Can. J. Math. Vol. **45** (1), 1993 pp. 88–103

NON-LOCAL LIE PRIMITIVE SUBGROUPS OF LIE GROUPS

ARJEH M. COHEN AND ROBERT L. GRIESS JR.

ABSTRACT Borovik found a Lie primitive subgroup of $E_8(\mathbb{C})$ isomorphic to $(\mathrm{Alt}_5 \times \mathrm{Sym}_6) \cdot 2$ In this note, we provide a short proof of existence and his result that the conjugacy class of this subgroup is the only one among those of non-local Lie primitive subgroups of finite dimensional simple complex Lie groups having a socle with more than one simple factor

1. **Introduction and statement of results.** In [CoGr 1987], the isomorphism types of finite nonabelian simple subgroups of the complex Lie groups $E_7(\mathbb{C})$ and $E_8(\mathbb{C})$ were studied. We define a *Lie primitive subgroup* of a complex Lie group to be a subgroup which is not contained in any proper, positive dimensional Zariski closed subgroup. In any group, a *local subgroup* is the normalizer of a nonidentity p-subgroup, for some prime number p. In [Aleks 1974] and, later, with different methods in [CLSS 1989], the local Lie primitive subgroups of complex simple Lie groups of exceptional type were classified.

Here, we continue the study of Lie primitive subgroups of a complex simple Lie group G of exceptional type. We show that any finite nonlocal Lie primitive subgroup of G normalizes a nonabelian simple subgroup, which, apart from a single exception found by Borovik, is unique up to conjugacy. Thus, we establish:

THEOREM 1.1. *Let G be an adjoint simple complex Lie group. Suppose L is a finite Lie primitive subgroup of G. Then either L is contained in a finite local subgroup or its socle is a nonabelian simple subgroup or* $G = E_8(\mathbb{C})$ *and* $\mathrm{soc}\, L$ *is isomorphic to* $\mathrm{Alt}_5 \times \mathrm{Alt}_6$. *Conversely there exists a subgroup of* $E_8(\mathbb{C})$ *isomorphic to* $\mathrm{Alt}_5 \times \mathrm{Alt}_6$ *which is Lie primitive and such a group is unique up to conjugacy.*

The above group of the form $\mathrm{Alt}_5 \times \mathrm{Alt}_6$ is called the *semisimple Borovik group* and its normalizer is called the *Borovik group*. The Borovik group contains the semisimple Borovik group with index 4 and it contains $\mathrm{Alt}_5 \times \mathrm{Sym}_6$ with index 2. More details on this group are given in § 4.

A more general version of this theorem (arbitrary characteristic of the ground field) has been announced by [Borov 1989] and, later, by [LiSe 1989]. We obtained these results independently and our treatment is relatively elementary and more detailed. The result (2.7), given by [LiSe 1989], considerably shortened an earlier version of this paper.

REMARK. The result is known for classical groups, for instance, by [Aschb 1984]. In fact, he points out a distinguished list of closed subgroups such that every finite group

Received by the editors January 9, 1991
AMS subject classification 22E20, 22E46

whose socle is not nonabelian simple is a subgroup of one of them. The members of that list are infinite except for normalizers of abelian subgroups, which come from nonabelian groups in the universal cover.

In a letter to one of us, Borovik exhibited a Lie primitive subgroup of $E_8(\mathbb{C})$ isomorphic to $(\mathrm{Alt}_5 \times \mathrm{Sym}_6) : 2$. Our construction of this group can be found in § 4.

We take this opportunity to report that the simple group Sz(8) with a ? should be on Table 2 of [CoGr 1987]. First of all, Sz(8) is in a 2-local subgroup of the sporadic group Ru. There is an embedding of Ru in $E_7(5)$ [KMR 1989]. Hence, by [Gr 1991, Appendix 2], the Borel subgroup of Sz(8), being of order prime to 5, lifts to $E_7(\mathbb{C})$ (an error is in (5.6.2) of [CoGr 1987]). From [GrRy 1992], we know that Sz(8) is contained in $E_7(K)$ for a field K if and only if char(K) is 2 or 5; the possibility that Sz(8) is embedded in $E_8(\mathbb{C})$ remains. Also, $U_3(8) : 12$ is now known to be embedded in $E_7(\mathbb{C})$ [GrRy 1992] and Ru is embedded in $E_7(5)$ [GrRy 1992] [KMR 1989]. An embedding of $L(2, 61)$ in $E_8(\mathbb{C})$ was proved recently [CoGrLi 1992]. Also, Lemma (3.5) of [CoGr] does not suffice to eliminate $L(4, 5)$, though its nonembedding in $E_8(\mathbb{C})$ follows trivially from the nonembedding of a $PSp(4, 5)$-subgroup. Finally, the second argument given to show the nonembedding of F_3 in $E_8(\mathbb{C})$ is not valid since the indicated element of order 3 need not have trace 5.

Another correction should be made to part (ii) of Theorem 1.1 of [CoGr 1987]; the groups SL(2, 31) and SL(3, 4) should be removed from the list, and the group $2 \cdot L(3, 4)$ should be inserted. The error is just a misstatement of our correct results (5.3.1) and (5.2.7) (which are correctly reported in Table 2).

A consequence of the above remarks, Theorem 1.1 of this article, [CoGr 1987] and [CoWa 1983, 1989] is that the isomorphism types of semisimple Lie primitive subgroups of exceptional Lie groups G are known, except for the few specific cases listed in [CoGr 1987] and [CoWa 1989].

2. **The setup.** Throughout this article, we shall denote by L a finite Lie primitive subgroup of G whose socle is denoted soc L and which is a direct product of finite nonabelian simple groups. Let N be a nonidentity normal subgroup of L such that $N \leq \operatorname{soc} L$. Then there exist $t \in \mathbb{N}$ and nonabelian simple subgroups N_i $(1 \leq i \leq t)$ such that $N = N_1 \times N_2 \times \cdots \times N_t$.

We assume that $t > 1$ and prove that N is the semisimple Borovik group; see (3.6) and § 4.

NOTATION 2.1. The adjoint module of G is denoted by $\mathbf{g}$, and the corresponding character of G by χ. By $E_7(\mathbb{C})$ we mean the adjoint group; its universal cover will be denoted by $2E_7(\mathbb{C})$. Similar notations for central extensions apply to the other simple Lie groups. By $1_a, 8_b, \ldots$ we mean an irreducible module of dimension 1, 8, *etc.* for some group or Lie algebra. The subscripts distinguish nonisomorphic modules of a given dimension. When the group is finite and essentially simple, we use the notation of [Atlas 1985]; otherwise, the symbols stand for well-known modules of the group, *e.g.*, $8_a, 8_b, 8_c$ stand for the complete set of 8-dimensional irreducibles for the Lie algebra or simply connected

https://doi.org/10.4153/CJM-1993-005-7 Published online by Cambridge University Press

Lie group of type D_4. The type of an element of finite order at most 7 in $E_8(\mathbb{C})$ is the label given to its conjugacy class in [CoGr 1987, Table 4].

Any irreducible representation of N is the tensor product of representations of the N_ι $(1 \le i \le t)$. Thus, if $\psi_0^{(\iota)}, \ldots, \psi_{s_\iota}^{(\iota)}$ are the irreducible characters of N_ι, the irreducible characters of N are of shape $\psi_{\iota_1}^{(1)} \otimes \psi_{\iota_2}^{(2)} \otimes \cdots \otimes \psi_{\iota_t}^{(t)}$, where $\otimes$ denotes character multiplication for a tensor product of modules for a direct product of groups. Hence there are non-negative numbers $a_{\iota_1, \ldots, \iota_t}$ such that

$$(*) \qquad \chi|_N = \sum_{\iota_1, \iota_2, \ldots, \iota_t} a_{\iota_1, \ldots, \iota_t} \psi_{\iota_1}^{(1)} \otimes \psi_{\iota_2}^{(2)} \otimes \cdots \otimes \psi_{\iota_t}^{(t)}.$$

In using this kind of decomposition, we will write the characters as in [Atlas 1985].

