# 9

# Loop representation: further developments

## 9.1 Introduction

In the previous chapter we discussed the basics of the loop representation for quantum gravity. We obtained expressions for the constraints at both a formal and a regularized level and discussed generalities about the physical states of the theory. In this chapter we would like to discuss several developments that are based on the loop representation. We will first discuss the coupling of fields of various kinds: fermions using an open path formalism, Maxwell fields in a unified fashion and antisymmetric fields with the introduction of surfaces. These examples illustrate the various possibilities that matter couplings offer in terms of loops. We then present a discussion of various ideas for extracting approximate physical predictions from the loop representation of quantum gravity. We discuss the semi-classical approximation in terms of weaves and the introduction of a time variable using matter fields and the resulting perturbation theory. We end with a discussion of the loop representation of $2+1$ gravity as a toy model for several issues in the $3+1$ theory.

## 9.2 Inclusion of matter: Weyl fermions

As we did for the Yang–Mills case, we now show that the loop representation for quantum gravity naturally accommodates the inclusion of matter. In the Yang–Mills case, in order to accommodate particles with Yang–Mills charge one needed to couple the theory to four-component Dirac spinors. A Dirac spinor is composed of two two-component spinors that transform under inequivalent representations of the group. This made the addition of matter complicated and one had to resort to the staggered fermion techniques. Although one could couple Dirac fermions to gravity, in the gravitational case the simplest and most natural kind



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of matter to couple would be uncharged spinning particles. These are described by two-component Weyl fermions. From a particle point of view we will be studying the coupling of neutrinos to general relativity, which is described by the Einstein–Weyl fermion theory. The first discussion of this system in terms of the new variable formulation is due to Morales-Técotl and Rovelli [149].

In order to describe Weyl fermions we need to use two-component spinors. We recall some basic definitions of these mathematical objects, a more complete treatment is in the appendix of reference [2]. Consider a two-dimensional complex vector space. Consider a two-form in that space $\epsilon_{AB}$ and its inverse $\epsilon^{AB}$, defined by $\epsilon_{AB}\epsilon^{BC} = \delta_A{}^C$. The linear mappings $L_A{}^C$ which preserve the two-form $\epsilon_{AB}$ must have unit determinant, i.e., they are elements of $SL(2,C)$. The two-forms $\epsilon$ provide an isomorphism between the two-dimensional vector space and its dual, which we can denote by raising and lowering of indices with the following conventions (care should be exercised because of the antisymmetry of $\epsilon_{AB}$): $\eta^A = \epsilon^{AB}\eta_B$ and $\eta_B = \eta^A\epsilon_{AB}$. Since the elements of this vector space are complex, a natural notion arises of the vector space of the complex conjugate elements and its dual. A vector in the complex conjugate space is denoted by a prime in its index $\eta^{A'}$ or $\eta_{A'}$ if it is in its dual. Primed indices are raised and lowered with the matrix $\bar{\epsilon}^{A'B'}$ and $\bar{\epsilon}_{A'B'}$.

In terms of two spinors one can define a vector space $V$ of objects of the form $\beta^{AA'}$ such that $\bar{\beta}^{AA'} = -\beta^{AA'}$. It is straightforward to check that this has the structure of a four-dimensional real vector space equipped with a natural metric $\epsilon_{AB}\epsilon_{A'B'}$ of signature $(-,+,+,+)$. Consider now a four-dimensional spacetime and a fiber bundle over it with fibers isomorphic to the two-dimensional vector space introduced above. It is natural to identify the tangent space at each point of the spacetime with $V$,

$$\sigma^a_{AA'}\beta^{AA'} \equiv \beta^a \tag{9.1}$$

in such a way that the metric of $V$ is mapped to the metric of spacetime, $g^{ab} = \sigma^a_{AA'}\sigma^b_{BB'}\epsilon^{AB}\bar{\epsilon}^{A'B'}$. If $\sigma$ exists globally on the spacetime we say that it admits an $SL(2,C)$ spinor structure. Objects of the form $\eta^A$ are called unprimed spinors and those of the form $\eta^{A'}$ primed spinors; the matrices $\sigma$ are called soldering forms. The role of the soldering forms is the analogue in spinor language of the role of the tetrad fields in tetradic language. Both entities carry enough information to reconstruct the spacetime metric and they are determined by the metric up to local transformations ($SO(3,1)$ in the case of the tetrads, $SL(2,C)$ in the case of the soldering forms). Their relation can be explicitly written $\sigma^a_{AB'} = e^a_I\sigma^I_{0\,AB'}$ where the $\sigma^I_{0\,AB'}$ matrices are constant $SL(2,C)$ matrices. A basis of such matrices is given by $\sigma^I = (\mathbf{1},\tau^i)$, where $\tau^i$ are $i\sqrt{2}/2$ times the Pauli matrices [148].

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The Lagrangian for general relativity coupled to Weyl fermions in terms of self-dual first order variables was independently introduced by Jacobson [147] and Ashtekar, Romano and Tate [133]. It is given by

$$S(e, A, \bar{\psi}, \psi) = \int d^4x \left[ e\, e^a_J e^b_K F^{JK}_{ab} + \sqrt{2}\, e\, e^a_I \sigma^I_{0\,AA'} \bar{\psi}^{A'} D_a \psi^A \right], \quad (9.2)$$

where the notation is the same as in section 7.3. The fields $\psi^A$ and $\bar{\psi}^{A'}$ are Grassmann-valued (anticommuting) $SL(2, C)$ spinors. The covariant derivative on spinors is defined as $D_a\psi^A \equiv \partial_a \psi^A + A_{a}{}^{A}_{B}\psi^B$ and the self-dual connection is defined in terms of the connection defined in section 7.3.3 by $A_a{}^A{}_B \equiv A_{aIJ}\sigma_0^{I\,A}{}_{C'}\sigma_0^{J\,C'}{}_B$.

One can perform a canonical decomposition of this action along the same lines as that performed in chapter 7 for pure gravity. We will not give the details here (they are discussed in reference [2]). The main point is that the introduction of the unit normal $n^a$ introduces an isomorphism in the spinor space that casts the formalism in terms of $SU(2)$ spinors.

$SU(2)$ spinors are defined in the same fashion as $SL(2, C)$ spinors but one introduces an additional structure, a Hermitian inner product among spinors defined by $< \psi|\phi > \equiv \bar{\psi}^{A'} G_{A'A}\phi^A$ with $\bar{G}_{AA'} = G_{AA'}$. It follows that the transformations that leave invariant *both* $\epsilon_{AB}$ and $G_{AA'}$ are $SU(2)$ transformations. The metric $G$ defines an operation "†" relating the primed and unprimed spinors $(\beta^A)^\dagger \equiv -\epsilon^{AB}G_{BA'}\bar{\beta}^{A'}$. If one now considers the space $H$ of objects $\beta^A{}_B$ such that $\beta^A{}_A = 0$ and $(\beta^\dagger)^A{}_B = \beta^A{}_B$, it turns out that it has the structure of a three-dimensional vector space equipped with a positive definite metric $(\beta, \eta) = -\beta^A{}_B\eta^B{}_A$. It can then be made isomorphic to the tangent space of a curved three-manifold with metric $q^{ab} \equiv -\sigma^{aA}{}_B\sigma^{bB}{}_A$, the matrices $\sigma$ again are called soldering forms and are related to the (undensitized triads) by $\sigma^{aA}{}_B = E^a_i\tau^i$ with $\tau^i$ as defined above.

Continuing with the discussion of the canonical decomposition of the action, the introduction of the unit normal $n^a$ and its associated $SL(2, C)$ spinor $n^{AA'} = i\sqrt{2}n^a\sigma^{AA'}_a$ gives the matrix $G^{AA'}$ that implements the Hermitian inner product that introduces the $SU(2)$ spinors in the formalism. The three- and four- dimensional soldering forms are related by $\sigma_{a}{}^{A}_{B} \equiv q^c_a\sigma^{AA'}_c G_{A'B}$. The canonical variables end up being $A^i_a$ and $\psi^A$ and the corresponding canonically conjugate momenta are $\tilde{E}^a_i$ and $\tilde{\pi}_A = -ie(\psi_A)^\dagger$. The theory has the same constraints as usual general relativity (the theory is invariant under the same symmetries) but the constraints are appropriately modified to generate the corresponding transformations in the fermionic variables. The constraints are given by

$$\mathcal{G}^i = D_a\tilde{E}^{ai} - \frac{i}{\sqrt{2}}\tilde{\pi}_A\psi_B\sigma_0{}^{iAB}, \quad (9.3)$$

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$$C_a = \tilde{E}_i^b F_{ab}^i + \frac{i}{\sqrt{2}} \tilde{\pi}_A D_a \psi^A, \tag{9.4}$$

$$\mathcal{H} = \epsilon_{ijk} \tilde{E}_i^a \tilde{E}_j^b F_{ab}^k - 2\tilde{E}_i^a \tilde{\pi}_A D_a \psi_B (\sigma_0)^{iAB}. \tag{9.5}$$

The Weyl field is Grassmann-valued, so the canonical Poisson bracket between $\tilde{\pi}_A$ and $\psi^B$ has a + sign.

