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On uniformly distributed orbits of certain horocycle flows

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Abstract. Let

$$G = \mathrm{SL}(2, \mathbb{R}), \qquad \Gamma = \mathrm{SL}(2, \mathbb{Z}), \qquad u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

(where $t \in \mathbb{R}$) and let μ be the G-invariant probability measure on G/Γ. We show that if x is a non-periodic point of the flow given by the (u_t)-action on G/Γ then the (u_t)-orbit of x is uniformly distributed with respect to μ; that is, if Ω is an open subset whose boundary has zero measure, and l is the Lebesque measure on $\mathbb{R}$ then, as $T \to \infty$, $T^{-1} l\{0 \le t \le T \mid u_t x \in \Omega\}$ converges to $\mu(\Omega)$.

Let $G = \mathrm{SL}(2, \mathbb{R})$, the special linear group of 2×2 matrices, and let $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ be the subgroup consisting of integral matrices in G. The homogeneous space G/Γ carries a unique G-invariant probability measure which we shall denote by μ. Let (u_t) be the one-parameter subgroup of G defined by $u_t = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ for all $t \in \mathbb{R}$. Let P be the subgroup of G consisting of all upper triangular matrices in G.

Consider the action of (u_t) on G/Γ. It is well-known that for any $g \in P\Gamma$ the (u_t)-orbit of $g\Gamma$ in G/Γ is periodic. Further, if $g \notin P\Gamma$ then the (u_t)-orbit of $g\Gamma$ is dense in G/Γ. The object of this paper is to show that each of these dense orbits is uniformly distributed on G/Γ with respect to μ; that is, if $g \notin P\Gamma$ and Ω is an open subset of G/Γ whose boundary has zero μ-measure then as $T \to \infty$,

$$T^{-1} \int_0^T \chi_\Omega(u_t g\Gamma)\, dt$$

converges to $\mu(\Omega)$ (χ_Ω is the characteristic function of Ω). Similarly, we prove that the orbit under (iterates of) $u = u_1$ of $g \notin P\Gamma$ is also uniformly distributed in the sense that for Ω as above

$$\frac{1}{n} \sum_{j=0}^{n-1} \chi_\Omega(u^j g\Gamma)$$

converges to $\mu(\Omega)$ as $n \to \infty$ (cf. theorem 6.1). It may be noted that these results extend in a natural way to any subgroup of finite index in Γ.

In § 6 we also discuss the dynamical significance of the result and an application to number theory.

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1. *Preliminaries*

Let $\mathbb{R}^2$ be the two-dimensional Euclidean space. We denote by $\{e_1, e_2\}$ the standard basis of $\mathbb{R}^2$. Let $\langle\ ,\ \rangle$ denote the inner product on $\mathbb{R}^2$ with e_1, e_2 as an orthonormal basis and let $\|\cdot\|$ be the corresponding norm on $\mathbb{R}^2$. Also let m be the Lebesgue measure such that

$$m\{se_1+te_2 \mid 0\leqq s\leqq 1, 0\leqq t\leqq 1\}=1.$$

A *lattice* Λ in $\mathbb{R}^2$ is a discrete co-compact subgroup (that is, $\mathbb{R}^2/\Lambda$ is compact). Given a lattice Λ any measurable subset F such that $\{\Lambda+F\}_{\lambda\in\Lambda}$ is a partition of $\mathbb{R}^2$ (a fundamental domain) is of the same measure; the common value is called the determinant of Λ and shall be denoted by $d(\Lambda)$. We shall denote by $\mathscr{L}$ the set of all lattices Λ in $\mathbb{R}^2$ such that $d(\Lambda)=1$. We note that the lattice $\Lambda_0=\mathbb{Z}^2$ consisting of elements with integral coordinates belongs to $\mathscr{L}$.

In the sequel, we shall denote by G the (topological) group SL $(2, \mathbb{R})$ of real 2×2 matrices of determinant 1. The natural action of G on $\mathbb{R}^2$ induces a G-action on $\mathscr{L}$. It is straightforward to verify that this G-action on $\mathscr{L}$ is transitive and that the isotropy subgroup of the lattice Λ_0 is precisely the subgroup SL $(2, \mathbb{Z})$ consisting of all integral matrices in $G=\mathrm{SL}(2, \mathbb{R})$. We shall write Γ for SL $(2, \mathbb{Z})$. The map associating $g\Lambda_0$ to $g\Gamma$ for all $g\in G$ defines a canonical 1–1 correspondence of G/Γ onto $\mathscr{L}$. In the sequel we shall often identify $\mathscr{L}$ with G/Γ via the above correspondence. In particular, we shall consider $\mathscr{L}$ to be equipped with the topology arising from the identification with G/Γ, the latter having the topology as a homogeneous space of $G=\mathrm{SL}(2, \mathbb{R})$.

Let Λ be a lattice in $\mathbb{R}^2$. A non-zero element λ of Λ is said to be primitive in Λ if Λ does not contain any element of the form $t\lambda$ where $0<t<1$. We shall denote the set of all primitive elements of a lattice Λ by $\mathscr{P}(\Lambda)$. We need the following lemma which is well known and easy to prove.