We recall

LEMMA 2.2 (*cf.* [CoGr 1987]). *A nontrivial normal subgroup of L has zero fixed point subalgebra on* $\mathbf{g}$.

PROOF. Let M be a nontrivial normal subgroup of L. The connected component C of the identity of the centralizer of M (for short: the *connected centralizer* of M) in G is normalized by the normalizer in G of M, whence by L. If M has nonzero fixed vectors in $\mathbf{g}$ then $C_{\mathbf{g}}(M)$ is a nontrivial subalgebra of $\mathbf{g}$; therefore $N_G(C)$ is a closed complex Lie subgroup of positive dimension containing L, contradicting Lie primitivity of L. ∎

We remark that, for N_ι non-normal (so $t > 1$), (2.2) does not exclude $C_{\mathbf{g}}(N_\iota) \neq 0$, although eventually we shall see that this does not happen. Besides the connected centralizer of N, the lemma below gives another closed subgroup which is trivial.

LEMMA 2.3. *The subgroup $\Big(C_G\big(C_G(N_\iota)^{(\infty)}\big)^{(\infty)}\Big)^\circ$ is trivial, for all i.*

PROOF. Take distinct $i,j \in \{1, \ldots, t\}$. Clearly, $\prod_{k \neq \iota} N_k \le C(N_\iota)^{(\infty)}$, so $1 < N_\iota \le L_\iota := C\big(C(N_\iota)^{(\infty)}\big)^{(\infty)} \le C(\prod_{k\neq\iota} N_k)^{(\infty)}$, which is proper in G since $t > 1$. Similarly, $L_j \le C(\prod_{k\neq j} N_k)^{(\infty)} \le C(N_\iota)^{(\infty)}$, whence $L_\iota \le C(L_j)$. Thus, $\prod_k L_k$ is a proper algebraic subgroup of G normalized by L, so must be finite. Hence $L_k^\circ = 1$ for each $k \in \{1, \ldots, t\}$. ∎

COROLLARY 2.4. $\Big(C_G\big(C_G(S)^{(\infty)}\big)^{(\infty)}\Big)^\circ = 1$ *for any subgroup S of N_ι, for each i.*

PROOF. Immediate from $C_G\big(C_G(S)^{(\infty)}\big)^{(\infty)} \le C_G\big(C_G(N_\iota)^{(\infty)}\big)^{(\infty)}$ and the above lemma. ∎

LEMMA 2.5. *If, for each i, the group $C_G(N_\iota)$ has a solvable component group, the subgroup $C_G(C_G(N_\iota)^\circ)$ is finite.*

PROOF. As in the previous lemma, it can be shown that the subgroups $C_G\big(C_G(N_\iota)^\circ\big)^\circ$ of G commute and that their product is normalized by L. ∎

LEMMA 2.6. *Let S be a finite simple subgroup of G. Then the component group of $C_G(S)$ is solvable or we are in an exceptional group and $C_G(S)$ is finite and nonsolvable.*

PROOF. Without loss, we may alter G by a convenient central extension or quotient.

If G is of non-exceptional type, consider the standard representation on a complex vector space V, and the decomposition

$$V = \sum_{\iota \in I} V_\iota$$

of V into isotypical components V_ι ($i \in I$). If G has type A_n then $C_G(S)$ is an algebraic group between a direct product of groups $\mathrm{GL}(V_\iota)$ and its commutator subgroup, whence the result. Suppose, next, that G is the commutator subgroup of the group stabilizing a nondegenerate alternating or symmetric bilinear form f. For $i \in I$, denote by i' the index in I for which V_ι is contragredient to $V_{\iota'}$, and set $J = \{i \in I \mid |\{i, i'\}| = 2\}$. Then $C_G(S)$ is a subgroup containing the commutator subgroup of a direct product of the groups $\mathrm{GL}(V_\iota)$ (one for each pair $\{i, i'\} \in J$) and classical groups associated to the forms obtained by restricting f to the spaces V_ι ($i \in I - J$), whence the result.

From now on, assume G is of exceptional type. Let S be a counterexample. Define $C := C_G(S)$. Then $R := C^{(\infty)} > C^\circ \cap C^{(\infty)}$ and C is infinite. Note that C is reductive (the centralizer of the reductive subgroup S) and that R is an algebraic group (equal to $C^{(k)}$, for sufficiently large k) and satisfies $C^{\circ\prime} \leq R$. Consequently, $R \cap C^\circ = Z_k C^{\circ\prime}$, for k sufficiently large, where $Z_k := R^{(k)} \cap Z(C^\circ)$. Since Z_k is an algebraic subgroup of a torus, it is reductive. Therefore, $R \cap C^\circ$ is reductive, whence so is R. Observe that if the reductive group $C^\circ_C(R)$ is not 1, it contains nontrivial semisimple elements outside $Z(G)$. We consider cases to obtain a contradiction.

CASE 1. $C_{C^\circ}(R)$ has a semisimple element $t \in C_{C^\circ}(R), t \notin Z(G)$. Then, $C_G(t)$ has solvable component group and has dimension less than that of G, so we finish by induction on the dimension upon passing to a quasisimple component Y of $C_G(t)^\circ$ such that $C_Y(S)/C_Y(S)^\circ$ is nonsolvable.

CASE 2. $C_{C^\circ}(R) = Z(G), C^\circ$ has quasisimple components and R has a nontrivial orbit on the set of quasisimple components. The components in this orbit must consist of groups H_ι of type A_1 for $i \in J$, an index set of cardinality $n \geq 5$. Thus, G has type E_6, E_7 or E_8. Embed G in a group X of type E_8, altering G by a central extension or quotient if necessary. Since the 2-rank of G is at most 9 (by [Adams 1986], [CoSe 1987], [Gr 1991]), each H_ι is isomorphic to $\mathrm{SL}(2, \mathbb{C})$. Let $H := \langle H_\iota \mid i \in J \rangle$ and let z_ι be the central involution of H_ι. Since R is perfect and $n \leq 8$, the action on the set of z_ι is primitive. So, either the z_ι are pairwise distinct or all equal.

We claim that the H_ι are fundamental $\mathrm{SL}(2, \mathbb{C})s$ in X.

CASE 2a. z_ι has type $2A$. If $Z(H)$ contains a four group, V, of type AAA, H lies in $C(V) \cong T_2E_6.2$, a contradiction to $V \leq H^{(\infty)}$. If $Z(H)$ contains a four group, V, of type AAB, H lies in a natural subgroup of type $A_1A_1D_6$. Without loss, we assume that there is no four group of type AAA in $Z(H)$. Embed a maximal torus of H in T, a maximal torus of X. With respect to the natural quadratic form on $\{x \in T \mid x^2 = 1\}$, $Z(H)$ is singular with respect to the bilinear form, but not the quadratic form, so has rank at most 4 and $Z(H) \cap 2A$ is the nontrivial coset of a codimension one subspace. On the other hand, it supports a group of automorphisms which is transitive on the n distinct z_ι, so has rank at least 4, whence exactly 4. Therefore, from [CoGr], (3.8), we get that $C\big(Z(H)\big) \cong 2^4 A_1^8$. Since $|J| \geq 5$, at least one, hence all, of the H_ι are fundamental $\mathrm{SL}(2, \mathbb{C})$s. Finally, we suppose that the z_ι are equal and seek a contradiction. Then, $H \leq C(z_\iota) \cong 2A_1E_7$. If H contains the A_1 factor, the factor must be normal in H and so must be one of the H_ι, as required. So, we may assume that H does not contain the A_1 factor and so its image in the simple E_7 quotient is a direct product of n $\mathrm{PSL}(2, \mathbb{C})$s. This implies that the 2-rank of adjoint E_7 is at least 10, in contradiction to [Gr 1991], (9.8.ii).