We now define an algebra of gauge invariant operators for the theory. Apart from the usual $T$ variables constructed with the connection $A_a^i$ one can define the following variables, based on open paths such as the ones defined in chapter 1:

$$Z(\eta_x^y) = \psi^A(x) U_A{}^B(\eta_x^y) \psi_B(y), \tag{9.6}$$

$$Y(\eta_x^y) = \tilde{\pi}^A(x) U_A{}^B(\eta_x^y) \psi_B(y). \tag{9.7}$$

These objects form a closed algebra under Poisson brackets with the $T$ variables. One could define two other variables, one by considering $\psi$ and $\tilde{\pi}$ in the reverse order in $Y$ and another with two $\tilde{\pi}$s at the ends. Although one does not need these variables to write the Hamiltonian they are needed to write other gauge invariant quantities.

The open path variables satisfy a series of identities; first of all notice that the dependence is on a path, in the sense of chapter 1, so retraced portions do not contribute. Moreover, they satisfy the relations

$$Z(\eta_y^x) = Z(\eta_x^y), \tag{9.8}$$

$$Z(\eta_x^y \circ \beta_y^z) T^0(\gamma) = Z(\eta_x^y \circ \gamma \circ \beta_y^z) + Z(\eta_x^y \circ \gamma^{-1} \circ \beta_y^z), \tag{9.9}$$

$$Z(\eta_x^{y_1}) Z(\eta_x^{y_2}) Z(\eta_x^{y_3}) = 0. \tag{9.10}$$

The first identity (retracing) stems from the fact that the spinor fields are Grassmanian and as a consequence $U(\gamma)^A{}_B = U(\gamma^{-1})_B{}^A$ and also $U(\gamma)_{AB} = U(\gamma^{-1})_{AB}$. The second identity is the Mandelstam identity for open paths. In that identity the loop $\gamma$ is connected by a tree to the point $y$ to connect with the open path. The third identity, which is also valid for three open paths ending at the same point comes from the fact that the spinor fields are Grassmanian and being two-component objects one cannot have more than two at a given point. These identities are the same as those we found in chapter 6 for Yang–Mills theories coupled to fermions, with the exception of the retracing identity, which was absent in that case.

The algebra of these quantities is

$$\{Z(\eta_x^y), Z(\gamma_z^w)\} = 0, \tag{9.11}$$

$$\{Y(\eta_x^y), Z(\gamma_z^w)\} = \delta(x-w) Z(\gamma_z^w \circ \eta_x^y) + \delta(x-z) Z(\gamma_w^z \circ \eta_x^y), \tag{9.12}$$

$$\{Y(\eta_x^y), Y(\gamma_z^w)\} = \delta(x-w) Y(\gamma_z^w \circ \eta_x^y) + \delta(y-z) Y(\eta_x^y \circ \gamma_w^z), \tag{9.13}$$

and their commutators with the $T$ variables can be seen in reference [149]

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but are similar to the commutators of the $T$ variables with $T^0$ since the fermionic parts do not contribute. The algebra of the $Z$ and $Y$ operators can be viewed, as in the Yang–Mills case, as a set of rules of fusion and splitting of paths.

The diffeomorphism and Hamiltonian constraint can be written purely in terms of the $T$ variables and the $Y$ variable but we will not present a detailed derivation here.

A quantum representation of this algebra can be obtained in terms of operators that act on a space of wavefunctions of loops and open paths. One can reduce the loop dependence, using Mandelstam identities, to a single loop and a series of open paths $\beta_{x_1}^{y_1}, \ldots, \beta_{x_n}^{y_n}$. The quantum representation is given by

$$\hat{Z}(\pi_x^y)\Psi(\beta_{x_1}^{y_1}, \ldots, \beta_{x_n}^{y_n}, \gamma) = \Psi(\pi_x^y, \beta, \gamma), \tag{9.14}$$

$$\begin{aligned} \hat{Y}(\pi_x^y)\Psi(\beta, \gamma) = i \sum_{k=1}^{n} \Big[ &\delta(x - x_k)\Psi(\ldots, \beta_{y_k}^{x_k} \circ \pi_x^y, \ldots, \gamma) \\ &+ \delta(x - y_k)\Psi(\ldots, \beta_{x_k}^{y_k} \circ \pi_x^y, \ldots, \gamma)\Big] . \end{aligned} \tag{9.15}$$

The operator $Z$ simply appends the open path it has as argument to the wavefunction. The operator $Y$ appends its open path argument at the beginning and at the end of each of the open paths on which the wavefunction depends.

The quantum constraints can be written in a straightforward fashion. We will not discuss in detail the realization of the diffeomorphism constraint. The effect is the expected geometric one: the loops and paths are deformed along the diffeomorphism flow. The additional terms in the constraint take care of moving the end points of the open paths. It is immediate to construct the solution space to that constraint in a geometric fashion, much in the same spirit as in the purely gravitational case. The solution space is composed of wavefunctions of the generalized knot classes, the sets of knots and open paths that are related by the orbits of the diffeomorphism group. The concept of knotting when open paths are involved is non-trivial due almost only to the possible presence of intersections. If no intersections (or self-intersections) are present, all open paths are equivalent under diffeomorphisms. The "almost only" accounts for the fact that configurations with non-intersecting paths can be diffeomorphism inequivalent if the number of paths is different.

We will not present in an exhaustive fashion the general action of the Hamiltonian on a wavefunction of a multipath with arbitrary intersections and self-intersections, since it resembles very closely the case of pure gravity when written in terms of multiloops [138]. The action of the operator can be found by writing it in terms of the algebra of gauge invariant operators that we introduced above or in terms of the loop transform,

$$\Psi(\beta_{x_1}^{y_1},\ldots,\beta_{x_n}^{y_n},\gamma)=\int dA\int d\psi\Psi[A,\psi]Z(\beta_{x_1}^{y_1})\times\cdots\times Z(\beta_{x_n}^{y_n})T^0(\gamma). \tag{9.16}$$

We would, however, like to illustrate the new contributions that arise due to the fermionic parts by considering the action of the Hamiltonian on a state dependent on a single, possibly self-intersecting, path. In order to compute this, we consider the action of the fermionic part of the Hamiltonian on one of the $Z$s that appear in the loop transform,

$$\begin{aligned}\hat{H}_{\text{Weyl}}(z)Z(\pi_x^y)&=-2\hat{\tilde{E}}_i^a(z)\hat{\tilde{\pi}}_A(z)D_a\hat{\psi}_B(z)(\sigma_0)^{iAB}Z(\pi_x^y)\\&=-2D_a\psi_B(z)(\sigma_0)^{iAB}\frac{\delta}{\delta A_a^i(z)}\frac{\delta}{\delta\psi_A(z)}Z(\pi_x^y).\end{aligned} \tag{9.17}$$

The result is

$$\hat{H}_{\text{Weyl}}(z)Z(\pi_x^y)=2(\delta(z-x)X^{ax}(\pi)D_a^x-\delta(z-y)X^{ay}(\pi)D_a^y)Z(\pi_x^y), \tag{9.18}$$

where $D_a^y$ is the Mandelstam covariant derivative we introduced in chapter 1, with the generalization that it acts not only at the end point of the open path but also at the beginning. From this result we can read off the action of the Weyl part of the Hamiltonian on a state dependent on a single open path,

$$\hat{H}_{\text{Weyl}}(z)\Psi(\pi_x^y,\gamma)=2(\delta(z-x)X^{ax}(\pi)D_a^x-\delta(z-y)X^{ay}(\pi)D_a^y)\Psi(\pi_x^y,\gamma). \tag{9.19}$$

The geometric meaning of the Weyl part of the Hamiltonian is to translate the ends of the open paths in the direction of the tangent vector at those points. It is remarkable that the action of the purely gravitational Hamiltonian we discussed in the previous section on non-intersecting loops has a rather similar effect, in the sense that it can be interpreted as a diffeomorphism along the loop. In this sense, if one considers the purely gravitational Hamiltonian in terms of loops and extends naturally its action to open paths one is automatically left with the Einstein–Weyl fermion theory, without the need to input details about the Weyl Hamiltonian. In this sense the loop representation of quantum gravity naturally "predicts" the Dirac equation for fermions [149].

As in the previous section, one must regularize and renormalize the operators; the techniques involved are similar so we omit a detailed discussion. Morales Técotl and Rovelli [149] study the issue in detail using regularization ideas that we will discuss in section 9.5.1 in the context of pure gravity using a matter clock.

## 9.3 Inclusion of matter: Einstein–Maxwell and unification

Once a theory is cast in terms of a connection we can build a loop representation for it. We have done so for Yang–Mills theories and also for general relativity. What happens when one couples such theories? The obvious answer is to consider a mixed loop representation with some loops associated with the connection of a certain theory and others to the other. Such an approach can be pursued for all gauge fields that are coupled in gauge invariant fashion, as are all Yang–Mills fields coupled to gravity. In essence, the resulting description is faithful to the spirit of this book in which each gauge field has been treated as quantizable in its own right. For many years, however, the trend in particle physics has been towards viewing the different gauge fields as different low energy manifestations of a single unified theory that is apparent only at high energies. The question therefore arises: are loop descriptions estranged from unification ideas or can they be made compatible to a certain extent? Such a subject is largely unexplored at present. What we would like to show in this section is that the seeds for a unified description of gauge fields in terms of loops may be present. We will illustrate the idea with the simplest possible example, that of Einstein–Maxwell theory. However, we will see that the idea goes through largely unchanged if one replaces Maxwell theory with a Yang–Mills field.