(1.1) LEMMA. *Let $\Lambda\in\mathscr{L}$. A sequence $\{\Lambda_k\}$ in $\mathscr{L}$ converges to Λ in $\mathscr{L}$ if and only if for all $\varepsilon>0$ and $M>0$ there exists k_0 such that for all $k\geq k_0$ the following assertions hold*: (*a*) *for any $\lambda\in\mathscr{P}(\Lambda)$ satisfying $\|\lambda\|\leq M$ there exists $x\in\mathscr{P}(\Lambda_k)$ such that $\|x-\lambda\|<\varepsilon$ and* (*b*) *for any $x'\in\mathscr{P}(\Lambda_k)$ satisfying $\|x'\|\leq M$ there exists $\lambda'\in\mathscr{P}(\Lambda)$ such that $\|x'-\lambda'\|<\varepsilon$.*

For any subset E of $\mathbb{R}^2$ we put

$$W(E)=\{\Delta\in\mathscr{L} \mid \mathscr{P}(\Delta)\cap E \text{ is non-empty}\}.$$

(1.2) LEMMA. *If E is an open subset of $\mathbb{R}^2$ then $W(E)$ is open in $\mathscr{L}$. If E is a closed bounded subset of $\mathbb{R}^2$ then $W(E)$ is closed. If E is a bounded subset of $\mathbb{R}^2$ such that* 0 *is not a limit point of E then $W(E)$ is a bounded subset of $\mathscr{L}$.*

Proof. The first assertion is obvious. Next let E be a closed bounded subset of $\mathbb{R}^2$. Let $\{\Lambda_k\}$ be a sequence in $W(E)$ converging to a lattice Λ in $\mathscr{L}$. By lemma 1.1 for any $\varepsilon>0$, $\mathscr{P}(\Lambda)$ contains an element within distance ε from some element of E. Since $\mathscr{P}(\Lambda)$ is discrete and E is compact this implies that $\mathscr{P}(\Lambda)\cap E$ must be non-empty. Hence $\Lambda\in W(E)$, thus proving the second assertion. The last assertion follows from the well-known Mahler criterion (cf. [**8**, corollary 10.9]). □

(1.3) LEMMA. *Let C be a closed convex subset of $\mathbb{R}^2$ containing 0. Suppose that $m(C)<\frac{1}{2}$. If $\Lambda \in \mathscr{L}$ and*

$$\lambda \in \mathscr{P}(\Lambda) \cap C$$

then $\mathscr{P}(\Lambda) \cap C$ is contained in $\{\pm\lambda\}$; that is, $\mathscr{P}(\Lambda) \cap C$ does not contain two linearly independent elements.

Proof. Let $\Lambda \in \mathscr{L}$ and $\lambda, \lambda' \in \mathscr{P}(\Lambda) \cap C$ and suppose that $\lambda' \neq \pm\lambda$. Then the parallelogram formed by $0, \lambda, \lambda'$ and $\lambda + \lambda'$ contains a fundamental domain for Λ and consequently its area must be at least 1. Hence the area of the triangle formed by $0, \lambda$ and λ' must be at least $\frac{1}{2}$. But clearly the triangle is contained in C and consequently its area is less than $\frac{1}{2}$, which is a contradiction. Hence $\lambda' = \pm\lambda$. □

For any subset E of $\mathbb{R}^2$ let $C(E)$ denote the smallest closed convex subset containing E and $\{0\}$. Also for any subset Ω either of $\mathbb{R}^2$ or of $\mathscr{L}$ let $\partial\Omega$ denote the (topological) boundary of Ω in the respective space.

(1.4) PROPOSITION. *Let E be a bounded open subset of $\mathbb{R}^2$ such that $m(C(E))<\frac{1}{2}$. Suppose that $-E$ and ∂E are disjoint. Then*

$$\partial(W(E)) = W(\partial E).$$

Proof. Let $\Lambda \in W(\partial E)$. There exists

$$\lambda \in \mathscr{P}(\Lambda) \cap \partial E.$$

Let $\{\lambda_k\}$ be a sequence in E converging to λ. It is easy to see that one can construct a sequence Λ_k in $\mathscr{L}$ converging to Λ and such that

$$\lambda_k \in \mathscr{P}(\Lambda_k).$$

Hence Λ is contained in the closure of $W(E)$ which in view of lemma 1.2 coincides with

$$W(E) \cup \partial(W(E)).$$

By lemma 1.3 $\mathscr{P}(\Lambda) \cap C(E)$ is contained in $\{\pm\lambda\}$. Since $E \subset C(E)$ and neither λ nor $-\lambda$ can be contained in E we deduce that $\mathscr{P}(\Lambda) \cap E$ is empty. Thus $\Lambda \notin W(E)$. Consequently $\Lambda \in \partial(W(E))$. Thus