CASE 2b. z_ι has type $2B$. If all z_ι are distinct, then [CoGr 1987], (3.7) implies that H is in a group of type A_7 or D_4^2. If A_7, we get a contradiction by rank considerations. So, we may assume that $Z(H)$ contains no four group of type ABB. Thus, in any maximal torus T containing $Z(H)$, $Z(H)$ is a maximal isotropic subspace of $\{x \in T \mid x^2 = 1\}$ under the natural quadratic form. If D_4^2, we argue as in Case 2a to get H in a natural $2^4A_1^8$ and then verify the claim. Now assume that the z_ι are all equal. We obtain a contradiction in this last case. Reindex to arrange $J = \{1, \ldots, n\}$. Let $P \cong \mathrm{Alt}_4$ be diagonally embedded in H_1H_2. If the involutions of $O_2(P)$ are of type $2B$, then $C_X\big(O_2(P)\big) \cong 2^2D_4^2 : 2$ and $H_3 \cdots H_nS$ is embedded in a product of at most two groups of type G_2 or A_2 (see [Tits 1959] or look ahead to (3.2)). Since these two groups have Lie rank at most two, at most two H_ι project to a given factor and so, as $n \geq 5$, there is a pair i,j such that $H_\iota \cap H_J = 1$, a contradiction. If these involutions are of type $2A$, $H_3 \cdots H_nS$ is in Y, a natural E_6-subgroup. Since $H_3 \cdots H_nS$ contains $2^{1+2(n-2)} \times 2^2$, which has 2-rank $n - 1 + 2 \geq 6$, it follows from [Gr 1991] that if E is a subgroup of $H_3 \cdots H_nS$ of rank at least 6, it is toral of rank 6 and is maximal elementary abelian in Y and that $C_X(E)^{\circ\prime}$ is a natural $3A_2$-subgroup. Since H_1H_2 is not embeddable in $\mathrm{SL}(3, \mathbb{C})$, we have a contradiction.

We now have that the H_ι are fundamental $\mathrm{SL}(2, \mathbb{C})$s. From [CoGr, 1987], (3.7), we know that the centralizer in X of two distinct such H_ι has shape $2^2D_6.2$ and so the structure of D_6 implies that the connected centralizer of five such is a product of three fundamental $\mathrm{SL}(2, \mathbb{C})$s (and lies in the subgroup $2^4A_1^8$ of [CoGr 1987], (3.8.i)). Since S is simple and $C(H)'$ contains the finite simple group S and is a *direct* product of at most three fundamental $\mathrm{SL}(2, \mathbb{C})$s, we have a contradiction to the classification of finite subgroups of $\mathrm{SL}(2, \mathbb{C})$.

CASE 3. $C_{C^\circ}(R) = Z(G)$, C° has quasisimple components and R has only trivial orbits on the set of quasisimple components. Thus, $R = C^{\circ\prime} \circ C_R(C^{\circ\prime})$, a central product. We get a contradiction by replacing G with a quasisimple component of $C_G(t)^\circ$, for some $t \in C^{\circ\prime} - Z(G)$ and using induction on the dimension; see the last remark in Case 1.

CASE 4. $C_{C^\circ}(R) = Z(G)$, C° has no quasisimple components, so is a torus. Set $T := C^\circ$. Since C is infinite, $d := \dim(T) > 0$. By Case 1, $C_T(R) = Z(G)$. Let $D := C_G(T)$; if we embed T in a maximal torus T_0 and let Π be a root system, then D is generated by $N_G(T_0) \cap C_G(T)$ and those root groups centralizing T. Thus, D is connected and D' contains S, whence $\mathrm{rank}(D') \geq 1$. Also, the action of R on T corresponds to a subgroup of the Weyl group of G acting trivially on the subsystem Π' of roots associated to D'. Thus, R acts on T_0 as a subgroup of the Weyl group associated to Π'', the set of roots in Π perpendicular to those roots in Π'. Since R acts on T as a nontrivial perfect group, Π'' must have a connected component which contains an A_4 subsystem and $R/C_R(T)$ contains an element h of order 5. It follows that D' is generated by root groups in a natural simply connected subgroup $H = C_G([T_0, h])$ of type A_m, for some $m > 0, m \leq 4$.

In particular, D' is a nonempty direct product of at most two $\mathrm{SL}(n, \mathbb{C})$s and $Z(D') \cap Z(G) = 1$. Thus, R centralizes the nontrivial finite group $Z(D')$, which is in T but not in $Z(G)$, a contradiction. ∎

We owe part (i) of the following simple but powerful lemma to [LiSe 1989].

LEMMA 2.7. *Denote by n the product of all primes dividing the coefficients of the highest root when expressed as a linear combinations as fundamental roots. Thus $n = 30$ if $G = E_8(\mathbb{C})$ and $n = 6$ if $G = E_7(\mathbb{C})$, $E_6(\mathbb{C})$, $F_4(\mathbb{C})$ or $G_2(\mathbb{C})$. If G has type A_n, $n = 1$ and otherwise $n = 2$.*

(i) If $x \in G$ is an element of finite order not equal to a coefficient of the highest root (in particular, if the order is prime to n), then the connected center $Z\big(C_G(x)\big)^{\circ}$ of $C_G(x)$ is nontrivial.

(ii) If X is a subgroup of G such that $C_G(X)^{\circ} = 1$, then each element $x \in C_G(X)$ satisfies $Z\big(C_G(x)\big)^{\circ} = 1$. In particular, $|x|$ divides 60.

(iii) For $E_8(\mathbb{C})$, the classes of finite order elements x such that $Z\big(C_G(x)\big)^{\circ} = 1$ are the following (below which are the component types of the centralizer):

$1A$	$2A$	$2B$	$3A$	$3B$	$4A$	$4C$	$5C$	$6F$
E_8	A_1E_7	D_8	A_8	A_2E_6	A_7A_1	A_3D_5	A_4A_4	$A_5A_2A_1$.

PROOF. (i) Let $l = \mathrm{rank}(G)$ and let $(a_0, \ldots, a_l)$ be the labels of the extended Dynkin diagram ($a_0 = 1$ and the other a_i are coefficients of the highest root; see [Kac 1985], Chapter 4, Table Aff 1 for this and Chapter 8 for what follows). Elements of order m in $\mathrm{Inn}(G)$, up to conjugacy in $\mathrm{Aut}(G)$, are given, modulo diagram automorphisms, by assignments $(m_0, \ldots, m_l)$ of nonnegative integers to the nodes which generate the unit ideal of $\mathbb{Z}$ and satisfy $m = \sum_i a_i m_i$. Furthermore, the semisimple part of the centralizer of such an automorphism has as Dynkin diagram that subdiagram of the extended diagram which is supported at the set of those $i \in \{0, \ldots, l\}$ where $m_i = 0$. If x is an element of order m such that $Z(C_G(x))^{\circ}$ is trivial, this index set must have cardinality l, and if i is the unique index where m_i is nonzero, then (by the unit ideal condition) $m_i = 1$. Thus, $m = a_i$.

(ii) For $x \in C_G(X)$, we have $Z\big(C_G(x)\big) \leq C_G\big(C_G(x)\big) \leq C_G(X)$ whence $Z\big(C_G(x)\big)^{\circ} \leq C_G(X)^{\circ} = 1$.