The Einstein–Maxwell theory in the canonical formulation based on Ashtekar's new variables is described in terms of the usual variables for the gravitational part plus a $U(1)$ vector potential $a_a$, its associated field tensor $f_{ab} = \partial_{[a} a_{b]}$ and the electric field $\tilde{e}^a$. The constraint equations are

$$\partial_a \tilde{e}^a = 0, \tag{9.20}$$

$$D_a \tilde{E}^{ai} = 0, \tag{9.21}$$

$$\tilde{E}^a_i F^i_{ab} + i\frac{\sqrt{2}}{4}\tilde{e}^a f_{ab} = 0, \tag{9.22}$$

plus a Hamiltonian constraint. The first equation is the $U(1)$ Gauss law of Maxwell theory, the second set is the gravitational Gauss law and the third set is the diffeomorphism constraint. Due to the fact that the Gauss laws for both gauge groups appear separately one could build, as argued above, a loop representation based on two separate sets of loops, one associated with the $U(1)$ invariance and other with the $SU(2)$ invariance. In that loop representation, each set of loops would operate independently and be subject to separate Mandelstam identities.

We now show that the above gauge symmetries can be cast in a unified fashion, suitable for the introduction of a loop representation based on a *single* kind of loop that still captures the information of the two interacting

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theories. Let us introduce a $U(2)$ connection $\mathcal{A}_a$ in the following way,

$$\mathcal{A}_a \equiv A_a^i \tau_i + i a_a \mathbf{1}, \tag{9.23}$$

where $\tau_i$ are the Pauli matrices with our usual conventions and $\mathbf{1}$ is the identity matrix in two dimensions. One can similarly introduce a $U(2)$ electric field $\tilde{\mathcal{E}}^a$ and from the $U(2)$ connection build a field tensor $\mathcal{F}_{ab}$ and a covariant derivative $\mathcal{D}_a$.

The remarkable fact is that the constraints we wrote above can now be written

$$\mathcal{D}_a \tilde{\mathcal{E}}^a = 0, \tag{9.24}$$

$$\mathrm{Tr}(\tilde{\mathcal{E}}^a \mathcal{F}_{ab}) = 0, \tag{9.25}$$

and the Hamiltonian constraint can also be written in terms of these variables, though we will not need its particular expression here. We refer the reader to reference [97] for more details. The point is that at the kinematical level, the theory looks *exactly the same as vacuum general relativity* but with an enlarged gauge group, $U(2)$ instead of $SU(2)$. This construction can also be carried out for general relativity coupled to a Yang–Mills field with gauge group $G^{YM}$, the resulting group is $SU(2) \times G^{YM}$ [150].

Therefore one can now construct a loop representation based on a single kind of loop for the $U(2)$ symmetry. In such a representation the unified Gauss law (9.24) is automatically solved. The wavefunctions are functions of multiloops subject to the $U(2)$ Mandelstam constraints,

$$\Psi(\gamma_1 \circ \gamma_2) = \Psi(\gamma_2 \circ \gamma_1), \tag{9.26}$$

$$\Psi(\gamma_1, \gamma_2, \gamma_3) = \Psi(\gamma_1 \circ \gamma_2, \gamma_3) + \Psi(\gamma_2 \circ \gamma_3, \gamma_1) + \Psi(\gamma_3 \circ \gamma_1, \gamma_2) - \Psi(\gamma_1 \circ \gamma_2 \circ \gamma_3) - \Psi(\gamma_1 \circ \gamma_3 \circ \gamma_2). \tag{9.27}$$

Two comments are in order. First notice that there is no retracing identity, $\Psi(\gamma) \neq \Psi(\gamma^{-1})$. Second, notice that the second Mandelstam identity is considerably different from that of $SU(2)$. In the $SU(2)$ case the second Mandelstam identity allowed us to express a wavefunction of $n$ loops as a combination of wavefunctions of $n-1$ loops and could be used recursively to reduce any wavefunction of a multiloop to a single-loop wavefunction. In the present case, the identity allows us to reduce a wavefunction of $n$ loops to a combination of wavefunctions of $n-1$ and $n-2$ loops. This implies in particular that one can only reduce a wavefunction of an arbitrary multiloop to a wavefunction of *two* loops.

Remarkable we therefore come to the conclusion that wavefunctions in the unified loop representation depend on two loops, exactly as if we had built two independent representations for gravity and electromagnetism. There is an important difference: in the unified case there is no distinction

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between the two loops and the Mandelstam identities for both of them are the same. We therefore see that a unified setting arises as a consequence of going to the language of loops. Similar considerations hold for the case in which the group is not $U(2)$ but $SU(2) \times G^{YM}$ though the minimum number of loops involved is higher.

There are several aspects of this unification that are interesting enough to merit investigation. We will only briefly discuss them here since the subject is largely unexplored. By inspection we can tell what the diffeomorphism constraint of the unified theory implies in the loop representation. Since it has exactly the same form as the usual diffeomorphism constraint of general relativity, we know it will require that the wavefunctions be invariant under smooth deformations of the loops. Therefore we know how to solve that constraint: we just need to consider functions of two loops that are invariant under deformations of the loops. Notice that if one had pursued a loop representation based on separate loops for both gauge invariances, the action of this constraint would be considerably less geometrical and more involved. Some of the results we introduced in the connection representation for gravity in the previous chapter apply to the unified model. For instance, if one constructs a state based on the exponential of the Chern–Simons form of the unified connection, such a state solves all the constraints of the theory with cosmological constant. As we will see in chapters 10 and 11, such a state has importance in the loop representation, being related to the Jones polynomial. The same relationship appears for the unified model. If one considers the inclusion of fermions in the unified model, one would have to proceed as in the previous section by introducing open paths. However, in the unified model, opening the loops implies introducing not only a charge at the level of the gravitational Gauss law (spin) but also one for the Maxwell Gauss law (electric charge). This means that the most natural form of matter in the unified model has an electric charge if it has spin.

Finally, what happens with the Hamiltonian? The Hamiltonian of Einstein–Maxwell can be written in terms of the unified variables but its form is slightly cumbersome (it still is polynomial) and differs from that of vacuum gravity. There is nothing to prevent us from realizing it in the loop representation and studying its solutions, though this issue is as yet unexplored. In an interesting development Chakraborty and Peldán [150] have noticed that one can write a Hamiltonian in terms of the unified variables that looks quite similar to that of vacuum gravity. The resulting theory is not Einstein–Maxwell but reduces to it in the weak field limit. The loop representation of such a model could appear quite naturally and lead to new insights of the unified theory.

## 9.4 Kalb–Ramond fields and surfaces

Kalb–Ramond fields [171] are antisymmetric second rank tensor gauge fields. They found physical application in the field theories that arise as low energy limits of string theory [151] and as models of dark cosmological matter[152]. We will discuss Abelian Kalb–Ramond fields here because they couple very simply to gravity and because their gauge symmetry makes them associated with surfaces which are a natural higher dimensional generalization of loops. They are suited to a geometric quantum formulation completely analogous to the loop representation for usual gauge symmetries, but based on surfaces instead of loops. These surfaces can later be used in quantum gravity to measure properties of the metric, as was argued by Smolin [172] and which we will see in section 9.5.1. The first analysis of antisymmetric tensor fields in terms of surfaces was discussed by Arias, Di Bartolo, Fustero, Gambini and Trias [155]. Although non-Abelian antisymmetric tensor fields have been considered [154], it has not been possible to give them a geometric formulation.

Let us start with a brief discussion of the properties of the Abelian group of surfaces and then relate it to the loop representation of an Abelian Kalb–Ramond field.

### *9.4.1 The Abelian group of surfaces*

Consider the set $S$ of closed two-dimensional oriented surfaces in $\mathbf{R}^3$. For each surface $s$ we denote by $\bar{s}$ the reverse-oriented surface. We define the following product,

$$s_1 \circ s_2 \equiv s_1 \cup s_2, \qquad s_1, s_2 \in S, \tag{9.28}$$

which is associative and commutative, but lacks an inverse element. In order to define this we introduce, in the same spirit as for loops, the notion of a tree. We define as trees all elements of $S$ such that the integral of all scalar functions on them is zero. We introduce an equivalence relation in $S$ by identifying two elements if their composition is a tree. The quotient set is an Abelian group. We denote it by $\Sigma$ and its elements by $\sigma_i$. This structure is easily generalizable to the set $\Lambda$ of surfaces with boundary. One can naturally view the group of closed surfaces without boundary as the group of deformations of surfaces with boundaries, very much as loops can be viewed as deformations of paths. We will not discuss open surfaces here, details can be seen in references [155, 172]. We will see that from these group structures we can recover all the kinematical content of a Kalb–Ramond theory, in the same sense as the group of loops contained all kinematical information of usual gauge theories. In order to unravel

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this connection we proceed as we did for gauge theories, by introducing the infinitesimal generators of the group.

Consider the following two infinitesimal elements in $S$. The first one which we call $\delta\sigma(x, \vec{u}, \vec{v}, \vec{w})$ is an infinitesimal three-dimensional parallelepiped with vertex at the point $x$ and sides along the three vectors $\vec{u}, \vec{v}, \vec{w}$. The second element which we call $\delta\sigma(\pi_o^x, \vec{u}, \vec{v})$ is defined by an infinitesimal parallelogram similar to the one we used to define the loop derivative in chapter 1 attached to a path going from a basepoint to the point $x$.