$$W(\partial E) \subset \partial(W(E)).$$

Next let $\Lambda \in \partial(W(E))$. Since by lemma 1.2 $W(E \cup \partial E)$ is closed,

$$\Lambda \in W(E \cup \partial E).$$

Thus $\mathscr{P}(\Lambda) \cap (E \cup \partial E)$ is non-empty. Since by lemma 1.2 $W(E)$ is open, $W(E)$ and $\partial(W(E))$ are disjoint. Hence $\Lambda \notin W(E)$ and consequently $\mathscr{P}(\Lambda) \cap E$ is empty. Therefore $\mathscr{P}(\Lambda) \cap \partial E$ is non-empty. Hence $\Lambda \in W(\partial E)$, which shows that

$$W(\partial E) = \partial(W(E)).$$ □

(1.5) PROPOSITION. *Let $\{E_k\}_1^\infty$ be a sequence of subsets of $\mathbb{R}^2$ such that $E_{k+1} \subset E_k$ for all k. Suppose that E_1 is bounded. Then*

$$W\left(\bigcap_i^\infty E_k\right) = \bigcap_1^\infty W(E_k).$$

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Proof. Evidently $W(\bigcap_1^\infty E_k)$ is contained in $\bigcap_i^\infty W(E_k)$. Now let $\Lambda \in \mathscr{L}$ be such that $\Lambda \in W(E_k)$ for all $k \in \mathbb{N}$; that is, $\mathscr{P}(\Lambda) \cap E_k$ is non-empty for all k. Since E_1 is bounded and $\mathscr{P}(\Lambda)$ is discrete the set $\mathscr{P}(\Lambda) \cap E_1$ is finite. Therefore

$$\mathscr{P}(\Lambda) \cap \left(\bigcap_1^\infty E_k\right)$$

cannot be empty unless $\mathscr{P}(\Lambda) \cap E_k$ is empty for all large k. Since the latter contradicts our supposition

$$\mathscr{P}(\Lambda) \cap \left(\bigcap_1^\infty E_k\right)$$

must be non-empty, i.e.

$$\Lambda \in W\left(\bigcap_1^\infty E_k\right). \qquad \square$$

2. *Invariant measures of the horocycle flow*

Let the notations be as in § 1. Further, let (u_t) be the one-parameter subgroup of G defined by

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Also let P be the subgroup of G consisting of all upper triangular matrices. The following lemma describes the set of periodic points of the flow defined by the action of (u_t) on G/Γ, on the left.

(2.1) LEMMA. *The element $g\Gamma \in G/\Gamma$, where $g \in P\Gamma \subset G$, is a periodic point of the flow defined by the action of (u_t) on G/Γ.*

Proof. Since $u_1 \in \Gamma$, as an element of G/Γ, Γ is a periodic point of the flow. Next let $g = p\gamma$ where $p \in P$ and $\gamma \in \Gamma$. Then for any t we have

$$u_t p\gamma\Gamma = u_t p\Gamma = p(p^{-1}u_t p)\Gamma = pu_{\alpha t}\Gamma$$

where α is a certain non-zero real number depending only on p. This shows that $u_\beta g\Gamma = g\Gamma$, where $\beta = |\alpha|^{-1}$. Thus $g\Gamma$ is a periodic point whenever $g \in P\Gamma$. $\square$

Conversely, it is known that for any $g \notin P\Gamma$ the orbit of $g\Gamma \in G/\Gamma$ under the action of (u_t) is dense in G/Γ and in particular not periodic (cf. [**4**] for a more general result). In the sequel, we shall however not need this information; we show independently that the orbits in question are uniformly distributed which is clearly a stronger assertion.

Recall the identification of G/Γ with $\mathscr{L}$ as in § 1. It is straightforward to verify that under the identification the set

$$\{g\Gamma \in G/\Gamma \mid g \in P\Gamma\}$$

corresponds to the subset $\mathscr{L}_0$ defined by

(2.2) $$\mathscr{L}_0 = \left\{\Lambda \in \mathscr{L} \,\middle|\, \begin{matrix}\text{There exists } \lambda \in \Lambda, \lambda \neq 0 \\ \text{such that } u_t\lambda = \lambda \text{ for all } t \in \mathbb{R}\end{matrix}\right\}.$$

That is, $\mathscr{L}_0$ is the set of those lattices which have some non-zero element common with the 'x-axis', the latter being the set of points fixed by any u_t, $t \neq 0$.

Lemma 2.1 may thus be restated as follows.

(2.3) LEMMA. *Any $\Lambda \in \mathscr{L}_0$ is a periodic point for the action of (u_t) on $\mathscr{L}$.*

The proof of uniform distribution depends on the following classification of (u_t)-invariant measures.

