollary 13 and Proposition 16, namely that any two are conjugate and they satisfy the double centralizer condition, are reminiscent of the two corresponding properties of n-dimensional full linear subalgebras over F ($\cong F_n$ for fixed n), with centre F, of a given central simple finite-dimensional algebra over F. (The latter properties follow from the classical Noether-Skolem and Double Centralizer Theorems for such algebras.)

5. Chains of full linear subrings. Throughout this section Q denotes a simple, Type III, right self-injective ring with centre F.

The manner in which one countably-infinite dimensional full linear ring over F embeds in another is well-known (see Lemma 10(2)).

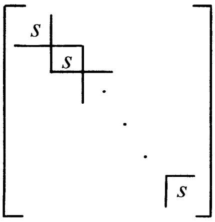
PROPOSITION 17. *Let F be a field and let $T = F_\infty$ be a countable-dimensional full linear ring over F.*

(1) *The conjugacy classes of T-full-linear subrings are*

$$\mathscr{C}_1, \mathscr{C}_2, \dots, \mathscr{C}_n, \dots, \mathscr{C}_\infty$$

where $\mathscr{C}_n$ consists of those S in which primitive idempotents of S split into n primitives in T.

(2) *For $S \in \mathscr{C}_n$ ($n = 1, 2, \dots, \infty$) the embedding $S \hookrightarrow T$ is equivalent to the diagonal embedding of F_∞ into $(F_\infty)_n \subseteq F_\infty$*

$s \mapsto$ 

n blocks

An alternative view of (2) is that $S \hookrightarrow T$ is a $\mathscr{C}_n$ embedding if and only if there exists a complete set $N = \{n_{ij}: i, j = 1, \dots, n\}$ of n^2 matrix units of T such that

$$S = C_T(N) = D_T(N) \quad \text{(diagonal of } T \text{ relative to } N).$$

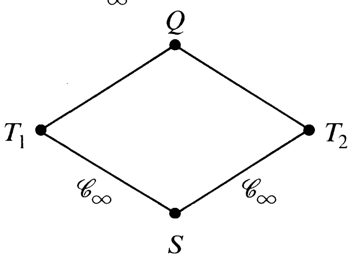
Here, by "complete" we mean $\{n_{ii}\}_1^n$ is a complete set of orthogonal idempotents of T.

The following lemma proves very useful when we examine chains of Q-full-linear subrings.

LEMMA 18. *Suppose S, T_1, T_2 are Q-full-linear subrings with $S \subseteq T_1$, $S \subseteq T_2$ and such that both embeddings*

$$S \hookrightarrow T_1, \quad S \hookrightarrow T_2$$

are in class $\mathscr{C}_\infty$:



Then there exists a unit $c \in Q$ such that c centralizes S and $T_1^c = T_2$.

Proof. Let $M = \{m_{ij}\}_{i,j=1,\dots,\infty}$ be a complete set of matrix units for S with each $e_i = m_{ii}$ a primitive idempotent of S. Since $S \hookrightarrow T_1$ and $S \hookrightarrow T_2$ are C_∞ embeddings, there exist orthogonal primitive idempotents f_{ij} of T_1 and orthogonal primitive idempotents g_{ij} of T_2 such that

$$e_iT_1 = E\left(\bigoplus_{j=1}^{\infty} f_{ij}T_1\right), \quad e_iT_2 = E\left(\bigoplus_{j=1}^{\infty} g_{ij}T_2\right)$$

and

$$m_{ji}(f_{ik}T_1) = f_{jk}T_1, \quad m_{ji}(g_{ik}T_2) = g_{jk}T_2 \quad \forall i, j, k.$$

We can find isomorphisms

$$\alpha_{lk}^{(i)}:f_{ik}T_1 \to f_{il}T_1, \quad \beta_{lk}^{(i)}:g_{ik}T_2 \to g_{il}T_2$$

with

$$\alpha_{kk}^{(i)} = f_{ik}, \ \beta_{kk}^{(i)} = g_{ik}, \ \alpha_{jl}^{(i)}\alpha_{lk}^{(i)} = \alpha_{jk}^{(i)}, \ \beta_{jl}^{(i)}\beta_{lk}^{(i)} = \beta_{jk}^{(i)}$$