Consider now a representation of the Abelian group that associates to each element $\sigma_i$ a complex number $U(\sigma_i)$, acting on a space of functions on the group of surfaces $\Psi(\sigma_i)$,

$$U(\sigma)\Psi(\sigma') = \Psi(\sigma \circ \sigma'), \tag{9.29}$$

$$U(\sigma_1 \circ \sigma_2) = U(\sigma_1)U(\sigma_2), \tag{9.30}$$

and we fix $U(\sigma_0) = 1$ with $\sigma_0$ the identity element in $\Sigma$.

To find the infinitesimal generators of the group we assume the representation is differentiable in the sense that the following expansions exist:

$$\Psi(\delta\sigma(z, \vec{u}, \vec{v}, \vec{w}) \circ \sigma) = (1 + u^a v^b w^c \Delta(z)_{abc})\Psi(\sigma), \tag{9.31}$$

$$\Psi(\delta\sigma(\pi_o^z, \vec{u}, \vec{v}) \circ \sigma) = (1 - u^a v^b \delta(\pi_o^x)_{ab})\Psi(\sigma). \tag{9.32}$$

As in the case of loops, we now have differential operators and we would like to find relations among them. Exactly like when we proved the Bianchi identity for loop derivatives, we start from an identity in the space of surfaces,

$$\begin{aligned}\delta\sigma(x, \vec{u}, \vec{v}, \vec{w}) = {} & \delta\sigma(\pi_o^x, \vec{u}, \vec{v}) \circ \delta\sigma(\pi_o^x, \vec{u}, \vec{w}) \circ \delta\sigma(\pi_o^x, \vec{v}, \vec{w}) \\ & \circ\delta\sigma(\pi_o^{x+u}, \vec{w}, \vec{v}) \circ \delta\sigma(\pi_o^{x+v}, \vec{u}, \vec{w}) \circ \delta\sigma(\pi_o^{x+w}, \vec{v}, \vec{u}),\end{aligned} \tag{9.33}$$

simply stating that the parallelepiped can be obtained by joining together six parallelograms. Introducing the parallel derivative acting on the path dependence of functions,

$$(u^a \partial_a)\delta(\pi_o^x) = \delta(\pi_o^{x+u}) - \delta(\pi_o^x), \tag{9.34}$$

the geometric identity (9.33) implies

$$\Delta_{abc}(z) = \partial_a \delta_{bc}(\pi_o^z) + \partial_b \delta_{ca}(\pi_o^z) + \partial_c \delta_{ab}(\pi_o^z). \tag{9.35}$$

This relation is the analogue for surfaces of the identity between the connection and the loop derivatives that we proved in chapter 1. As in that case, we have an identity that relates path dependent objects with path independent ones. If one seeks a description that is path independent, one must associate with each point $x$ a prescription of a fiducial

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path $\pi_o^x$, with which we can identify $\delta(x) \to \delta(\pi_o^x)$. Modifications of the fiducial path prescription amount to gauge transformations.

Representations of the Abelian group of surfaces in $U(1)$ yield as a result the kinematical structure of the Kalb–Ramond fields. The two derivatives $\Delta_{abc}$ and $\delta_{ab}$ become the field tensor and antisymmetric tensor potential. Let us now discuss the usual formulation of a Kalb–Ramond field to make explicit contact with the ideas that we have introduced in this subsection.

### *9.4.2 Kalb–Ramond fields and surface representation*

An Abelian Kalb–Ramond field is a two-form $A_{ab} = -A_{ba}$. Its field strength is a three-form $F_{abc} = \partial_{[a}A_{bc]}$. The action for these fields is defined in analogy with the Maxwell action,

$$S_{KR} = \int d^4x\sqrt{-g}F_{abc}F^{abc}, \tag{9.36}$$

where indices are raised with a spacetime metric $g^{ab}$ that defines the coupling to gravity. The action is invariant under gauge transformations

$$A_{ab} \to A'_{ab} = A_{ab} + \partial_{[a}\Lambda_{b]}. \tag{9.37}$$

It is easy to introduce a Hamiltonian formulation of the theory. The canonical variables are the pull-back to the three-surface of the Kalb–Ramond field $A_{ab}$ and its canonically conjugate momentum $\tilde{\pi}^{ab}$. The theory has a Gauss law constraint associated with the gauge symmetry

$$\partial_c\tilde{\pi}^{cd} = 0, \tag{9.38}$$

and the coupling to gravity is achieved by adding the following terms to the usual Hamiltonian and diffeomorphism constraints of vacuum general relativity:

$$\tilde{C}_a = \underset{\sim}{\epsilon}_{abc}\tilde{\pi}^{bc}\tilde{\epsilon}^{cde}F_{cde}, \tag{9.39}$$

$$\mathcal{H} = \tfrac{1}{2}(\tilde{\epsilon}^{abc}F_{abc})^2 + \tfrac{1}{2}\tilde{\pi}^{ab}\tilde{\pi}^{cd}\underset{\sim}{\epsilon}_{abe}\underset{\sim}{\epsilon}_{cdf}\tilde{\tilde{q}}^{ef}. \tag{9.40}$$

One can build a quantum representation for the joint Einstein–Kalb–Ramond system coordinatized by loops and surfaces. We start by constructing a non-canonical algebra of quantities associated with the Kalb–Ramond field, to supplement the usual $T$ algebra for gravity. To each surface we associate the gauge invariant quantity,

$$U(\sigma) = \exp\left(i\int_\sigma d^2S^{ab}A_{ab}\right), \tag{9.41}$$

which is the analogue of the Wilson loop for Kalb–Ramond fields and materializes the representation of the group of surfaces we discussed in

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the previous subsection. This object forms an algebra with the gauge invariant quantity $\tilde{\pi}^{ab}$,

$$\{U(\sigma), \tilde{\pi}^{ab}\} = i \int d^2S^{ab}(y)\delta(x-y)U(\sigma). \tag{9.42}$$

One can construct a representation of this non-canonical algebra in terms of functions of the group of surfaces $\Psi(\sigma)$; it is given by

$$\hat{W}(\sigma')\Psi(\sigma) = \Psi(\sigma' \circ \sigma), \tag{9.43}$$

$$\hat{\tilde{\pi}}^{ab}\Psi(\sigma) = \int d^2S^{ab}(y)\delta(x-y)\Psi(\sigma). \tag{9.44}$$

A joint representation for gravity and Kalb–Ramond fields can be obtained by considering states that are functions of loops and surfaces $\Psi(\gamma, \sigma)$. We will see in the next section how to build diffeomorphism invariant quantum observables for such a system. Finally, the formalism involving open surfaces is useful for representing the coupling of Kalb–Ramond fields to matter, in particular to Abelian one-form fields. Details can be seen in references [155, 172, 156]. It is a complete analogue of what happened with usual gauge theories and coupling to fermions, which was achieved through the introduction of open paths on which loops acted as deformations.

## 9.5 Physical operators and weaves

In canonical quantization, as we outlined in chapter 3, after finding the space of physical states one needs to introduce an inner product under which the observables are self-adjoint operators. One can then compute expectation values and make measurable physical predictions.

Several difficulties prevent us from completing these steps for general relativity. Although we discussed some possible solutions to the constraints and in the next two chapters we will introduce further ones, one is far from knowing at present the space of all solutions to the constraints. On the other hand, for general relativity on compact spatial manifolds, we do not know at present a single observable in the Dirac sense. As a consequence, we are far from knowing a suitable inner product. In spite of these drawbacks, one would like to know if the structures that we have developed in these chapters have any connection, even at a kinematical or formal level, with possible physical ideas. An example of this was the argument presented in chapter 7 concerning the value of the determinant of the metric operator on the space of loops without intersections. This was not an argument based on a Dirac observable and yet it allowed us to draw conclusions about the space of states discussed. A further motivation for this kind of studies is that by discussing the action of operators in loop space one gains knowledge that may be useful whenever operators that commute with the constraints are found. Apart from this, one is, in general, interested in making a connection between the quantum description of gravity in terms of loops and the classical picture of general relativity as a theory of a metric.

All these considerations lead us to discuss in some detail some operators in the space of loop states. These operators will not commute with the constraints and therefore are not observables. We will see, however, that some of them could be related to observables (or at least commute with some of the constraints) if matter sources are introduced. They will allow us to give a notion of classical geometry associated with a certain set of loop states that play the role of "semi-classical states". We will see the crucial role that diffeomorphism invariance plays in the regularization of these operators.

### *9.5.1 Measuring the geometry of space in terms of loops*

Consider the metric in terms of the new variables

$$\tilde{\tilde{q}}^{ab} = \tilde{E}^a_i \tilde{E}^b_i. \tag{9.45}$$

It would be easy using the technology introduced in chapters 7 and 8 to promote this to an operator in loop space. One can regularize and renormalize the expression but it will depend on the particular details of the background metric introduced in order to regularize it. This is a general result. One regularizes expressions that involve products of Dirac delta functions, as the metric operator does. The renormalization procedure always amounts to replacing the product of two delta functions by a single one. The problem is that the Dirac delta function $\delta(x-y)$ is not only a distribution but also a density*. Therefore any procedure that converts the product of two deltas into one has to supply a factor with appropriate density weight. Since there are no natural scalar densities defined in a manifold without a metric, one is forced to introduce a density weight constructed with an external metric structure. Therefore renormalized expressions will always depend on a background metric.