(2.4) THEOREM. *Let π be a (u_t)-invariant ergodic measure on G/Γ. Then either π is G-invariant or it is a (u_t)-invariant measure supported on the periodic orbit of an element $g\Gamma$ where $g \in P\Gamma$. If π is a (u_t)-invariant measure such that $\pi(P\Gamma/\Gamma) = 0$ then π is G-invariant.*

The first part of the assertion is simply the particular case of theorem 6.1 in [**3**] for $G = \mathrm{SL}(2, \mathbb{R})$, and $\Gamma = \mathrm{SL}(2, \mathbb{Z})$; it may be noted in this connection that in our present special case $Pq\Gamma = P\Gamma$ for any rational matrix q in $G = \mathrm{SL}(2, \mathbb{R})$. The second part of the assertion may be deduced from the first, using theorem 4.1 in [**2**] and ergodic decomposition of a finite (u_t)-invariant measure as a direct integral of ergodic invariant measures. We also note that a proof of theorem 2.4 for a *finite* (u_t)-invariant (actually this is enough for the purpose of the present paper) is also essentially contained in [**1**]. □

(2.5) THEOREM. *Let π be a (u_t)-invariant measure on $\mathscr{L}$ such that $\pi(\mathscr{L}_0) = 0$ then π is G-invariant.*

Proof. This follows from theorem 2.4 and the fact that under the identification of G/Γ with $\mathscr{L}$ the set $P\Gamma$ corresponds to $\mathscr{L}_0$. □

3. *Time averages of continuous functions*

Let X be the one-point compactification of $\mathscr{L}$, the extra point being denoted by ∞. The action of (u_t) on $\mathscr{L}$ extends to a continuous flow on X with ∞ as a fixed point. We shall denote the flow by (ϕ_t); thus for all $t \in \mathbb{R}$ $\phi_t(\Lambda) = u_t\Lambda$ for all $\Lambda \in \mathscr{L}$ and $\phi_t(\infty) = \infty$. Also in the sequel the notation $W(E)$, $E \subset \mathbb{R}^2$ as in §1, shall be considered modified to include ∞ in $W(E)$ whenever 0 is a limit point of E. The main part of the proof of uniform distribution lies in proving the following.

(3.1) THEOREM. *Let $\Lambda \in \mathscr{L} - \mathscr{L}_0 \subset X$. Then for any continuous function f on X, as $s \to \infty$*

$$\frac{1}{s}\int_0^s f(\phi_t\Lambda)\,dt \to \int_X f\,d\mu$$

where μ is the probability measure on X such that $\mu(\{\infty\}) = 0$ and the restriction to $\mathscr{L}$ is the G-invariant probability measure on $\mathscr{L}$.

The proof of the theorem is divided into several steps.

(3.2) LEMMA. *Let $\{\sigma_j\}$ be a sequence of probability measures on a compact second countable space Z, converging in the weak* topology to a probability measure σ. Let Ω be an open subset of Z and let $\partial\Omega$ be its boundary. Suppose that $\sigma(\partial\Omega) = 0$. Then as $j \to \infty$, $\sigma_j(\Omega)$ converges to $\sigma(\Omega)$.*

Proof. Recall that convergence of σ_j to σ in weak* topology means that for any continuous function f on Z, $\int f\,d\sigma_j$ converges to $\int f\,d\sigma$. Let $\varepsilon>0$ be arbitrary. By inner regularity of σ there exists a continuous function f such that

$$0\le f(z)\le 1 \quad \text{for all } z\in Z, \qquad f(z)=0 \quad \text{for all } z\in Z-\Omega$$

and

$$\sigma(\Omega)\le\int f\,d\sigma+\varepsilon.$$

Thus

$$\begin{aligned}\sigma(\Omega)&\le\int f\,d\sigma+\varepsilon\\ &=\lim_{j\to\infty}\int f\,d\sigma_j+\varepsilon\\ &\le\liminf_{j\to\infty}\sigma_j(\Omega)+\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary we get that

$$\sigma(\Omega)\le\liminf_{j\to\infty}\sigma_j(\Omega).$$

Since this is true for

$$\Omega'=X-\bar{\Omega}=X-(\Omega\cup\partial\Omega)$$

in the place of Ω and $\sigma(\partial\Omega)=0$ we have

$$\begin{aligned}\sigma(\Omega)=1-\sigma(\Omega')&\ge 1-\liminf_{j\to\infty}\sigma_j(\Omega')\\ &\ge\limsup_{j\to\infty}\,(1-\sigma_j(\Omega'))\\ &\ge\limsup_{j\to\infty}\sigma_j(\Omega).\end{aligned}$$

The last inequality follows from the fact that for any j,

$$\sigma_j(\Omega)+\sigma_j(\Omega')=\sigma_j(\Omega\cup\Omega')\le 1.$$

Combining the two inequalities for $\sigma(\Omega)$ we deduce the assertion in the lemma. □

Now for any $s>0$ let π_s be the probability measure on X such that for any continuous function f on X

$$\int_X f\,d\pi_s=\frac{1}{s}\int_0^s f(\phi_t\Lambda)\,dt \tag{3.3}$$

where Λ is a fixed lattice in $\mathscr{L}-\mathscr{L}_0$, as in the statement of theorem 3.1. Recall that the space $\mathscr{M}(X)$ of probability measures on X, equipped with weak* topology, is a compact second countable space. Thus for any $j\in\mathbb{N}$,

$$L_j=\overline{\{\pi_s\,|\,s>j\}}$$

(bar overhead denotes closure with respect to the weak* topology) is a compact subset of $\mathscr{M}(X)$. Further L_j is a decreasing sequence and consequently $L=\bigcap L_j$ is a non-empty compact subset of $\mathscr{M}(X)$.