(iii) Use (ii), [CoGr 1987] and the coefficients of the highest roots [Bour 1968]. ∎

COROLLARY 2.8. *$G = E_8(\mathbb{C})$, and, for all i, the order of N_ι has no prime divisors greater than 5 and the centralizer of every element of N_ι has trivial connected center. Furthermore, a Sylow 5-group of N_ι has order 5 and there exists an involution of N_ι inverting it under conjugation.*

PROOF. We first claim that every element of N_ι has trivial connected centralizer. By Lemmas 2.5 and 2.6, $X := C_G(N_\iota)^\circ$ is trivial or has a finite centralizer. Suppose that $C_G(X)$ is finite. Then Lemma 2.7(ii) applies yielding that $Z\big(C_G(x)\big)$ is finite for each $x \in N_\iota$. According to Lemma 2.7(i), this implies that the order of N_ι is as stated. Now assume that $X = 1$ and that the claim is false. There is an element $x \in N_\iota$ such that $Z := Z\big(C_G(x)\big)$ is nontrivial. Thus, for every index $j \neq i$, N_j centralizes Z and so $C_G(\langle N_j \mid j \neq i\rangle)$ is a positive dimensional closed subgroup. It is normalized by L (since $X = 1$) and we have a contradiction to Lie primitivity of L. The claim implies that $G = E_8(\mathbb{C})$ since the order of a nonabelian simple group requires at least three primes.

A Sylow 5-group has exponent 5, so it suffices to show that it does not contain a subgroup of the form 5×5. Suppose that A is such a group of order 25. Since A is a 2-generator finite abelian group, it is toral, so its centralizer has dimension at least 8. Orthogonality relations and the fact that traces of elements of order 5 here are all -2 (by (2.7.iii)) lead to a connected centralizer of dimension 8 exactly which therefore must be a torus, say T. Inspection of the centralizer of such an element of order 5 (shape $5A_4A_4$) shows that $C_G(A) \cong T : 5$, a solvable group. This is a contradiction since, for $j \neq i$, $N_j \leq C_G(A)$. (At this point, one could quote [Brauer 1968], which classifies finite simple groups of order 2^a3^b5 ($a, b \in \mathbb{N}$). The argument we choose in this article is more elementary.)

Burnside's famous normal p-complement theorem implies that, if P is a Sylow 5-group of N_ι, there is $x \in N_{N_\iota}(P)$ which acts nontrivially on P. Since Aut(P) is cyclic of order 4 and $P = C_{N_\iota}(P)$, we may take x to be an involution. ■

LEMMA 2.9. *Suppose that $x_1, \ldots, x_n$ are involutions from a torus of $G = E_8(\mathbb{C})$ and that each x_ι is in 2A. Assume further, for each i, that S_ι is a fundamental* SL(2, $\mathbb{C}$)-*subgroup containing x_ι in its center (it is just the* SL(2, $\mathbb{C}$)-*factor in $C_G(x_\iota)$) and that, for each pair of indices $i \neq j$, $[S_\iota, S_j] = 1$. If the product $x_1 \cdots x_n$ is an involution, it is in 2B iff n is even.*

PROOF. Use the interpretation of involutions in the torus T as isotropic or anisotropic vectors in the vector space $\{x \in T \mid x^2 = 1\}$, according to whether they are in class 2B or 2A. Under the natural bilinear form, two anisotropic vectors are orthogonal iff $[S_\iota, S_j] = 1$. Our hypotheses imply that the x_ι generate a subspace of $\{x \in T \mid x^2 = 1\}$ which is totally singular with respect to the bilinear form. The products of evenly many x_ι form a subgroup of index 2 consisting of the identity and the singular vectors. ■

COROLLARY 2.10. *For each i, N_ι contains no element of order 6 and, for some i, N_ι contains a subgroup isomorphic to* Alt_4.

PROOF. Suppose that N_ι contains an element x of order 6. Then, x, x^2 and x^3 are in 6F, 3B and 2A, respectively. Let $j \neq i$ and let $D := \langle h, u\rangle$ be a subgroup of N_j which is dihedral of order 10, with $|h| = 5$ and $|u| = 2$; by (2.8), it is available. The centralizer of h has shape $5A_4A_4$ and u induces on each factor an outer automorphism whose fixed points form a copy of $SO(5, \mathbb{C})$. Let F_1 and F_2 be the two factors of type $5A_4$. For each index $l \neq j$. Each F_k meets N_l trivially, or else simplicity of N_l implies that $N_l \leq F_k$ and that a subgroup of order 5 in N_l meets $F_{k'}$ ($\{k, k'\} = \{1, 2\}$) trivially, against (2.7.iii). Thus, each N_l injects into each $F_k/Z(F_k)$ under the natural maps. By considering the natural 5-dimensional module for F_k, which contains $\langle N_l \mid l \neq j \rangle$, we conclude that $t = 2$. Suppose that N_i is normal in L. Since $C_G(x) \cong 6A_1A_2A_5$, (2.2) implies that N_j projects nontrivially to each factor, whence the classficiation of finite subgroups of $SL(2, \mathbb{C})$ implies that $N_j \cong$ Alt_5. But then, its image in the $6A_5$-factor is a reducible subgroup of the group $6A_5$ in its action on a 6-dimensional irreducible module and so $C(N_i)^\circ \neq 1$, against (2.2). We conclude that N_1 and N_2 are conjugate in L and so both contain elements of order 6. Thus, N_j centralizes Y, the A_1-factor in $C_G(x)$. Letting $D \leq N_j, D \cong Dih_{10}$ as above, we get that $C_G(D) \cong SO(5, \mathbb{C})^2$ and that, under one of the projections, the central involution z of Y maps to 1 or an involution conjugate to $\mathrm{diag}(-1, -1, -1, -1, 1)$ in $C_{F_i}(t) \cong SO(5, \mathbb{C})$ due to the invariant *symmetric* bilinear form. Thus, z is a product of evenly many $2A$ involutions as in (2.9) (the fundamental $SL(2, \mathbb{C})s$ come from $C(D') \cong 5A_4^2$) and so is in $2B$; however, the structure of $C_G(x)$ implies that it is in $2A$ since Y is a fundamental $SL(2, \mathbb{C})$. This contradiction proves that no N_i has an element of order 6.

We now prove that one of the N_i contains a copy of Alt_4. Since N_i is simple, it has no normal 2-complement, so by an old theorem of Frobenius, [Gor 1968] (7.4.5), there is a nonidentity 2-subgroup, Q, and an element u of odd order which normalizes but does not centralize Q. The possibilities here are $|u| = 3$ or 5. If 3, we are done, since $\langle u, t \rangle \cong \mathrm{Alt}_4$ for any involution $t \in Q$. So, we may assume that 3 does not occur this way for any i. The fact that N_i has no elements of order 10 means that u is fixed point free on Q. We may assume that Q is elementary abelian of order 16. Then, in the notation of the previous paragraph, every involution of Q is a product of involutions from the two factors F_i.

CASE 1. For each involution of Q, both components from the F_i are conjugate to either $\mathrm{diag}(-1, -1, 1, 1, 1)$ or $\mathrm{diag}(-1, -1, -1, -1, 1)$. In either case, every involution of Q is the product of central involutions from n pairwise commuting fundamental $SL(2, \mathbb{C})s$, where n is even and positive. Thus, involutions of Q are in $2B$, by (2.9). It follows from (3.8.ii) of [CoGr] that $C_G(Q)^\circ$ is a maximal torus and $C_G(Q)$ has component group 2^{1+6}. Since $C_G(Q)$ is solvable but contains N_j, for $j \neq i$, we have our contradiction.