There is a way out of this general objection that is based on the definition of operators that, unlike the metric, are well defined via a regularization procedure but *without a renormalization.* Let us give an example of such an operator, first introduced by Ashtekar, Rovelli and Smolin [167].

* In reference [153] this is emphasized by writing it as $\delta(x, y)$ since the expression $\delta(x-y)$ only makes sense when a background metric is defined. Then $\delta(x, y)$ is a density in *one* of its arguments.

Consider the classical expression

$$Q(\omega) = \int d^3x \sqrt{\tilde{E}^a_i(x)\tilde{E}^b_i(x)\omega_a(x)\omega_b(x)}, \tag{9.46}$$

where $\omega$ is an arbitrary smooth one-form. This quantity associates a real number to any one-form and it evidently contains information about the spatial metric in the sense that one can reconstruct the spatial metric from knowledge of $Q(\omega)\,\forall\omega$. To construct the quantum version of this operator we regulate the classical expression, $Q(\omega) = \lim_{\epsilon\to 0} Q^\epsilon(\omega)$, where

$$Q^\epsilon(\omega) = \int d^3x \sqrt{\int d^3y \int d^3z f_\epsilon(x,y) f_\epsilon(x,z)\tilde{E}^a_i(y)\tilde{E}^b_i(z)\omega_a(y)\omega_b(z)}. \tag{9.47}$$

We now promote the quantity under the square root to an operator in the loop representation. This is accomplished in a straightforward fashion using the calculational techniques of the previous chapter. The result is

$$\begin{aligned} &\int d^3y \int d^3z f_\epsilon(x,y) f_\epsilon(x,z)\hat{\tilde{E}}^a_i(y)\hat{\tilde{E}}^b_i(z)\omega_a(y)\omega_b(z)\Psi(\gamma) = \\ &2\int d^3y \int d^3z f_\epsilon(x,y) f_\epsilon(x,z)\omega_a(y)\omega_b(z) \\ &\times \oint_\gamma dv^a \oint_\gamma dw^b \delta(z-v)\delta(y-w)\left[2\Psi(\gamma^w_z \circ \gamma^w_{zo}) + \Psi(\gamma)\right]. \end{aligned} \tag{9.48}$$

Notice that this operator, when acting on a state $\Psi(\gamma)$ with support on loops without intersections like the ones we discussed in the previous chapter, returns a contribution proportional to $\Psi(\gamma)$. Therefore the square root is well defined, as we shall see. If the wavefunction has support on loops with intersections, the definition of the square root is more involved. It turns out that one can ignore the intersections in the definition of the operator. To see this, notice that the argument of the square root is a function of $x$ that coincides with its value on loops without intersections at all $x$s except at a finite number of $x_i$s, the intersection points. Therefore if one assumes that the value of the square root at those points is finite, one can ignore their contribution to the integral defining the operator $\hat{Q}(\omega)$.

Therefore at points without intersections one can explicitly compute the square root that appears in the definition of $Q(\omega)$. The result is, in the limit in which $\epsilon \to 0$,

$$\left|\sqrt{6}\oint_\gamma dy^a f_\epsilon(x,y)\omega_a(y)\right|\Psi(\gamma) + O(\epsilon), \tag{9.49}$$

which gives for the operator $Q(\omega)$, after noting that it is the integral of a

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positive quantity with support along the loop,

$$\hat{Q}(\omega)\Psi(\gamma) = \sqrt{6}\oint_\gamma |dx^a\omega_a(x)|\Psi(\gamma). \tag{9.50}$$

The operator so defined is finite and independent of the background structure that was used to define the regulator. No renormalization was needed in its definition.

Let us now define another quantity in the same spirit as the previous one, also suggested by Ashtekar, Rovelli and Smolin [167]. This quantity is associated with a surface $S$ with normal vector $n^a$. Its infinitesimal element of area is given by $d\mathcal{A} = d^2S^{ab}\sqrt{h}n^c\epsilon_{abc}$ where $h = \tilde{\tilde{q}}^{ab}n_a n_b$. From here we can give a polynomial expression in terms of the new variables for the square of the infinitesimal area element,

$$d\mathcal{A}(S) = dS^{ab}dS^{cd}\underset{\sim}{\epsilon}_{abe}\underset{\sim}{\epsilon}_{cdf}\tilde{E}^e_i\tilde{E}^f_i. \tag{9.51}$$

Additional comments on this formula for the area can be seen in reference [166]. From the above expression one can compute the area of the surface partitioning it into a countable number $N$ of small area elements and writing

$$\mathcal{A}(S) = \lim_{N\to\infty}\sum_{i=1}^{N}\sqrt{\mathcal{A}^2_{\text{approx}}(S_i)}, \tag{9.52}$$

where the quantity

$$\mathcal{A}^2_{\text{approx}}(S_i) = \int_{S_i} d^2S^{ab}_i(x)\int_{S_i} d^2S^{cd}_i(y)\epsilon_{abe}\epsilon_{cdf}T^{ef}(\eta^y_x, \eta^x_y) \tag{9.53}$$

approximates the infinitesimal element of area in the limit in which $S_i$ shrinks to a point and therefore the points $x$ and $y$ coincide and $\eta$ (a loop contained in $S_i$ that passes through $x$ and $y$) shrinks to a point as well. Recall that when the loop $\eta$ shrinks to a point $T^{ab}(\eta^y_x, \eta^x_y) \to \tilde{E}^a_i(x)\tilde{E}^b_i(x)$. We could have also proceeded as for the $\hat{Q}$ operator and introduced the limit in a gauge non-invariant fashion. We do it here in terms of the $T^2$ for illustration purposes.

The above expression for $\mathcal{A}^2_{\text{approx}}(S_i)$ is immediately promoted to a quantum operator in the loop representation replacing the $T$ by its corresponding quantum operator, which we introduced in formula (8.13). Assuming we act on a wavefunction $\Psi(\gamma)$ of a loop $\gamma$ without intersections on the surface $S_i$ the action of $\hat{T}^2$ gives four terms that in the limit in which $S_i$ shrinks to a point are, after Mandelstam rearrangements, identical,

$$\lim_{\eta\to\iota}\hat{T}^{ab}(\eta^y_x, \eta^x_y)\Psi(\gamma) = 6X^{ax}(\gamma)X^{bx}(\gamma)\Psi(\gamma). \tag{9.54}$$

and in the limit $\eta \to \iota$ the point $x \to y$.

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On the other hand, the integral $\int_{S_i} d^2S^{ab}(x)\epsilon_{abc}X^{ax}(\gamma)$ gives as a result the intersection number of the surface $S_i$ and the loop $\gamma$, $I[S_i,\gamma]$, meaning that it counts the (oriented) number of times the loop $\gamma$ pierces the surface $S_i$. As a result, we can write for the approximate area element in the limit in which it shrinks to a point,

$$\hat{\mathcal{A}}^2_{\text{approx}}(S_i)\Psi(\gamma) = 6I[S_i,\gamma]^2\Psi(\gamma). \tag{9.55}$$

We can therefore write as a result the expression for the area operator,

$$\hat{\mathcal{A}}(S)\Psi(\gamma) = \sqrt{6}I^+[S,\gamma]\Psi(\gamma), \tag{9.56}$$

where $I^+[S,\gamma]$ is the unoriented intersection number of the loop $\gamma$ and the surface $S$. That is, the area operator counts the (unoriented) number of times the loop pierces the surface $S$.

In contrast to the case of the $\hat{Q}$ operator, one cannot simply ignore intersections in the definition of the $\hat{\mathcal{A}}$ operator. If the wavefunction on which it acts has support on intersecting loops, it is no longer an eigenstate of $\hat{\mathcal{A}}^2$ and the square root cannot be taken. Smolin [145] proposed an extension of the $\mathcal{A}(S)$ operator to the intersecting case using a constructive procedure. This is based on the application of the Mandelstam identity to infer the value of $\mathcal{A}(S)$ on loops with intersections. Some simple examples are given in reference [145] but a complete definition can only be introduced through the use of spin network states [146].

Rovelli and Smolin[146] have introduced a regularized definition of the volume element along the same lines as the operators we have introduced here. In that case the relevant states on which the operator is non-vanishing have intersections with three independent tangents. The attractive feature of this is that as a consequence its eigenstates can be naturally described by spin networks. Spin networks also seem to simplify the definition of $\mathcal{A}(S)$ for intersecting loops. Further discussion of the $\hat{Q}$ and $\hat{\mathcal{A}}$ operators can be found in the Les Houches lectures of Ashtekar [3] as well as in the previously quoted references.

In both the $\hat{Q}$ and $\hat{\mathcal{A}}$ operators we saw that in order to define the square roots it was simpler if the states on which they operated had support on loops without intersections. As we discussed in chapter 8 it is apparently inconsistent with the Mandelstam identities to consider such states. Therefore it is useful to introduce the notation of bras, in terms of which it is well defined to say that

$$< \alpha|\hat{Q}(\omega) = \sqrt{6}\oint_\alpha |dx^a\omega_a(x)| < \alpha|, \tag{9.57}$$

$$< \alpha|\hat{\mathcal{A}}(S) = I^+[S,\alpha] < \alpha|, \tag{9.58}$$

if the loop $\alpha$ is smooth and non-intersecting.