Arguing as in the standard proof of the Markov–Kakutani theorem it is easy to verify that each π in L is a (ϕ_t)-invariant measure on X. In what follows, through a sequence of steps we shall show that L consists of only one element; namely μ as in the statement of theorem 3.1. The theorem readily follows once the last assertion is proved.

Now let π be an arbitrarily chosen element of L. Then evidently there exists an increasing sequence $\{s_j\}$ of positive real numbers such that $s_j \to \infty$ and $\pi_{s_j} \to \pi$ in the weak* topology. In the sequel the sequence $\{s_j\}$ shall be considered fixed.

In the sequel, we shall use the following notation. Let $\langle e_1 \rangle$ be the subspace of $\mathbb{R}^2$ generated by the basis vector e_1; i.e. the 'x-axis'. By an interval I on $\langle e_1 \rangle$ we mean a set of the form

$$\{\alpha e_1 | a \le \alpha \le b\}$$

where $a, b \in \mathbb{R}$ and $a \le b$; in this case $b - a$ is called the length of I and is denoted by $l(I)$. For any interval

$$I = \{\alpha e_1 | a \le \alpha \le b\}$$

and $\delta > 0$ we put

$$B(I, \delta) = \{\alpha e_1 + \beta e_2 | a - \delta < \alpha < b + \delta \text{ and } |\beta| < \delta\},$$
$$Q(I, \delta) = \{x \in \mathbb{R}^2 | x \notin B(I, \delta) \text{ and } u_t x \in B(I, \delta) \text{ for some } t > 0\},$$
$$R(I, \delta) = \mathbb{R}^2 - (B(I, \delta) \cup Q(I, \delta)).$$

For any set S in $\mathbb{R}^2$ we shall denote by χ_S the characteristic function of S on $\mathbb{R}^2$. For any $x \in \mathbb{R}^2$ we shall denote by $\xi(x)$ and $\eta(x)$ the e_1 and e_2 coordinates of x, respectively; that is,

$$x = \xi(x)e_1 + \eta(x)e_2.$$

(3.4) LEMMA. *Let I be an interval on $\langle e_1 \rangle$ and $\delta > 0$. Let $\{x_k\}$ be a sequence in $\mathscr{P}(\Lambda)$ and $\{t_k\}$ be a sequence in $\mathbb{R}$ such that $t_k \to \infty$ as $k \to \infty$. Suppose that*

$$u_{t_k}(x_k) \in B(I, \delta) \quad \textit{for all } k.$$

Then

$$t_k |\eta(x_k)| \to \infty \quad \textit{as } k \to \infty.$$

Proof. Let $a, b \in \mathbb{R}$ be such that

$$I = \{\alpha e_1 | a \le \alpha \le b\}.$$

Since $u_{t_k}(x_k) \in B(I, \delta)$ we have

$$a - \delta < \xi(x_k) + t_k \eta(x_k) < b + \delta \quad \text{and} \quad |\eta(x_k)| < \delta. \tag{3.5}$$

Hence to prove the lemma clearly it is enough to prove that $|\xi(x_k)| \to \infty$ as $k \to \infty$. Suppose this is false; then passing to a subsequence if necessary, we may assume that $|\xi(x_k)|$ is bounded, say $|\xi(x_k)| \le M$ for all k. Then by (3.5), $|t_k \eta(x_k)|$ must also be bounded and since $t_k \to \infty$ we have $|\eta(x_k)| \to 0$. Both coordinates being bounded, $\{x_k\}$ must be contained in a compact subset of $\mathbb{R}^2$. Since $\{x_k, k \in \mathbb{N}\}$ is also contained in the discrete set $\mathscr{P}(\Lambda)$ it must be finite. Since $\mathscr{P}(\Lambda)$ does not contain any element on $\langle e \rangle_1$, in particular this contradicts the fact that $|\eta(x_k)| \to 0$. Hence the lemma is proved. □

(3.6) LEMMA. *For any interval I on $\langle e_1\rangle$ there exists $\varepsilon(I)>0$ such that the following assertions hold*:

(i) *if* $0\notin I$ *then* $0\notin B(I, 2\varepsilon(I))$,

(ii) $m(C(B(I,\varepsilon(I))))<\frac{1}{2}$,

(iii) *for any* $\Delta\in\mathscr{L}$, $\mathscr{P}(\Delta)\cap B(I,\varepsilon(I))\subset\{\pm x\}$ *for some* x.