CASE 2. Case 1 does not hold for either value of i. In either case, we may assume that the image of the natural map of Q to the F_i lies in the diagonal group, whose involutions are in $2A$ iff they are conjugate to $\mathrm{diag}(-1, -1, 1, 1, 1)$; see (2.9). Since $\langle u \rangle$ has three orbits on $Q^\#$, we deduce from knowing the three orbits of a 5-cycle permutation matrix on the diagonal group and from our being in Case 2 that exactly one orbit of $\langle u \rangle$ on Q consists of elements of $2B$. An inner product calculation with (2.7.iii) gives that $\dim C_G(\langle Q, u \rangle) =$ 4. Thus, $C_G(\langle Q, u \rangle)$ is of type T_1^4 or A_1T_1. This forces N_j to be Alt_5, which contains an Alt_4 subgroup, and so we are done. ∎

3 **The proof.** Recall that L is a finite Lie primitive subgroup of G with socle $N = N_1 \times \quad \times N_t$, a direct product of t nonabelian simple subgroups In this section, we shall assume $t \geq 2$ From this, we derive that $N \cong \mathrm{Alt}_5 \times \mathrm{Alt}_6$, and describe $\chi|_N$ According to (2 10), $G = E_8(\mathbb{C})$ and there is an index, k, such that N_k contains a subgroup isomorphic to Alt_4

LEMMA 3 1 *Let E be a four group in G all of whose involutions are conjugate Set $Y = C_G(E)^{(\infty)}$ Then E is conjugate to a subgroup of T, Y is connected, and one of the following holds*

(i) *All involutions in Y are of type 2B, Y is of type D_4D_4 and $E \leq Z(Y)$*

(ii) *All involutions in Y are of type 2A, Y is of type E_6 and $E \cap Y = 1$ Moreover, $C_G(Y)^{(\infty)}$ is a Lie subgroup of type A_2*

PROOF See [CoGr 1987], (3 8) and (3 9) The statement about the centralizer of Y in (ii) follows from the fact that Y contains a conjugate of T ■

LEMMA 3 2 *Let S be a subgroup of G isomorphic to* Alt_4 *all of whose involutions have type* 2B *Then $C_G(S)^{(\infty)}$ has type A_2A_2, A_2G_2, or G_2G_2 according as the trace of an order 3 element of S on* $\mathbf{g}$ *equals* -4, 5, *or* 14 *Moreover, $C_G\big(C_G(S)^{(\infty)}\big)^{(\infty)}$ is finite only in the first two cases, while in the last case, the centralizer is a subgroup of type A_1*

PROOF Let E be the four group in S By Lemma 2 4, $C = C_G(E)^{(\infty)}$ is of type D_4D_4 It acts on $\mathbf{g}$ with character

$$(**) \qquad 8_*^{2-} \otimes 1_a + 1_a \otimes 8_*^{2} \ + 8_* \otimes 8_* + 8_* \otimes 8_* + 8_* \otimes 8_*$$

Choose an element $y \in S$ of order three It induces an outer automorphism on C, which, by [CoGr 1987] is nontrivial on both factors D_4 By classical results on triality (*cf* [Tits 1959]), the centralizer subgroup in each factor must then be of type A_2 or G_2, the centralizer of type A_2 acting irreducibly on each irreducible 8-dimensional module for D_4 Thus, $Y = C_G(S)$ is a closed subgroup of C of type A_2A_2, A_2G_2, or G_2G_2, as claimed Moreover, the dimension of this subgroup is 16, 22, 28 in the respective cases and must equal

$$(1_a, \chi|_S) = \frac{1}{12}\big(248 + 3 \quad (-8) + 8 \quad \chi(y)\big)$$

Hence y has trace -4, 5, 14 in the respective cases

On any 8-dimensional module for D_4, the triality subgroups of type A_2 and G_2 have restrictions 8_a and $1_a + 7_a$, respectively On the Lie algebra for D_4, they have restrictions $8_a^{2-} = 8_a + 10_a + 10_b$ and $(1_a + 7_a)^{2-} = 2 \quad 7_a + 14_a$, respectively Straightforward character computations show that the trivial character occurs in $\chi|_Y$ only if Y has type G_2G_2 (coming from the triple $1_a \otimes 1_a$ part) Conversely, the centralizer subgroups of type G_2 have centralizer of type F_4 Thus, if Y has type G_2G_2, the centralizer F of one factor is isomorphic to $F_4(\mathbb{C})$ and contains the other factor, whence $C_G(Y) \geq C_F(Y)$, a subgroup of type A_1 ■

LEMMA 3.3. *We have $t = 2$. Let $\{i,j\} = \{1,2\}$ and let E be a four subgroup of N_i all of whose involutions are G-conjugate. Then E is of the kind described in (i) of Lemma 3.1. Suppose furthermore that S is a subgroup of N_i isomorphic to* Alt_4. *Then N_j projects nontrivially into both factors of $Y = C_G(S)^{(\infty)}$ as in the previous lemma. In particular, N_j embeds in* $\mathrm{PSL}(3,\mathbb{C})$ *and so is isomorphic to one of* Alt_5, Alt_6.

PROOF. By definition of k, such an E is available in N_k, at least. Let $j \neq i$. If (ii) of Lemma 3.1 holds for some $E \leq N_i$, then, the subgroup $C_G\big(C_G(E)^{(\infty)}\big)^{(\infty)}$ is a group of type A_2 contradicting Corollary 2.4 above. Hence E is as described in (i) of Lemma 3.1. Again by Corollary 2.4, $C_G\big(C_G(S)^{(\infty)}\big)^{(\infty)}$ must be finite. By Lemma 3.2 this implies that Y has a factor of type A_2. The group N_j must project nontrivially on each factor of Y, for otherwise N_j lies in an $\mathrm{PSL}(3,\mathbb{C})$-subgroup of a D_4 factor which is irreducible on a natural 8-dimensional representation. The D_4-factor is isomorphic to $\mathrm{Spin}(8,\mathbb{C})$; its involutions form two conjugacy classes, one central (in the G-class 2B) and one noncentral (in the G-class 2A); it follows that the involutions of such N_j are of type 2A. In particular, N_j would have a four group as described by (ii) of Lemma 3.1, contradicting the first assertion of this lemma. Hence N_j embeds in both factors. Since at least one of them is of type A_2, the centralizer of N_1N_2 has trivial projection on at least one factor. Therefore, $t \leq 2$. Since Alt_5 and Alt_6 are the only simple $\{2,3,5\}$-subgroups of $\mathrm{PSL}(3,\mathbb{C})$ [Blich 1917], we need only reverse the roles of i and j to establish the lemma. ∎

LEMMA 3.4 (ELEMENTS OF ORDER 3 IN TRIALITY SUBGROUPS OF D_4D_4). *Let Y_1, Y_2 be triality subgroups of D_4 (of type A_2 or G_2) such that $C_G(S)^{(\infty)} = Y_1Y_2$, and suppose $y_i \in Y_i$ is an element of order 3 ($i = 1,2$) in Y_i lifting to an element of order 3 in the covering group of Y_i.*

(i) If Y_i has type A_2, y_i has trace -1 on an 8-dimensional module for D_4.

(ii) If Y_i has type G_2, y_i has trace -1 or 2 on an 8-dimensional module for D_4.

(iii) The product $y = y_1y_2$ satisfies $\chi(y) = 5$ if both y_i have trace -1 on the 8-dimensional D_4-modules and $\chi(y) = -4$ if one has trace -1 and the other trace 2 on the 8-dimensional D_4-modules.