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A few comments are in order about the above expressions. The integrals that appear in the right-hand side of these equations are, *prima facie*, not well defined as functions of loops, in the sense of chapter 1. If one adds a tree to the loop $\gamma$ their value changes. In the first integral this is due to the absolute value and in the second one it is due to the fact that we are considering the unoriented intersection number (which can be arbitrarily changed adding trees that pierce the surface). In both cases the problem can be traced back to the definition of an operator as the square root of a square, which introduces a sign ambiguity. This difficulty can be remedied by defining the operators in the following way. Consider the action of the operator on a loop $\gamma$. Then define a curve $p$ obtained by stripping all trees from a representative curve of $\gamma$ and compute the integrals in the right-hand side of the definitions of the operators using the curve $p$. The result is a function of loops.

As we mentioned above the definition introduced for $\hat{\mathcal{A}}(S)$ is only valid for smooth non-intersecting loops and a more elaborate one is needed for intersections. It seemed that the definition of $\hat{Q}$ was free of these kinds of complications since intersections only constituted a set of measure zero in the integral on the loop that appeared in the definition of the operator. This, however, assumes that intersections only appear at isolated points. If one considers loops with lines traversed several times each of them would contribute a non-negligible amount to the integral and in those cases the operator that appears inside the square root is no longer in an eigenstate. It is possible that this complication can be handled in the same fashion as for the $\hat{\mathcal{A}}$ operator with the use of spin networks. Loops with multiple lines are inevitable in any formulation that preserves the Mandelstam identities as we discussed in chapter 8, figure 8.1.

The idea of considering operators that are naturally densities of order one through the introduction of a square root in order to define them without renormalization is a general one. It could be applied to any operator with those characteristics. An interesting point would be to apply this idea to the Hamiltonian constraint as a means to overcome the regularization ambiguities and background dependences discussed in chapter 8. The Hamiltonian constraint is naturally a density of weight two. Its square root is a density of weight one and its integral on the three-manifold is likely to be well defined without the introduction of additional background structures. The trouble is that the canonical formulation of vacuum general relativity requires that the constraint vanish point by point and not only as an integral.

There is a context, however, in which the integral of the square root of the Hamiltonian constraint arises naturally. This has been explored by Rovelli and Smolin [140]. Suppose one couples general relativity to a massless minimally coupled scalar field $T(x)$. In the canonical formulation a new canonical pair arises: $T(x)$ and $\tilde{P}_T$. The constraints are

$$C_a(x) = C_a^0(x) + P_T(x)\partial_a T(x) = 0, \tag{9.59}$$
$$H(x) = H^0(x) + \tilde{P}_T^2 + \tfrac{1}{4}\tilde{E}_i^a \tilde{E}_i^b \partial_a T(x) \partial_b T(x) = 0, \tag{9.60}$$

where we have used a zero superscript to denote the usual diffeomorphism and Hamiltonian constraints of general relativity.

One can now use the scalar field to keep track of time in this problem by considering a foliation of spacetime defined by leaves in which $T(x)$ is constant. Such a foliation, in general, only exists locally but the approach is to explore the resulting quantum theory as long as such a foliation is acceptable. In that case the terms involving gradients of the fields drop out from the constraints and one can solve the Hamiltonian constraint for $P_T$ classically,

$$P_T = \int d^3x \sqrt{-H(x)}, \tag{9.61}$$

where we have kept the same notation for $P_T$ though it is now a constant due to the gauge fixing for the field $T$.

We therefore see that the square root of the integral of the Hamiltonian appears naturally in this context. One can now construct a quantum theory in which states are functions of loops parametrized by the "time" $T$, $\Psi(\gamma, T)$, and satisfying a Schrödinger-like equation,

$$i\frac{\partial}{\partial T}\Psi(\gamma, T) = \int d^3x \sqrt{-\hat{H}(x)}\Psi(\gamma, T), \tag{9.62}$$

in which the integral of the square root of the usual Hamiltonian constraint in the loop representation which we discussed in chapter 8 arises naturally. Rovelli and Smolin [140] studied this operator and found that it can be defined without the introduction of a background metric for smooth and intersecting loops without kinks at the intersections or elsewhere. The resulting Hamiltonian for the theory, being background independent, can be purely formulated as a set of topological operations on the space of knots. This requires setting a prescription for how one adds the infinitesimal loop that arises in the definition of the usual Hamiltonian constraint. There is an infinite-fold ambiguity on how to add the infinitesimal loop in the space of knots with intersections. Also, the definition of the addition of an infinitesimal loop is problematic in the space of knots, since the addition of the infinitesimal loop changes the knot character and therefore may lead to discontinuities unless one requires special properties of the wavefunctions. It is not even clear that there exists, in general, a diffeomorphism invariant assignment of a small loop. Moreover, viewing the Hamiltonian as the square root of a matrix may be problematic since the basis of knots with intersections is, in general, not a countable basis†.

This approach has recently led to an important amount of activity with the introduction of topological Feynman rules for the interaction of gravity and matter, when combined with the formulation of fermions of Morales-Técotl and Rovelli that we discussed at the beginning of this chapter.

### 9.5.2 Semi-classical states: the weave

Usually, in a quantum theory, in order to extract physical information it is necessary to make some kind of correspondence with classical physics. This is accomplished through a semi-classical limit. This implies picking a preferred set of states that "approximate" the classical behavior with "small" quantum corrections. Typically what is meant by "approximate" and "small" refers to quantum fluctuations, which implies considering expectation values of observable quantities. Evidently, we cannot take this route here since we do not have an inner product or observables to compute such expectation values.

Ashtekar, Rovelli and Smolin [167] have suggested a different strategy. They consider a set of loop states based on collections of loops characterized by a certain (macroscopic) length scale $L$ and a (microscopic) length scale $l$, associated with a classical geometry $h$. We will consider the operators introduced in the previous subsection and study their eigenvalues. We will see that if one considers the functions that parametrize the operators ($w_a$ and the surface $S$) as smoothly varying with respect to $L$ there is a unique value of the parameter $l$ for which the eigenvalues coincide with the classical value of the quantities $Q$ and $\mathcal{A}$ as calculated in the classical geometry determined by $h$.

The states that we will consider are constructed in the following way. Given the three-geometry $h$ to which one wants to associate a state, sprinkle in a region of it a randomly distributed number of points $N$ with a density $n = N/L^3$, where $L$ is a certain (macroscopic) length scale. At each point draw a circle of radius $l = (1/n)^{1/3}$ with a random orientation. This results in a set of curves which we can consider a representative element of a multiloop $< \Delta|$. For the moment we consider multiloops without intersections, for simplicity, though the circles involved can be linked. One then considers, for instance, the action of the $\hat{Q}(w)$ operator on such a state with $w$ a fixed one-form that is smooth on the scales de-

† An intuitive way of seeing that it is not countable is to consider quintuple intersections. In the extension of the braid group to intersections one needs to consider a two-parameter family of new elements with quintuple intersections, parametrized by the angles one of the tangents forms with the other three.

termined by $L$ and smaller. Applying the results of the previous section we get

$$< \Delta|Q(\omega) = \sqrt{6}\sum_{i=1}^{N}\oint_{\alpha_i}|dy^a\omega_a| < \Delta|, \tag{9.63}$$

where $\alpha_i$ is the circle located at the $i$th point.

Since $w_a$ is slowly varying the integral at each circle can be computed, and averaging over all possible directions, we get

$$< \Delta|\hat{Q}(\omega) = \left(\sqrt{6}\pi\left(\frac{1}{l^2}\right)Q(w)[h] + O\left(\frac{l}{L}\right)\right) < \Delta| \tag{9.64}$$

where $Q(w)[h]$ is the value of the quantity $Q$ evaluated for the one-form $w_a$ on the classical three-geometry $h$, which we have assumed to be flat for the particular form of the result presented. We therefore see that for the eigenvalue of the operator $Q$ to approximate its classical value on the three geometry that we want to associate with the state $< \Delta|$ to order $l/L$ we need the separation of the loops to be fixed to a value of order unity ($l^2 = \sqrt{6}\pi$), in natural units. That is, the loops need to be separated by a length approximately equal to the Planck length.

This result is remarkable in the sense that we have derived a natural cutoff scale in quantum gravity in order to recover classical physics. The *a priori* feeling would have been that the weave would have approximated the classical geometry better as the separation of the loops was smaller. The detailed calculation shows that this is true until one reaches scales of the order of the Planck scale. Then, beyond a certain value, the approximation is worse the finer the weave is. The appearance of a natural cutoff in a detailed calculation of quantum gravity opens up the possibility that the theory could be made finite by its own dynamics and shows a significant departure from the usual behavior of quantum field theories. It is also remarkable that the Planck length appears naturally as the cutoff, especially since the theory is a diffeomorphism invariant one and one has the expectation that no scales would be privileged in such theories. Natural cutoffs appear in various contexts in quantum gravity. For a discussion see the paper by Garay [173].

To consider that we have approximated a classical geometry because a single quantity — which does not commute with the constraints — approximates its classical value is clearly insufficient. One can repeat the above construction for the area operator, but it is clear that these kinds of calculations are at the moment only indicative of the kind of physics one should expect when one is able to perform similar calculations with genuine observables of the theory.

It is possible that this kind of calculation could be performed for general relativity coupled to matter, where one can construct quantities that commute with (at least some of) the constraints. For instance, Smolin [172] has considered general relativity coupled to Kalb–Ramond fields in a surface formulation similar to the one we discussed in section 9.4.2. In that context he can use the surfaces that characterize the Kalb–Ramond field to construct with the gravitational variables an operator similar to $\mathcal{A}(S)$. The difference now is that the surface $S$ is determined by the fields of the theory, i.e., it is dynamical. As a consequence the operator defined is invariant under diffeomorphisms. A similar construction of diffeomorphism invariant quantities was performed for the Maxwell field by Husain [169], and for topological field theories by Rovelli [170].