Proof. Existence of $\varepsilon(I)$ satisfying conditions (i) and (ii) is obvious. Condition (iii) follows from condition (ii) and lemma 1.3. □

An interval I on $\langle e_1\rangle$ is said to be *admissible* if either

$$I=\{\alpha e_1 | a\le\alpha\le b\}$$

where $0<a\le b$ or $I=\{0\}$. We shall denote by $\mathscr{A}$ the set of all admissible intervals on $\langle e_1\rangle$.

(3.7) LEMMA. *Let $I\in\mathscr{A}$ and let $\varepsilon(I)>0$ be as in lemma 3.6. Then the following conditions hold*:

(iv) *if* $I\ne\{0\}$ *then for any* $\Delta\in\mathscr{L}$,

$$\mathscr{P}(\Delta)\cap B(I,\varepsilon(I))$$

contains at most one element,

(v) *if* $I=\{0\}$ *then for any* $0<\delta<\varepsilon(I)$,

$$\mathscr{P}(\Delta)\cap B(I,\delta)$$

is either empty or equals $\{\pm x\}$ *for some* x.

Proof. Condition (iv) follows from conditions (i) and (iii) as in lemma 3.6. Condition (v) follows from condition (ii) as in lemma 3.6 and the fact that

$$\mathscr{P}(\Delta)\cap B(I,\delta)$$

is symmetric (contains the negative of any of its elements). □

(3.8) LEMMA. *Let $I\in\mathscr{A}$ and $\varepsilon(I)>0$ be as in lemma 3.6. Then there exists a set $D(I)$ of positive real numbers such that the following conditions hold*:

(vi) $D(I)\subset[0,\varepsilon(I)]$ *and* $[0,\varepsilon(I)]-D(I)$ *is countable,*

(vii) *for any* $\delta\in D(I)$, $\pi(\partial W(B(I,\delta)))=0$. *(Note that though ∞ may belong to $W(B(I,\delta))$ it is never a boundary point of the set.)*

Proof. Observe that the sets

$$\{\partial B(I,\delta)\}_{0<\delta<\varepsilon(I)}$$

are pairwise disjoint. Further, for any $I\in\mathscr{A}$ and any $\delta_1<\delta_2<\varepsilon(I)$, $\partial B(I,\delta_1)$ is also disjoint from $-\partial B(I,\delta_2)$, the set of negatives. Hence by condition (ii) as in lemma 3.6 and lemma 1.3 the sets

$$\{W(\partial B(I,\delta))\}_{0<\delta<\varepsilon(I)}$$

are pairwise disjoint. Put

$$D(I)=\{\delta | 0<\delta<\varepsilon(I)\text{ and }\pi(W(\partial B(I,\delta)))=0\}.$$

Since π is a probability measure there could be only countably many mutually disjoint sets of positive π-measure. Hence in view of the above disjointness assertion, condition (vi) must hold. Again, for any δ clearly $B(I,\delta)$ and $-\partial B(I,\delta)$ are

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disjoint. Hence by condition (ii) in lemma 3.6 and proposition 1.4 we have

$$\partial W(B(I,\delta)) = W(\partial B(I,\delta)).$$

Hence for all $\delta \in D(I)$ condition (vii) holds. □

(3.9) PROPOSITION. *Let $I \in \mathcal{A}$ and $\delta \in D(I)$. Put $B = B(I,\delta)$, $Q = Q(I,\delta)$ and $R = R(I,\delta)$. Let χ_B, χ_Q and χ_R be the characteristic functions of B, Q and R respectively on $\mathbb{R}^2$. Let τ_I be the function on $\mathbb{R}^2 - \langle e_1\rangle$ defined by $\tau_I(x) \equiv 1$ if $I \neq \{0\}$ and*

$$\tau_{\{0\}}(x) = \tfrac{1}{2}(1 + |\eta(x)|^{-1}\eta(x)).$$

Then

$$\pi(W(B)) = \lim_{j\to\infty} \frac{(l(I)+2\delta)}{s_j} \sum_{x\in\mathcal{P}(\Lambda)} \tau_I(x)\chi_Q(x)\chi_R(u_{s_j}x)|\eta(x)^{-1}|.$$

Proof. By condition (vii) in lemma 3.8 and lemma 3.2 we have

$$\pi(W(B)) = \lim_{j\to\infty} \pi_{s_j}(W(B)) = \lim_{j\to\infty} \frac{1}{s_j} l(E_j) \tag{3.10}$$

where l is the standard Lebesgue measure on $\mathbb{R}$ and

$$E_j = \{t \mid 0 \le t \le s_j \text{ and } u_t\Lambda \in W(B)\}$$

for all $j \in \mathbb{N}$. For any $x \in \mathcal{P}(\Lambda)$ and $j \in \mathbb{N}$ put

$$E_j^x = \{t \mid 0 \le t \le s_j \text{ and } u_t x \in B\}.$$

From the definition of $W(B)$ we see that for each $j \in \mathbb{N}$,

$$E_j = \bigcup_{x\in\mathcal{P}(\Lambda)} E_j^x.$$

If $I \neq \{0\}$ then by condition (iv) in lemma 3.7 for each j the sets

$$\{E_j^x\}_{x\in\mathcal{P}(\Lambda)}$$

are pairwise disjoint. If $I = \{0\}$ then by condition (v) in lemma 3.7 for each j the sets

$$\{E_j^x\}_{x\in\mathcal{P}(\Lambda),\tau_I(x)=1}$$

are pairwise disjoint and cover E_j. Since $\tau_I(x) \equiv 1$ if $I \neq \{0\}$, in either case we have

$$l(E_j) = \sum_{x\in\mathcal{P}(\Lambda)} \tau_I(x) l(E_j^x) \tag{3.11}$$

for all $j \in \mathbb{N}$.