such that the diagrams

$$\begin{array}{ccc} f_{ik}T_1 & \xrightarrow{\alpha_{lk}^{(i)}} & f_{il}T_1 \\ {\scriptstyle m_{ji}}\downarrow & & \downarrow{\scriptstyle m_{ji}} \\ f_{jk}T_1 & \xrightarrow{\alpha_{lk}^{(i)}} & f_{jl}T_1 \end{array} \qquad \begin{array}{ccc} g_{ik}T_2 & \xrightarrow{\beta_{lk}^{(i)}} & g_{il}T_2 \\ {\scriptstyle m_{ji}}\downarrow & & \downarrow{\scriptstyle m_{ji}} \\ g_{jk}T_2 & \xrightarrow{\beta_{lk}^{(j)}} & g_{jl}T_2 \end{array}$$

commute $\forall i, j, k, l$. (This amounts to a free choice for $\alpha_{21}^{(1)}, \alpha_{32}^{(1)}, \dots, \alpha_{i+1,i}^{(1)} \dots$ and $\beta_{21}^{(1)}, \beta_{32}^{(1)}, \dots, \beta_{i+1,i}^{(1)}, \dots$, and the rest are then determined.) Since

$$xQ \cong yQ \quad \forall 0 \neq x, y \in Q,$$

we can obtain isomorphisms

$$\gamma_{ij}:f_{ij}Q \to g_{ij}Q$$

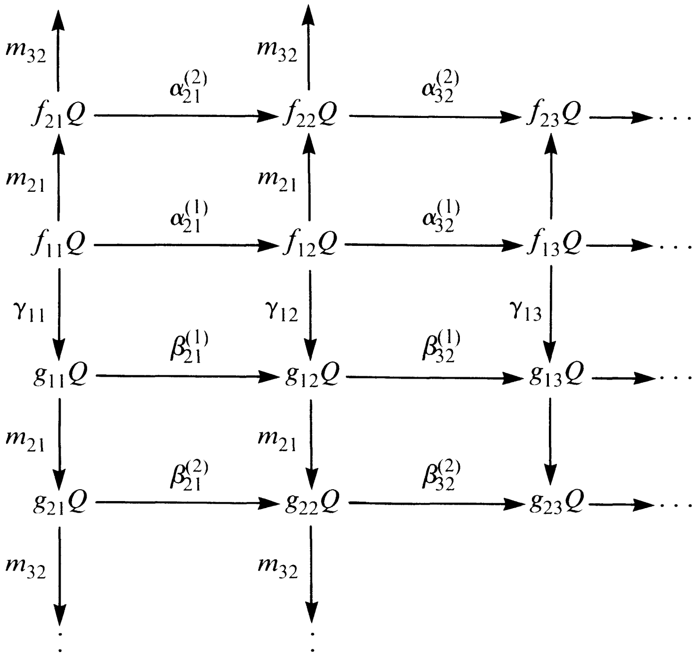
such that the diagram

$$\begin{array}{ccccc} f_{ik}Q & \xrightarrow{\gamma_{ik}} & g_{ik}Q & \xrightarrow{\beta_{lk}^{(i)}} & g_{il}Q \\ {\scriptstyle m_{ji}}\downarrow & & & & \downarrow{\scriptstyle m_{ji}} \\ f_{jk}Q & \xrightarrow{\alpha_{lk}^{(j)}} & f_{jl}Q & \xrightarrow{\gamma_{jl}} & g_{jl}Q \end{array}$$

commutes $\forall i, j, k, l$. This amounts to a free choice for

$$\gamma_{11}: f_{11}Q \to g_{11}Q$$

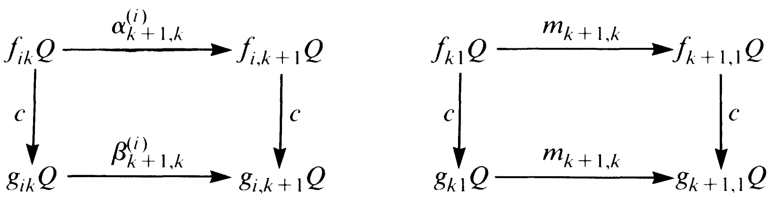
and a determination of the other γ_{ij} from the following commutative network of maps:



Since Q is right self-injective, there exists $c \in Q$ which induces the γ's, that is

$$cf_{ij} = \gamma_{ij} \quad \forall i, j.$$

Now $\{f_{ij}\}_{i,j=1,\ldots,\infty}$ and $\{g_{ij}\}_{i,j=1,\ldots,\infty}$ are respectively complete sets of orthogonal primitive idempotents for T_1 and T_2, and the diagrams



are commutative, whence by a simple variation of Lemma 10 (2) (note that from the above maps we can canonically derive Q-isomorphisms between any two $f_{ij}Q$ and between any two $g_{ij}Q$), c is a unit of Q and $T_2 = T_1^c$. Also, from commutativity of

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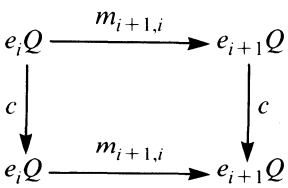


we conclude from Lemma 10(3) that c centralizes S.