A very important result is that of Iwasaki and Rovelli [168] who have carried this analysis one step further. They studied in detail the correspondence between the theory offered by the weave and its perturbations with the quantized theory of gravitational perturbations of a classical background (the usual linearized quantum gravity). They found a correspondence between sectors of the space of states of both theories such that now gravitons can be viewed as a particular family of perturbations of the weave.

## 9.6 2+1 gravity

General relativity in two spatial and one temporal dimensions offers a remarkable laboratory to test ideas of loop quantization. Because in three dimensions the Ricci tensor completely determines the Riemann tensor, the vacuum Einstein equations in three dimensions,

$$R_{\mu\nu} = 0, \tag{9.65}$$

imply that spacetime is locally flat. The only non-triviality of the Einstein theory in three spacetime dimensions comes from the topology of spacetime. The theory therefore does not have any local degrees of freedom. It has a finite number of topological degrees of freedom. Therefore the theory is exempt from the difficulties associated with the infinite number of degrees of freedom of field theories and yet it shares several features with the $3+1$ theory, foremost among which is the invariance under diffeomorphisms.

The Einstein action in $2+1$ dimensions is, written in first order form [157],

$$S(e,A) = \int d^3x \tilde{\epsilon}^{abc} e_{ai} F^i_{bc}, \tag{9.66}$$

where $e_{ai}$ is a set of triads, $A^i_a$ is the spin connection compatible with the triad and $F^i_{bc}$ is the curvature. The indices $i, j$ and $k$ are $SO(2,1)$ indices in the tangent space to the three-dimensional spacetime. Notice that this is just the Palatini action for the theory and all the variables are real. The equations of motion resulting from the variation with respect to the triad and the connection are

$$\frac{\delta}{\delta A_a^i} \quad : \quad {}^3D_{[a}e_{b]}^i = 0, \tag{9.67}$$

$$\frac{\delta}{\delta e_a^i} \quad : \quad F_{ab}^i = 0. \tag{9.68}$$

The first equation tells us that $A_a^i$ is the torsion free covariant derivative that annihilates the metric constructed with the triad. The second equation tells us that the curvature of the connection is zero and therefore spacetime is flat.

One can immediately perform a canonical decomposition of the action. The details can be seen in reference [153]. The resulting phase space consists of the pull-back of the connection $A_a^i$ to the two-dimensional surface and the canonically conjugate momentum is the pull-back of the cotriad $\tilde{E}_i^a = \tilde{\epsilon}^{ab} e_{bi}$, where $\tilde{\epsilon}^{ab}$ is the Levi–Civita density on the two-dimensional spatial surface. The other variables are Lagrange multipliers whose variation enforces the constraints,

$$D_a \tilde{E}_i^a = 0, \tag{9.69}$$

$$F_{ab}^i = 0, \tag{9.70}$$

which can be obtained by pulling back to the spatial slice the equations of motion.

The first equation is the usual Gauss law that tells us that the theory is invariant under $SO(2,1)$ triad rotations in the tangent space. The second equation contains in a joint form the diffeomorphism and Hamiltonian constraint of the $2+1$ theory. This form of the constraints was first introduced (in a slightly different way) by Witten [46]. The relationship with the usual form of the (3+1)-dimensional Ashtekar constraints can be made explicit [158] by contracting the second equation with $\tilde{E}_i^a$ for the diffeomorphism and with $f_{ijk}\tilde{E}_i^a \tilde{E}_j^b$, where $f_{ijk}$ are the structure constants of $SO(2,1)$. With these projections the form of the constraints becomes exactly the same as those in the $3+1$ theory (with the exception that the spatial indices run from 1 to 2 and the internal indices are $SO(2,1)$). One could choose either form of the constraints to study the theory, but it should be made clear that the two forms are inequivalent. The projections introduced to obtain the Ashtekar constraints from the Witten ones can become degenerate and therefore the former admit many more solutions. This inequivalence is far from academic. It can be shown that the Ashtekar form of the constraints classically allows configurations with an infinite number of degrees of freedom whereas the Witten one does not [159].

Counting the degrees of freedom, we have twelve variables in phase space and six first class constraints. The system has no local degrees of freedom. The constraints are either linear in momenta or independent of momenta. The situation is considerably simpler than in the $3+1$ theory. In particular we can find observables for the theory, i.e., quantities that commute with all the constraints, something we were unable to do in the $3+1$ theory. It is easy to see that the $T^0$ and $T^1$ quantities constructed with the canonical variables have vanishing Poisson brackets with the constraints (in the $2+1$ literature it is customary to integrate the $T^1$ along a loop using $T^1 = \int dy^a \underset{\sim}{\epsilon}_{ab} \text{Tr}(\tilde{E}^b(y)U(\gamma_y^y))$, we adopt this nomenclature for the rest of this section). These observables are closely related to those introduced by Martin [160] in the Witten language. Another difference with the $3+1$ case was that there the higher order $T$ variables were needed to express in a convenient fashion physical quantities of interest, such as the Hamiltonian constraint. Here we can write the constraints exclusively in terms of the small $T$ algebra. In order to gain an intuitive feeling for the meaning of the observables introduced, it is worthwhile mentioning that in three-dimensional gravity in the asymptotically flat case the metric of a point particle corresponds to a cone. The deficit angle of the cone is proportional to the mass of the particle. The $T^0$ measures that quantity. If the particle is spinning, the $T^1$ measures the rate of spin.

It may appear surprising that in a diffeomorphism invariant theory the $T$ variables are observables. After all, they are not in the $3+1$ theory. The key to this difference lies in the flatness of the connection. For a flat connection, a loop and the same loop shifted by a diffeomorphism yield the same holonomy. Therefore quantities based on holonomies, in spite of their dependence on an external structure — the loop —, can be diffeomorphism invariant. Because the connection is flat, two homotopic loops lead to the same holonomy. If the loops are homotopic to the identity, the holonomy is the identity. That implies that the only loops that yield non-trivial holonomies are those that wrap around topologically non-contractible paths.

We can now proceed to the quantization of the theory following the same steps as for the $3+1$ theory. Let us start in the connection representation. We pick wavefunctionals of the connection $\Psi[A]$. The constraints are easily promoted in an unambiguous fashion to quantum operators. The Gauss law simply requires that we consider wavefunctions that are gauge invariant. The constraint $\hat{F}^i_{ab}\Psi[A] = 0$ simply requires that the wavefunctions have support on flat connections. Together they demand that the wavefunctions considered be wavefunctions on the moduli space of flat connections. This space can be endowed with a simplectic structure, and an inner product that makes the $T$ operators self-adjoint [2].

Consider now the loop representation. We can find a representation of

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the $T$ operators in terms of loops in exactly the same fashion as we did for the $3+1$ theory (the formulae differ slightly since the gauge group is $SO(2,1)$ instead of $SO(3)$). If one wants to make more progress it is convenient to pick a particular spatial topology. Let us consider the simplest non-trivial example: that in which the spatial manifold has the topology of a two-torus.

There are two possible avenues one could take to construct the loop representation. One is to promote the constraint equations to operators in loop space and study the space of solutions. The other one is to take advantage of the knowledge that the physical states in the connection representation have support on the moduli space of flat connections and directly build a loop representation for the flat connection.

If we take the latter approach let us start by characterizing the moduli space of flat connections on a torus. In this case any flat connection can be characterized by the value $a_1, a_2$ of its holonomy along two preferred families of loops $\eta_1, \eta_2$, the loops that encircle the two generatrices of the torus. As we discussed above, homotopically equivalent loops yield the same holonomy when the connection is flat, so the holonomies are really functions of the homotopy classes of loops. Since the homotopy group of the torus is Abelian, the holonomies along the two different families of loops commute. That is, they correspond to $SO(2,1)$ rotations around the same axis. Depending on the null, time-like or spatial character of this axis in the $SO(2,1)$ manifold one has three distinct sectors of the theory.

Let us discuss in detail the time-like sector. For this case, the two non-trivial holonomies can be written,

$$U(a_b) = \exp(2a_b t^i \tau_i), \qquad b = 1, 2, \tag{9.71}$$

where $t^i$ is the time-like rotation axis in the internal space, $a_b$ are the rotation parameters and $\tau_i$ are the $SO(2,1)$ matrices. From here it is immediate to compute the value of the $T$ quantities,

$$T^0(\vec{n}) = 2\cos(\vec{n} \circ \vec{a}), \tag{9.72}$$

$$T^1(\vec{n}) = 2\sin(\vec{n} \circ \vec{a})\vec{n} \times \vec{p}, \tag{9.73}$$

where $\vec{n} = (n_1, n_2)$ and $\vec{n} \circ \vec{a} = n_1 a_1 + n_2 a_2$, $\vec{n} \times \vec{p} = n_1 a_2 + n_2 a_1$, and $\vec{p}$ are the variables canonically conjugate to the $\vec{a}$.