It is straightforward to verify, preferably by drawing a picture of the (u_t) orbits on $\mathbb{R}^2$, that for any

$$x \in \mathcal{P}(\Lambda) \subset \mathbb{R}^2 - \langle e_1\rangle$$

and $j \in \mathbb{N}$, $l(E_j^x)$ satisfies the following conditions:

$$\begin{aligned} l(E_j^x) &= (l(I)+2\delta)|\eta(x)^{-1}| && \text{if } x \in Q \cap (u_{-s_j}R) \\ &= 0 && \text{if } x \in \{Q \cap u_{-s_j}Q\} \cup R \end{aligned} \tag{3.12}$$

and

$$0 \le l(E_j^x) \le (l(I)+2\delta)|\eta(x)^{-1}| \quad \text{if } x \in B \cup (u_{-s_j}B). \tag{3.13}$$

Those enumerated in (3.12) and (3.13) indeed cover all the possibilities for $x \in \mathbb{R}^2$.

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Substituting (3.11) and (3.12) in (3.10) we get

$$\pi(W(B)) = \lim_{j\to\infty} \frac{1}{s_j}\left\{ \sum_{x\in\mathcal{P}(\Lambda)} \tau_I(x)\chi_Q(x)\chi_R(u_{s_j}x)(l(I)+2\delta)|\eta(x)^{-1}| + \sum_{x\in\mathcal{P}(\Lambda)\cap\{B\cup u_{-s_j}B\}} \tau_I(x)l(E_j^x)\right\}.$$

The proposition would therefore be proved if we show that as $j\to\infty$

$$\frac{1}{s_j}\left\{\sum_{x\in\mathcal{P}(\Lambda)\cap\{B\cup u_{-s_j}B\}} \tau_I(x)l(E_j^x)\right\}\to 0.$$

Since the contribution from the elements in $\mathcal{P}(\Lambda)\cap B$ is independent of j and $s_j\to\infty$, in view of (3.13) it is enough to prove the convergence to 0 as $j\to\infty$ of the sequence $\{\theta_j\}$ defined by

(3.14) $$\theta_j = \frac{1}{s_j}\sum_{x\in\mathcal{P}(\Lambda)} \tau_I(x)\chi_B(u_{s_j}x)|\eta(x)^{-1}|.$$

By conditions (iv) and (v) in lemma 3.7, for any $j\in\mathbb{N}$ there exists at most one element $x\in\mathcal{P}(\Lambda)$ such that $\tau_I(x)\neq 0$ and $u_{s_j}x\in B$. Let Z be the set of j for which such an element does exist and for $k\in Z$ let $x_k\in\mathcal{P}(\Lambda)$ be the unique element such that $\tau_I(x_k)=1$ and $u_{s_k}x_k\in B$. Then clearly

$$\theta_j = |s_j\eta(x_j)|^{-1} \text{ if } j\in Z$$

and $\theta_j=0$ otherwise. Thus if Z is bounded θ_j is indeed eventually 0. If Z is unbounded, by lemma 3.4 $\theta_k\to 0$ as $k\to\infty$ in Z. In either case $\theta_j\to 0$ as $j\to\infty$, thus proving the proposition. □

(3.15) PROPOSITION. *Let $I_1, I_2\in\mathcal{A}$ be such that*

$$l(I_1)=l(I_2)=c>0.$$

Then

$$\pi(W(I_1))=\pi(W(I_2)).$$

Proof. Since $l(I_1)=l(I_2)>0$ there exists an admissible interval $I_0\in\mathcal{A}$ such that $I_1\cup I_2\subset I_0$. Put

$$D=D(I_1)\cap D(I_2)\cap[0,\varepsilon(I_0)]$$

and let $\delta\in D$. For $i=0$, 1 and 2 let $B_i=B(I_i,\delta)$, $Q_i=Q(I_i,\delta)$ and $R_i=R(I_i,\delta)$. By proposition 3.9 we have for $i=1,2$,

(3.16) $$\pi(W(B_i)) = \lim_{j\to\infty}\frac{(c+2\delta)}{s_j}\sum_{x\in\mathcal{P}(\Lambda)}\chi_{Q_i}(x)\chi_{R_i}(u_{s_j}x)|\eta(x)^{-1}|.$$

It is straightforward to verify that the sets $Q_1\triangle Q_2$ and $R_1\triangle R_2$ (where $\triangle$ stands for symmetric difference of sets) are contained in B_0. Hence for any $x\in\mathcal{P}(\Lambda)$ and $j\in\mathbb{N}$ we have