Remark. A similar proof shows that the lemma also holds when the embeddings $S \hookrightarrow T_1$ and $S \hookrightarrow T_2$ are in class $\mathscr{C}_n$, for the same finite n.

THEOREM 19. *Suppose*

$$T_1 \subseteq T_2 \subseteq \ldots \subseteq T_n \subseteq \ldots$$

$$S_1 \subseteq S_2 \subseteq \ldots \subseteq S_n \subseteq \ldots$$

are two chains of Q-full-linear subrings with each $T_i \hookrightarrow T_{i+1}$, $S_i \hookrightarrow S_{i+1}$ *a* $\mathscr{C}_\infty$ *embedding* (*or a* $\mathscr{C}_{n_i}$ *embedding for the same* n_i). *Then there exist units* $c_1, c_2, \ldots, c_n, \ldots \in Q$ *such that for all* n, c_n *centralizes* S_{n-1} *and*

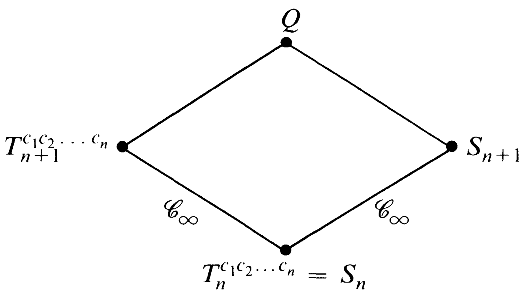
$$T_n^{c_1c_2\ldots c_n} = S_n.$$

In particular the two chains are equivalent as chains (*although they can sit in Q quite differently*).

Proof. By Corollary 13, there exists a unit $c_1 \in Q$ with

$$T_1^{c_1} = S_1.$$

Assume $c_1, \ldots, c_n$ have been constructed. By Lemma 18 applied to the diagram



there is a unit $c_{n+1} \in Q$ which centralizes S_n and satisfies

$$(T_{n+1}^{c_1c_2\ldots c_n})^{c_{n+1}} = S_{n+1}.$$

Hence the c_n can be constructed inductively.

Now the map

$$\psi: \cup T_i \rightarrow \cup S_i, \quad x \mapsto x^{c_1c_2\ldots c_n} \quad \text{if } x \in T_n,$$

is a well-defined ring isomorphism with

$$\psi(T_n) = S_n \quad \forall n.$$

Thus the two chains are equivalent.

Remark. $\cup T_i$ and $\cup S_i$ need not be conjugate in Q.

COROLLARY 20. *Let F be a given field and let*

$$R = \varinjlim_{i<\omega} T_i \quad \textit{and} \quad S = \varinjlim_{i<\omega} S_i$$

be any countable direct limits of countably-infinite dimensional full linear rings over F where the maps $T_i \rightarrow T_{i+1}$, $S_i \rightarrow S_{i+1}$ are monomorphisms making T_i (resp. S_i) a T_{i+1} (resp. S_{i+1})-full-linear subring in class $\mathcal{C}_{n_i}$ for all i (with n_i finite or infinite). Then $R \cong S$.

Proof. We can assume

$$R = \bigcup_1^\infty T_i \quad \text{and} \quad S = \bigcup_1^\infty S_i$$

where each T_i (resp. S_i) is a T_{i+1} (resp. S_{i+1})-full-linear subring in class $\mathcal{C}_{n_i}$, and that infinitely many $n_i > 1$. Let

$$Q_1 = Q_{\max}(R), \quad Q_2 = Q_{\max}(S).$$

Then Q_1 and Q_2 are simple, Type III, right self-injective rings (because R and S are right quotient rings of countable-dimensional, prime, non-singular algebras without uniform ideals; see [7]). Moreover

$$\text{centre }(Q_1) = \text{centre }(R) \cong F \text{ and centre }(Q_2) = \text{centre }(S) \cong F.$$

Since $(Q_1)_R$ and R_{T_i} are nonsingular, and the T_i are right Utumi rings, it follows that each $(Q_1)_{T_i}$ is nonsingular. Hence each T_i is a Q_1-full-linear subring. Similarly, each S_i is a Q_2-full-linear subring.