We now consider a quantum representation of this algebra on a space of functions $\Psi(a_1, a_2)$. This space can be endowed with an inner product such that $\vec{a}$ and $\vec{p}$ are self-adjoint operators. The measure is simply given by $da_1 da_2$. The $T$ operators are [157]

$$\hat{T}^0(\vec{n})\Psi(a_1, a_2) = 2\cos(\vec{n} \circ \vec{a})\Psi(a_1, a_2), \tag{9.74}$$

$$\hat{T}^1(\vec{n})\Psi(a_1,a_2) = 2i\sin(\vec{n}\circ\vec{a})\vec{n}\times\frac{\partial}{\partial\vec{a}}\Psi(a_1,a_2), \tag{9.75}$$

and with the inner product introduced above the $T$s become self-adjoint operators.

Let us now build a loop representation from this "connection representation". Recalling the expression for the $T^0$ variable, the loop transform then reads

$$\Psi(n_1,n_2) = \int_{-1}^{1}\int_{-1}^{1} da_1 da_2 \cos((n_1a_1+n_2a_2))\Psi(a_1,a_2), \tag{9.76}$$

and we see that it reduces to a familiar Fourier transform.

Wavefunctions in the loop representation are given by functions of two integers $n_1$ and $n_2$. The only Mandelstam identity left (because of the Abelian nature of the homotopy group of the torus) is given by $\Psi(n_1,n_2) = \Psi(-n_1,-n_2)$.

In this space of wavefunctions one can particularize the usual expressions for the action of the $T$ operators,

$$\hat{T}^0(\eta)\Psi(\gamma) \equiv \Psi(\gamma\circ\eta) + \Psi(\gamma\circ\eta^{-1}), \tag{9.77}$$

$$\hat{T}^a(\eta^x_x)\Psi(\gamma) \equiv i\sum_{\epsilon=-1}^{1}\epsilon\oint dy^a\delta(x-y)\Psi(\gamma\circ\eta^\epsilon), \tag{9.78}$$

to the following expressions for the basic operators associated with $\eta_1$,

$$\hat{T}^0(\eta_1)\Psi(n_1,n_2) = \Psi(n_1+1,n_2) + \Psi(n_1-1,n_2), \tag{9.79}$$

$$\hat{T}^1(\eta_1)\Psi(n_1,n_2) = in_2(\Psi(n_1+1,n_2) - \Psi(n_1-1,n_2)), \tag{9.80}$$

which are Hermitian in terms of the inner product

$$<\Psi|\Phi> = \sum_{n_{1,2}=-\infty}^{\infty}\bar{\Psi}(n_1,n_2)\Phi(n_1,n_2). \tag{9.81}$$

The loop transform (9.76) is symmetric in $a_{1,2}$. Therefore antisymmetric functions are mapped to zero. The transform defines an isomorphism between the space of symmetric functions of $a_{1,2}$ and the loop representation‡.

Up to now we have considered the case of holonomies that are $SO(2,1)$ rotations around a time-like internal vector. Let us now briefly consider the case of space-like rotations. In that case, the holonomy is again given by (9.71), which particularized to the space-like sector of $SO(2,1)$ (where

‡ As we mentioned in chapter 3, we are disregarding symmetries that are not generated by constraints. Peldán [161] discusses the role that large diffeomorphisms play in this sector of 2+1 gravity.

the $\tau$ matrices are anti-Hermitic) gives

$$T^0(\vec{n}) = 2\cosh(\vec{n} \circ \vec{a}), \tag{9.82}$$

$$T^1(\vec{n}) = 2\sinh(\vec{n} \circ \vec{a})\vec{n} \times \vec{p}. \tag{9.83}$$

Notice that the topology of the phase space is different from that in the time-like case: $|\vec{a}|$ is now unbounded.

From here on one could construct a loop representation in exactly the same fashion as in the time-like case. The quantum $T$ algebras are exactly the same and the inner product is the same.

It should be emphasized that the $T$ algebras can be completely realized in terms of either the space-like or the time sectors of the phase space (the null sector has also been considered in detail in references [19, 162]). Therefore these sectors correspond to separate — in principle, inequivalent — quantum theories. In fact, the theories are quite inequivalent. As pointed out before, for the time-like case the $T^0$ operator was bounded as a classical quantity, and being a multiplicative quantum operator, its eigenvalues are bounded. This is not the case for the space-like holonomies. Moreover, it is the space-like sector that is equivalent to the ADM quantization [162].

This leads to a disquieting picture: both the space-like and the time-like cases in the connection representation seem to give rise to the same loop representation; however, at the level of connections they are quite distinct. How could it be that two inequivalent representations in terms of connections give rise to the same loop representation?

The answer lies in the precise relationship between the connection and loop representations. For the time-like case we saw that the loop representation was isomorphic to the symmetric connection representation. We will see that for the space-like case it is not. This solves the contradiction. Let us discuss the situation in some detail. The loop transform for the spatial case reads

$$\Psi(n_1, n_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} da_1 da_2 \cosh((n_1 a_1 + n_2 a_2))\Psi(a_1, a_2), \tag{9.84}$$

and using the symmetries of the connections in the spatial sector which imply $\Psi(\vec{a}) = \Psi(-\vec{a})$, we can reduce it to a two-sided Laplace transform,

$$\Psi(n_1, n_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} da_1 da_2 \exp((n_1 a_1 + n_2 a_2))\Psi(a_1, a_2). \tag{9.85}$$

The problem is that this would be a usual Laplace transform if the parameters $n_1$ and $n_2$ were real numbers. Being integers, one immediately finds that the transform is not an isomorphism. It turns out that it has a large non-trivial kernel. To give an intuitive idea of the problem it is instructive to construct one of the elements of the kernel of the

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transform. Simply consider a function $f(s_1, s_2)$ with $s_{1,2} \in \mathbf{R}$ such that it vanishes for $s_{1,2}$ integer. Such a function is mapped by the inverse Laplace transform to a non-trivial element in the connection representation that has a vanishing image in the loop representation.

One may ask if this problem is just a technicality or if it is a serious drawback. It turns out that the kernel we found is dense in the space of connections [19]. That means that *any* wavefunction in terms of connections can be obtained as a limit of a sequence of elements that are all in the kernel. This is a serious problem: the sector of the theory which corresponds to usual geometrodynamic quantization is not properly described by the loop representation we introduced.

Does this example imply that loop representations are inadequate to describe theories based on non-compact gauge groups? At least it should be viewed as a warning. It turns out that the problem is not fatal and one can deal with it if proper care is exercised. Up to now, two solutions have been proposed: the use of a non-trivial measure in the transform and the use of extended loops. In a sense, this difficulty can be seen as added motivation for the consideration of such ideas in the case of gravity.

Ashtekar and Loll [163] pointed out that if one introduces a non-trivial measure in the space of connections, the resulting loop transform does not have a non-trivial kernel. The explicit construction of the measure has been found for surfaces of lower genus and it implies a non-trivial change in the quantum realization of the $T$ operators. This just corresponds to one of the many different choices one has when quantizing a field theory. For the case of the torus the non-trivial measure is

$$d\mu[A] = \exp(-c[T^0(\eta_1) + T^0(\eta_2)]) \tag{9.86}$$

where $c$ is a positive constant and $\eta_1$ and $\eta_2$ are two fixed loops belonging to independent homotopy classes. The measure depends quite non-trivially on the genus and apparently cannot be derived from any general principle in the loop representation. One has to know that the difficulty arises and then carefully examine the correspondence between the loop and the connection representation to construct a measure that solves it. This is somewhat discouraging since one does not hope to have all that information available in more complicated cases.

Another solution that has been proposed by Marolf [19] is the use of extended loops. From the point of view that is presented in this book, this would be a very satisfactory solution since it uses the same general principles that are advocated for use in the non-trivial theories. The idea has not been analyzed in great detail, but the basic concept is very simple: if one considers extended loops, the holonomies of a connection in the torus are labeled by two real numbers instead of integers. If one writes the extended loop transform in the spatial case it corresponds to

(9.85) but with $n_{1,2}$ real numbers, which defines an isomorphism between the spaces.

Many other issues have been explored in the (2+1)-dimensional theory. Our intention here was not to be exhaustive but to show that the program in general lines can be carried along in a theory in which one has control over the mathematical issues. We encountered expected difficulties that can be resolved with ideas that are being applied in more complicated cases. For more details see the reviews by Carlip [164] and Ashtekar [165] and the papers by Marolf and Louko [19, 162].

## 9.7 Conclusions

We have discussed several applications of the loop representation of general relativity coupled to matter fields and the definition of physically meaningful quantities in terms of loops. We have shown how to define regularized operators that could be used through matter couplings to give an idealized picture of the measurement process in quantum gravity. Evidently there is a long way to go before we can make actual physical predictions in quantum gravity. In particular it is expected that currently unknown approximation techniques will be crucially needed due to the complexity of the Einstein equations. The hope is that the ideas presented in this chapter may act as building blocks of such a measurement theory. We ended this chapter by discussing a rather disjoint application: general relativity in $2 + 1$ dimensions, which plays the important role of showing that the ideas being advocated in this book can actually be applied to a theory of quantum gravity in a simplified setting. The ideas in this chapter were mainly built either at a kinematical level or heuristically implemented in the space of smooth non-intersecting loops. In the following chapter we will be concerned with the dynamics of quantum gravity and therefore we will make progress towards finding physical states of the theory. Many, though not all, of the ideas of this present chapter go through for the intersecting solutions that we will present in the next two chapters. Some others will need a revision. We expect that in the next few years progress will be made in this direction.