PROOF. In Case (i), y_1 has trace 0 on the standard module 3_a for A_2 (as it has order 3 in the covering group) whence trace -1 on the adjoint module for A_2. In Case (ii) there are only two possibilities for y_1 up to conjugacy in $G_2(\mathbb{C})$, leading to trace -2 or 1 on the standard module 7_a for G_2 and hence trace -1 or 2 on a natural module 8_* for D_4. The lemma follows from use of these observations, the decomposition (**) of the adjoint module in the proof of (3.2). ∎

LEMMA 3.5. *If $N_1 \cong \mathrm{Alt}_6$, then $N_2 \cong \mathrm{Alt}_5$ and, up to automorphisms of $N = N_1N_2$,*

$$\chi|_N = 3_a \otimes 5_a + 3_b \otimes 5_b + 4_a \otimes (8_a + 8_b) + (3_a + 3_b) \otimes 9_a + 2 \cdot (5_a \otimes 10_a).$$

PROOF. First suppose $N_2 \cong \mathrm{Alt}_6$. Consider the group D of type D_4D_4 centralizing a subgroup of N_1 isomorphic to 2×2. Let $N_2 \leq X_1X_2$, where X_i is in the i-th factor of

D and $X_i \cong \mathrm{Alt}_6$ or $\mathrm{SL}(2,9)$. The fixed point subgroup of a triality automorphism on the i-th factor of D contains X_i. Therefore, $X_1 \cong X_2 \cong \mathrm{Alt}_6$. Consequently, the character of N_2 on the 8-dimensional modules for D may be identified with 8_a and 8_b for Alt_6. We use this to find $\chi|_{N_2}$ in terms of character values. We set $b_5 = \frac{-1+\sqrt{5}}{2}$ and write b_5^* for the algebraic conjugate $\frac{-1-\sqrt{5}}{2}$ so that

$$b_5 + b_5^* = -1, \quad b_5^2 = 1 - b_5, \quad b_5 b_5^* = -1.$$

Now, for elements of orders (1,2,3,3,4,5,5) the character values are:

$$8_a = (8, 0, -1, -1, 0, -b_5, -b_5^*)$$

and

$$8_b = (8, 0, -1, -1, 0, -b_5^*, -b_5)$$

Thus on the exterior square for 8_a:

$$8_a^{2-} = (28, -4, 1, 1, 0, -b_5, -b_5^*)$$

and on the tensor products

$$(64, 0, 1, 1, 0, 1 - b_5, 1 - b_5^*) \text{ in case } 8_a \otimes 8_a$$
$$(64, 0, 1, 1, 0, -1, -1) \text{ in case } 8_a \otimes 8_b$$

The full character on $\mathbf{g}$ is therefore

$$(248, -8, 5, 5, 0, 3 - 5b_5, 3 - 5b_5^*) \text{ in case } 8_a \otimes 8_a$$
$$(248, -8, 5, 5, 0, -2, -2) \text{ in case } 8_a \otimes 8_a.$$

An inner product computation shows

$$\dim C_{\mathbf{g}}(N_2) = \begin{cases} 3 & \text{in case } 8_a \otimes 8_a \\ 0 & \text{in case } 8_a \otimes 8_b \end{cases}.$$

If $\dim C_{\mathbf{g}}(N_2) > 0$, Lemma 2.2 gives that L must conjugate N_2 to N_1. But then in the case at hand, $N_1 \cong \mathrm{Alt}_6$ must act trivially on the 3-space $C_{\mathbf{g}}(N_2)$ (because there are no non-trivial 3-dimensional modules for Alt_6), whence $N_1 \times N_2$ centralizes $C_{\mathbf{g}}(N_2)$, contradicting Lemma 2.2. Consequently, the character of N_2 is $8_a \otimes 8_b$. Taking inner products with the irreducibles for Alt_6, we obtain

$$(*) \qquad \chi|_{N_2} = 3 \cdot (5_a + 5_b) + 4 \cdot (8_a + 8_b) + 6 \cdot 9_a + 10 \cdot 10_a.$$

Since Alt_6 does not have a 3-dimensional character without trivial constituents, use of $(*)$ yields $N_1 \ncong \mathrm{Alt}_6$.

Hence $N_1 \cong \mathrm{Alt}_5$. In particular, N_1 is normal in L, so by Lemma 2.3, $\chi|_{N_1}$ has no trivial constituents. According to [CoGr 1987] there is a unique character associated to fixed point free embedding of N_1 in $E_8(\mathbb{C})$; its character $\chi|_{N_1}$ is $14{\cdot}(3_a{+}3_b){+}16{\cdot}4_a{+}20{\cdot}5_a$. Apart from the character mentioned in the lemma there is only one other character compatible with both factors (*cf.* (*)):

$$\chi|_N = 3_a \otimes 5_b + 3_b \otimes 5_a + 4_a \otimes (8_a + 8_b) + (3_a + 3_b) \otimes 9_a + 2 \cdot (5_a \otimes 10_a).$$

(It helps to note that an irreducible for N_ι of degree divisible by the order of a Sylow p-group of N_ι vanishes on its p-singular elements, for $p = 3$ and 5). But this character is obtained from the one in the lemma by an automorphism of N induced by an automorphism of the abstract group Alt_6. ∎

LEMMA 3.6. *If* $N_2 \cong \mathrm{Alt}_5$, *then* $N_1 \cong \mathrm{Alt}_6$.

PROOF. If not, then by (3.3), $N_1 \cong \mathrm{Alt}_5$. We assume this and seek a contradiction.

We claim that the trace of an element of order 3 in each N_ι is 5. Let $\{i,j\} = \{1,2\}$. Take a subgroup S of N_ι, $S \cong \mathrm{Alt}_4$. Then $C := C_G(S)^{(\infty)}$ is of type A_2A_2 or A_2G_2 by Lemma 3.2. Let $y = y_1y_2$ be an element of order 3 in N_j, with y_1 in a factor of C of type A_2 and y_2 in the other factor. If C has type A_2A_2, then y has trace 5 on $\mathbf{g}$ by (2.5) and (3.2), while elements of order 3 in S have trace -4, so N_1 and N_2 are not conjugate. Moreover, each N_ι is normal in L. Since Alt_5 has a unique fixed point free character on $\mathbf{g}$, at least one N_ι has nonzero fixed points, a contradiction to (2.2). Therefore, C has type A_2G_2, and by Lemma 3.2 again, if $h \in S$ has order 3, $\chi(h) = 5$. Reversing the roles of N_ι and N_j, we get $\chi(y) = 5$ whence the claim.

From (3.4), we deduce that both y_1 and y_2 have trace -1 on a natural module for a D_4 factor. The character table for Alt_5 shows that the restriction to N_j of a character 8_* for the D_4 factor must be of the form $3_* + 5_a$. But then N_j does not embed in a G_2-subgroup of D_4, contradicting $N_2 \leq C$ and (3.3). ∎

The conclusion is that L must have a normal subgroup N as described in Lemma 3.6. This establishes the first part of Theorem 1.1.

4. **Borovik's group.** In this section we prove the second part of Theorem 1.1, *i.e.*, we supply an existence proof of the Lie primitive group with socle $\mathrm{Alt}_5 \times \mathrm{Alt}_6$ and of its uniqueness up to conjugacy. It differs from Borovik's original approach in that he begins with a particular subgroup isomorphic to $\mathrm{PSL}(2,\mathbb{C})$ from Dynkin's list of subgroups of $E_8(\mathbb{C})$ [Dynk 1957] and takes an icosahedral subgroup of it. We begin with a subgroup $S \cong \mathrm{Alt}_4$ whose involutions are in class $2B$ and such that $C_G(S) \cong A_2(\mathbb{C})wr2$; see (3.4) and [CoGr 1987]. Let h be an element of order 3 in S. Since $\dim C_G(S) = 16$, we have $\chi(h) = -4$, $C_G(h) \cong 3A_8(\mathbb{C})$. Thus, the embedding of $C_G(S)$ in $C_G(h)$ is explained by identifying the 9-dimensional standard module for $C_G(h)$ with the tensor product of a pair of 3-dimensional spaces. Consequently, an involution of $C_G(S)$ not in either A_2-factor has eigenvalues $\{-1^4, 1^5\}$ on the 9-dimensional module, hence, by (2.9), is in G-class $2B$.