Fix a Q_1-full-linear subring A, a Q_2-full-linear subring B, and a ring isomorphism $\psi:A \rightarrow B$. From Proposition 17(2), it is clear that we can find a chain $A_1 \subseteq A_2 \subseteq \dots$ of A-full-linear subrings with each $A_i \hookrightarrow A_{i+1}$ a $\mathcal{C}_{n_i}$ embedding. By the above argument this is also a chain of Q_1-full-linear subrings, whence by Theorem 19

$$\bigcup_1^\infty T_i \cong \bigcup_1^\infty A_i.$$

Also $\psi(A_1) \subseteq \psi(A_2) \subseteq \dots$ is a chain of Q-full-linear subrings with each

$$\psi(A_i) \hookrightarrow \psi(A_{i+1})$$

a $\mathcal{C}_{n_i}$ embedding, so another application of Theorem 19 yields

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$$\bigcup_1^\infty S_i \cong \bigcup_1^\infty \psi(A_i).$$

Hence, since

$$\bigcup_1^\infty A_i \cong \bigcup_1^\infty \psi(A_i),$$

we have

$$R = \bigcup_1^\infty T_i \cong \bigcup_1^\infty S_i \cong S.$$

In general, if $T_1 \subseteq T_2 \subseteq \ldots \subseteq T_n \subseteq \ldots$ is a chain of Q-full-linear subrings with each $T_i \hookrightarrow T_{i+1}$ a $\mathscr{C}_\infty$ embedding, how does the subring $\cup_1^\infty T_i$ sit in Q? If, for example, $Q = Q_{\max}(U)$ where

$$U = \varinjlim T_i,$$

then Q is a right quotient ring of $\cup_1^\infty T_i$ (and by Corollary 8, Q is a ring of fractions of U_∞). However, such a countable union of a chain of Q-full-linear subrings can never equal Q. This follows from the (presumably well-known):

PROPOSITION 21. *Let*

$$R = \varinjlim_{i<\omega} R_i$$

be a countable direct limit of prime, right self-injective rings R_i, where the maps $R_i \to R_{i+1}$ are monomorphisms. Then R is right self-injective if and only if the R_i are simple Artinian of bounded length or the embeddings $R_i \to R_{i+1}$ are isomorphisms for almost all i.

Proof. ($\Leftarrow$) This is clear.

($\Rightarrow$) Observe that if a right self-injective ring A is a subring of a prime ring B, and $Af = Bf$ for some $0 \neq f \in A$, then $A = B$: for $Bf = Af \leqq A$ implies $A_A \leq_e B_A$, whence $A = B$. Now suppose neither of the two stated conditions holds. Then, by relabeling if necessary, we can assume $R_1 \subset R_2 \subset \ldots$ and R contains a right ideal

$$I = \bigoplus_1^\infty f_i R$$

where $0 \neq f_i \in R_i$. Now using the above observation, for each n we can choose $a_n \in R_{n+1}$ such that $a_n f_n \notin R_n f_n$. Consider the R-map

$$\psi: I \to R$$

determined by

$$\psi(f_n) = a_n f_n.$$

Clearly ψ is not induced by left multiplication by any $a \in R_n$ because

$$\psi(f_n) = a_n f_n \neq a f_n.$$

This contradicts the injectivity of R_R.

A similar argument applies to a direct limit

$$R = \varinjlim_{\alpha < \omega_1} R_\alpha$$

of prime, regular, right self-injective rings R_α: R is right self-injective if and only if R contains no uncountable direct sums of nonzero right ideals or the embeddings $R_\alpha \to R_{\alpha+1}$ are isomorphisms for all but countably many α. It seems possible that for some Q, $Q = \cup T_\alpha$ for a suitable chain $T_1 \subseteq T_2 \subseteq \ldots \subseteq T_\alpha \subseteq \ldots$ of Q-full-linear subrings, that is, a suitable direct limit of full linear rings F_∞ may be right self-injective.

COROLLARY 22. *If* $Q = Q_{\max}(R)$ *for some countable ring* R *(or countable-dimensional algebra* R*), then*

$$Q \neq \bigcup_{\alpha \in \Omega} T_\alpha$$

for any chain of Q*-full-linear subrings* T_α.

Proof. Let the index set Ω be totally ordered by $\leqq$.

First suppose there exists a countable $I \subseteq \Omega$ such that I is unbounded. Then

$$\bigcup_{\alpha \in \Omega} T_\alpha = \bigcup_{i \in I} T_i$$

is not right self-injective by Proposition 21 unless it is full linear. Hence

$$\bigcup_{\alpha \in \Omega} T_\alpha \neq Q$$

in this case.