Up to conjugacy, there is a unique subgroup of $\mathrm{PSL}(3,\mathbb{C})$ isomorphic to Alt_6 (it is the image in $\mathrm{PSL}(3,\mathbb{C})$ of a subgroup $3\ \mathrm{Alt}_6$ of $\mathrm{SL}(3,\mathbb{C})$ and is self-normalizing). Thus, in $C_G(S)$, there is up to conjugacy, a unique group of the form $\mathrm{Alt}_6\, wr2$ and this group contains one conjugacy class of subgroups isomorphic to Sym_6. This is the only way to get a Sym_6-subgroup of $C_G(S)$. By the preceeding paragraph, the involutions in the derived group of any such Sym_6-subgroup are in class $2B$.

We claim that if J is any Sym_5-subgroup of B, $C_{C_G(S)}(J) = 1$. We observe first that if Y is a subgroup of $C_G(S)^\circ$ such that $C_{C_G(S)^\circ}(Y) = 1$, then $C_{C_G(S)}(Y)$ has order at most 2. This remark applies to $Y = J'$. Since $N_{C_G(S)^\circ}(J') = J'$ and $N_{C_G(S)}(J')$ contains J, the claim follows.

Now, write B for a Sym_6-subgroup obtained as above. We study $C_G(B)$, which certainly contains S. The module $\mathbf{g}$ for $C_G(h)$ decomposes as $80_a + 9_a^{3-} + 9_b^{3-}$, where $80_a = 9_a \otimes 9_b - 1_a$ is the adjoint representation of $C_G(h)$. The embedding of B in $C_G(h)$ lifts to an action of B on the 9-dimensional module which, by the character table for Sym_6, is irreducible and which leaves invariant a nondegenerate symmetric bilinear form (the only other possible characters have degrees $(5,1,1,1,1)$, which would force the involutions of B' to be in class $2A$, a contradiction). Consequently, we may deduce the G-class of every element of B (straightforward with the above decomposition of $\mathbf{g}$ and the formula $\phi^{3-}(g) = [\phi(g)^3 - 3\phi(g)\phi(g^2) + 2\phi(g^3)]/6$ for the exterior cube of the character ϕ; on classes of cycle shapes $1, 2^2, 3, 3^2, 42, 5, 2, 2^3, 4, 6, 123$, the respective values under χ are $248, -8, 5, 5, 0, -2, 24, 24, 0, -3, -3$) and we may, because of the invariant bilinear form on the 9-dimensional module, arrange for an element $x \in C_G(B)$ to invert h under conjugation. Observe that $C_G(\langle h, B\rangle) = \langle h\rangle$. We get $C_G(B)$ finite either using this observation or by an inner product calculation with the traces given above. Define $U := \langle S, x\rangle$. By definition of S and x, $U' \geq S$. Note that U is finite since $U \leq C_G(B)$. We want to show that $C_G(B) = U \cong \mathrm{Alt}_5$.

Let J be a Sym_5-subgroup of B. On a 9-dimensional natural projective representation of $C_G(h)$, J has irreducibles of dimensions (4,5); also, $C_{C_G(h)}(J) \cong T_1$ and $C_{C_G(\langle h,x\rangle)}(J) \cong 2$. A straightforward inner product calculation with the above information shows that $\dim C_G(J) = 3$. Let F be a Frobenius group of order 20 in J. Since $C_G(F)$ is (by (2.7.iii)) isomorphic to $\mathrm{SO}(5,\mathbb{C})$, the reductive subgroup $C_G(J)^\circ$ cannot be a rank three torus, so has type A_1. On the standard 5-dimensional module for $C_G(F)$, $C_G(J)^\circ$ has irreducibles of degrees 5, (1,1,3) or (2,2,1) since there is an invariant symmetric bilinear form. Only in Case (2,2,1) is $C_G(J)^\circ \cong \mathrm{SL}(2,\mathbb{C})$, which contradicts an above statement that $C_{C_G(S)}(J) = 1$. Therefore, (2,2,1) does not occur and so $C_G(J) \cong \mathrm{PSL}(2,\mathbb{C}) \times E$, where E is isomorphic to a finite subgroup of $O(2,\mathbb{C})$ via its action on the 0- or 2-dimensional fixed point space. Since $C_{C_G(h)}(J) \cong T_1$, the action of h on $C_G(J)$ fixes exactly a torus and h acts fixed point freely on E, whence $E \cong 2 \times 2$ or 1. We claim that $E = 1$. Suppose not. Then, the irreducibles for $C_G(J)$ have dimensions (1,1,3) and the action of h on E preserves its subgroup acting with determinant 1 on the 2-dimensional fixed point space of $C_G(J)$. This eliminates the possibility $E \cong 2 \times 2$ and so $E = 1$. So, $C_G(J) \cong \mathrm{PSL}(2,\mathbb{C})$ (and $h \in C_G(J)$). The hypotheses on S and x and the classification of finite subgroups of $\mathrm{PSL}(2,\mathbb{C})$ imply that $U \cong \mathrm{Alt}_5$ or Sym_4. If $U \cong \mathrm{Sym}_4$ then $U' = S$, $C_G(S) \cong A_2(\mathbb{C})wr2$ and either $C_G(U) \cong \mathrm{PSL}(2,\mathbb{C})wr2$ (in case x normalizes the two A_2-factors) or $C_G(U) \cong \mathrm{PSL}(3,\mathbb{C}) \times 2$ (in case x interchanges the two factors) and so $C_G(S)$ has no Sym_6-subgroup, a contradiction. Therefore, $U \cong \mathrm{Alt}_5$. Since $C_G(B)$ is a finite subgroup of $C_G(U)$ containing U, we conclude that $C_G(B) = U$.

To get the full normalizer of the finite semisimple group $N := U \times B$, we just recall the above remarks about $C_G(S)$ and $S \times B$ and use the fact that $N_G(S)$ has the shape $C_G(S)\langle S, r\rangle$, where r is an involution normalizing $C_G(S)$. We have $\langle S, r\rangle \cong \mathrm{Sym}_4$. A Frattini argument shows that r may be arranged to normalize B. Since the outer automorphism group of B has order 2 and $C_G(B) = U$, we have $N_G(N) = \langle r, U, B\rangle$ and $N_G(N)/U \cong \mathrm{Aut}(\mathrm{Alt}_6) \cong \mathrm{Alt}_6.2^2$. It follows from $\langle S, r\rangle \cong \mathrm{Sym}_4$ that $\langle U, r\rangle \cong \mathrm{Sym}_5$. We may choose r to be an involution which satisfies $C_B(r) \cong 5:4$. Since this is a subgroup of $C_G(S)$, it follows that r induces a graph automorphism on each A_2-factor of $C_G(S)$ (see remarks about the action of x in the previous paragraph).

We now verify Lie primitivity of N, which implies Lie primitivity for every subgroup between it and its normalizer. Suppose H is a closed Lie subgroup of G of positive dimension containing L. Then, we may assume that H is reductive and that N is Lie primitive in H. We prove $H = G$. If H° has a nontrivial central torus, N must act nontrivially on the connected center of H hence also on its Lie algebra, which has dimension at most 8. On the other hand, the character of Lemma 3.5 shows that the minimal dimension of a nonzero N-submodule of $\mathbf{g}$ is 15, a contradiction. Hence, H° is semisimple.