Next suppose all countable subsets of Ω are bounded, and that

$$\bigcup_{\alpha \in \Omega} T_\alpha = Q.$$

Let

$$R = \{a_1, a_2, \ldots, a_n, \ldots\}.$$

Choose n_i such that $a_i \in T_{n_i}$ and choose $\alpha_0 \geqq n_i \quad \forall i$. Then $R \subseteq T_{\alpha_0}$ so

$$Q = Q_{\max}(R) = Q_{\max}(T_{\alpha_0}) = T_{\alpha_0},$$

which is impossible. Thus

$$\bigcup_{\alpha \in \Omega} T_\alpha \neq Q$$

holds here as well.

A similar argument shows that if

$$Q = \bigcup_{\alpha \in \Omega} T_\alpha$$

for some chain of Q-full-linear T_α, then for every countable chain $T_1 \subseteq T_2 \subseteq \ldots$ of Q-full-linear subrings there is a Q-full-linear subring T containing all T_i. Conversely, if each such countable chain has a Q-full-linear subring upper bound, then we could construct a chain $\{T_\alpha\}_{\alpha < \omega_1}$ with each

$$T_\alpha \hookrightarrow T_{\alpha+1}$$

a $\mathscr{C}_\infty$ embedding. Hence if Q has no uncountable direct sums of right ideals, then the subring

$$\bigcup_{\alpha < \omega_1} T_\alpha$$

would be a simple, Type III, right self-injective ring (it cannot contain an uncountable direct sum of right ideals because it is a regular subring of Q).

6. Questions. Let Q be a simple, Type III, right self-injective ring with an algebraically closed centre F, and assume Q contains only countable direct sums of nonzero right ideals.

(1) How "locally full linear" is Q? For example when does $x \in Q$ belong to some Q-full-linear subring? (A necessary condition is that $C_Q(x)$ is directly infinite.) What about finite subsets? (c.f. Proposition 14).

(2) Given a Q-full-linear subring T and an idempotent $e \in Q$, when does there exist a Q-full-linear subring T' containing T and e (c.f. Proposition 11 and Corollary 12)? Such a T' need not exist in general, as can be shown by examining

$$Q = Q_{\max}\left(\varinjlim F_{2^n}\right).$$

However T' would always exist if Q is a union of a chain of Q-full-linear subrings, say

$$Q = \bigcup_{\alpha \in \Omega} T_\alpha,$$

because by Proposition 21 countable subsets of Ω would be bounded and hence some T_α would contain the given T and e. One reason for focusing on idempotents is that Q is always generated as a ring by idempotents. In particular, a positive answer here would imply that any finite subset does belong to some Q-full-linear subring, answering (1) for such Q.

(3) Can Q be a union of a chain of Q-full-linear subrings? Note that by Corollary 20, we can form an uncountable direct limit

$$R = \varinjlim_{\alpha < \omega_1} T_\alpha$$

of countably-infinite dimensional full linear rings over F such that $T_\alpha \rightarrow T_\beta$ is a $\mathscr{C}_\infty$ embedding whenever $\alpha < \beta$. Is R right self-injective, or equivalently, does R contain only countable direct sums of nonzero right ideals? If 'yes', we have our desired Q (and from Corollary 22, an easy example of non-isomorphic Q's).

(4) Let

$$R = \varinjlim F_{n_i}$$

where $n_1, n_2, \ldots$ is a factor sequence (for example $n_i = 2^i$). Is

$$Q_{\max}(R) \cong Q_{\max}(U)$$

where

$$U = \varinjlim T_i$$

is the countable direct limit of countable-dimensional full linear rings over F in which $T_i \rightarrow T_{i+1}$ makes T_i a T_{i+1}-full-linear subring in class $\mathscr{C}_\infty$? If 'yes', then, by Corollary 20, different factor sequences would give isomorphic maximal right quotient rings. Note that it would suffice to construct a chain $S_1 \subseteq S_2 \subseteq \ldots$ of Q-full-linear subrings of $Q = Q_{\max}(R)$ such that $S_i \hookrightarrow S_{i+1}$ is a $\mathscr{C}_\infty$-embedding and

$$R \subseteq \bigcup_1^\infty S_i.$$

(5) Let U be as in (4) and let R be a non-right-Ore domain with centre F and countable-dimensional over F. Let

$$Q_1 = Q_{\text{max-cl}}(U_\infty) \quad \text{and} \quad Q_2 = Q_{\text{max-cl}}(R_\infty).$$

By Corollaries 7 and 8, Q_1 and Q_2 are simple, Type III, right self-injective rings with only countable direct sums of right ideals. When is $Q_1 \cong Q_2$? (As remarked earlier, $U_\infty \lesssim R_\infty$ and $R_\infty \lesssim U_\infty$ as rings.)

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