We argue that N must be in H°. For otherwise, on the set of components there is a nontrivial orbit $\{H_i \mid i \in I\}$, $5 \le |I| \le 8$. Every such H_i must have rank just 1 and, since the 2-rank of $E_8(\mathbb{C})$ is 9 (*cf.* [Adams 1986], [CoSe 1987] or [Gr 1991]), each must be an $\mathrm{SL}(2,\mathbb{C})$. Since the minimal degree of a faithful permutation of N is 11, one of the factors, say N_j, operates faithfully as inner automorphisms on $H^* := \langle H_k \mid k \in I\rangle$, whence $N_j \cong \mathrm{Alt}_5$ and so, if $\{i,j\} = \{1,2\}$, $N_i \cong \mathrm{Alt}_6$ and $|I| = 6$. Since the actions of N_i and N_j on H^* commute, N_i centralizes a diagonal subgroup of H^* isomorphic to $\mathrm{SL}(2,\mathbb{C})$ or $\mathrm{PSL}(2,\mathbb{C})$, contradicting fixed point freeness of N_i. Therefore, $N \le H^\circ$.

We now have that N projects faithfully into each quasisimple factor of H, by fixed point freeness. By Lie primitivity of N in H, these projections are Lie primitive in the respective factors, which, by (2.8) are all $E_8(\mathbb{C})$. Therefore, $H = G$ and we are done.

5. **Remarks on isotypical alternating subgroups.** If L is a subgroup of G containing a normal subgroup $N_1 \cdots N_t$ whose factors are nonabelian simple subgroups which are L-conjugate, there exist a nonabelian finite simple group N_0 and group isomorphisms $\phi_i\colon N_0 \longrightarrow N_i$ such that $\phi_j\phi_i^{-1}\colon N_i \longrightarrow N_j$ coincides with the restriction to N_i of conjugation by an element of L for each $i,j \in \{1,\ldots,t\}$. In particular, if χ is a character of G, then $\chi \circ \phi_i = \chi \circ \phi_j$ for all i,j $(1 \le i,j \le t)$. We say that a subgroup M of G is *t-isotypical* if there is a subgroup M_0 of M and an isomorphism $\phi = (\phi_i)_{1\le i\le t}\colon M_0 \times M_0 \times \cdots \times M_0 \longrightarrow M$ such that $\chi \circ \phi_i = \chi \circ \phi_j$ for all i,j $(1 \le i,j, \le t)$, where χ is the adjoint character for E_8.

One might try to prove Theorem 1.1 via determination of characters of t-isotypical subgroups for $t > 1$, using feasible characters of simple subgroups [CoGr 1987] and [CoWa 1989] and Lemma 2.2.

For E_8 and $N_1 \cong \mathrm{Alt}_5$, so many 2-isotypical characters (with zero fixed points in $\mathbf{g}$) exist that this does not seem an efficient method.

The group Alt_6 has very few fixed-point-free 2-isotypical representations in $E_8(\mathbb{C})$: up to outer automorphisms and permutations of the factors, there are two:

$$1_a \otimes 8_a + 8_a \otimes 1_a + 2 \cdot 1_a \otimes 10_a + 2 \cdot 10_a \otimes 1_a + 3 \cdot 8_a \otimes 8_a$$

and

$$1_a \otimes 5_a + 5_a \otimes 1_a + 1_a \otimes 9_a + 9_a \otimes 1_a + 1_a \otimes 10_a + 10_a \otimes 1_a + 4 \cdot 5_a \otimes 5_a + 10_a \otimes 10_a.$$

In the respective cases, the fixed point space of N_1 in $\mathbf{g}$ has dimension 28 and 24. They lead to embeddings of N in D_4D_4 and A_4A_4. The character table of Alt_7 then rules out 2-isotypical representations of Alt_i for $i \geq 7$.

ACKNOWLEDGEMENTS. The first author is grateful to the University of Michigan for enabling a stay during which this paper has been written. The second author wishes to acknowledge the University of Michigan for released time for research and the National Science Foundation for financial support. We thank A. Borovik and G. M. Seitz for communicating their results to us.

REFERENCES

[Adams 1986] J F Adams, *2-tori in exceptional Lie groups,* preprint

[Aleks 1974] A V Alekseevskii, *Finite commutative Jordan subgroups of complex simple Lie groups,* Funct Anal and its Appl **8**(1974), 277–279

[Aschb 1984] M Aschbacher, *On the maximal subgroups of the finite classical groups,* Inventiones Math **76**(1984), 469–514

[Atlas 1985] J H Conway, R T Curtis, S P Norton, R P Parker and R A Wilson, *Atlas of finite groups* Clarendon Press, Oxford, 1985

[Blich 1917] H F Blichfeldt, *Finite Collineation Groups,* University of Chicago Press, 1917

[Borov 1989] A Borovik, *The structure of finite subgroups of simple algebraic groups,* Algebra and Logic **28**(1989), 249–279

[Bourb 1968] N Bourbaki, *Groupes et algebres de Lie, Chap 4,5 et 6,* Hermann, Paris, 1968

[Brauer 1968] R D Brauer, *On simple subgroups of order* 5 3^a 2^b, Bull Amer Math Soc **74**(1968) 900–903

[CLSS 1992] A M Cohen, M W Liebeck, J Saxl and G M Seitz, *The local maximal subgroups of the exceptional groups of Lie type,* Proceedings London Math Soc (1992), to appear

[CoGr 1987] A M Cohen and R L Griess, Jr, *On finite simple subgroups of the complex Lie group of type* E_8, Proc of Symposia in Pure Math **47**(1987), 367–405

[CoGrLi 1992] A M Cohen, R L Griess, Jr and Bert Lisser, *The group* $L(2, 61)$ *embeds in the Lie group of type* E_8, Comm in Algebra (1992), to appear

[CoSe 1987] A M Cohen and G M Seitz, *The r-rank of the groups of exceptional Lie type* Indagationes Math **90**(1987), 251–259

[CoWa 1983] A M Cohen and D B Wales, *Finite subgroups of* $G_2(\mathbb{C})$, Comm Algebra **11**(1983), 441–459

[CoWa 1989] ———*Finite subgroups of* $E_6(\mathbb{C})$ *and* $F_4(\mathbb{C})$, preprint

[Dynk 1957] E B Dynkin, *Semisimple subalgebras of semisimple Lie algebras,* Amer Math Soc Transl **6**(1957), 111-244

[Glau 1966] G Glauberman, *Central elements in corefree groups,* Jour of Algebra **4**(1966), 403–420

[Gor 1968] D Gorenstein, *Finite groups,* Harper and Row, New York, 1968

[Gr 1991] R L Griess, Jr, *Elementary abelian p-subgroups of algebraic Groups,* Geometria Dedicata **39** (1991), 253–305

[GrRy 1992] R L Griess, Jr and A J E Ryba, *Embeddings of* $U_3(8)$ Sz(8) *and the Rudvalls group in algebraic groups of type* E_7, (1992), submitted

[Kac 1985] V Kac, *Infinite Dimensional Lie algebras,* Cambridge University Press, Cambridge, 1985
[KMR 1989] P Kleidman, U Meierfrankenfeld & A Ryba *Construction of* HiS *and* Ru *in* $E_7(5)$, preprint
[LiSe 1989] M W Liebeck and G M Seitz, *Maximal subgroups of exceptional groups of Lie type, finite and algebraic,* Geom Dedicata **35**(1989), 353–387
[Tits 1959] J Tits, *Sur la trialité et certains groupes qui s'en déduisent,* Publ Math I H E S **2**(1959), 14–60

Centre for Mathematics and Computer Science
Kruislaan 413,
1098 SJ Amsterdam

Robert L Griess Jr
Department of Mathematics
University of Michigan
Angell Hall
Ann Arbor, MI 48104

https://doi.org/10.4153/CJM-1993-005-7 Published online by Cambridge University Press