(3.17) $$\begin{aligned}|\chi_{Q_1}(x)\chi_{R_1}(u_{s_j}x)-\chi_{Q_2}(x)\chi_{R_2}(u_{s_j}x)| &\le \chi_{Q_2\triangle Q_1}(x)+\chi_{R_1\triangle R_2}(u_{s_j}x)\\ &\le \chi_{B_0}(x)+\chi_{B_0}(u_{s_j}x).\end{aligned}$$

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Thus from (3.16) and (3.17) we get that

$$(3.18)\quad |\pi(W(B_1))-\pi(W(B_2))|\leq \liminf_{j\to\infty}\frac{(c+2\delta)}{s_j}\sum_{x\in\mathscr{P}(\Lambda)}|\eta(x)^{-1}|(\chi_{B_0}(x)+\chi_{B_0}(u_{s_j}x)).$$

Evidently

$$\frac{1}{s_j}\sum_{x\in\mathscr{P}(\Lambda)}|\eta(x)^{-1}|\chi_{B_0}(x)\to 0\quad \text{as } j\to\infty.$$

On the other hand since $\delta\leq\varepsilon(I_0)$ the same argument as was used to show that the sequence $\{\theta_j\}$ in (3.14) converges to 0, now shows that

$$\frac{1}{s_j}\sum_{x\in\mathscr{P}(\Lambda)}|\eta(x)^{-1}|\chi_{B_0}(u_{s_j}x)\to 0\quad \text{as } j\to\infty.$$

Hence (3.18) implies that

$$\pi(W(B_1))=\pi(W(B_2)).$$

That is,

$$\pi(W(B(I_1,\delta)))=\pi(W(B(I_2,\delta)))\quad \text{for all } \delta\in D.$$

Evidently D contains all but countably many positive numbers in some neighbourhood of 0. In particular, there exists a decreasing sequence $\{\delta_k\}$ in D such that $\delta_k\to 0$. Since for $i=1$ and 2,

$$W(I_i)=\bigcap_{k=1}^{\infty}W(B(I_i,\delta_k))$$

in view of proposition 1.5, we get

$$\pi(W(I_1))=\lim\pi(W(B(I_1,\delta_k)))=\lim\pi(W(B(I_2,\delta_k)))=\pi(W(I_2)).\qquad\square$$

(3.19) COROLLARY. $\pi(\mathscr{L}_0)=0$.

Proof. Since $\mathscr{L}_0$ may be expressed as a countable union of sets of the form $W(I)$, where $I\in\mathscr{A}$ and $I\neq\{0\}$ it is enough to prove that $\pi(W(I))=0$ for all $I\in\mathscr{A}$, $I\neq\{0\}$. Let $I\in\mathscr{A}$ and $I\neq\{0\}$ and let $c=l(I)$. For each $k\in\mathbb{N}$ put

$$I_k=\{v+2kce_1\,|\,v\in I\}.$$

Then for all $k\in\mathbb{N}$, $I_k\in\mathscr{A}$ and $l(I_k)=l(I)=c$. Further $\{I_k\}_{k\in\mathbb{N}}$ are pairwise disjoint. Evidently this implies that $\{W(I_k)\}_{k\in\mathbb{N}}$ are pairwise disjoint subsets of $\mathscr{L}_0$. But by proposition 3.15 for any $k\in\mathbb{N}$,

$$\pi(W(I_k))=\pi(W(I)).$$

Since π is a probability measure this is not possible unless $\pi(W(I))=0$. $\square$

(3.20) PROPOSITION. $\pi(\{\infty\})=0$.

Proof. Let $I_1=\{0\}\in\mathscr{A}$. By the Mahler criterion (cf. [**8**, corollary 10.9])

$$\{W(B(I_1,\delta))\}_{\delta>0}$$

is a fundamental system of neighbourhoods of ∞ in X. Hence for any decreasing sequence $\{\delta_k\}$ such that $\delta_k\to 0$,

$$\pi(W(B(I_1,\delta_k)))\to\pi(\{\infty\})\quad \text{as } k\to\infty.$$

Let $p = ae_1$ where $a > 0$ and let $I_2 = \{p\}$. Let I_0 be the interval

$$\{\alpha e_1 | 0 \le \alpha \le a\}.$$

Let

$$D = D(I_1) \cap D(I_2) \cap [0, \varepsilon(I_0)]$$

and $\delta \in D$. For $i = 0$, 1 and 2 let $B_i = B(I_i, \delta)$, $Q_i = Q(I_i, \delta)$ and $R_i = R(I_i, \delta)$. Let τ be the function on $\mathbb{R}^2 - \langle e_1 \rangle$ defined by $\tau(x) = 1$ if $\eta(x) > 0$ and $\tau(x) = 0$ if $\eta(x) < 0$. By proposition 3.9 we have

$$\pi(W(B_1)) = \lim_{j\to\infty} \frac{2\delta}{s_j} \sum_{x \in \mathcal{P}(\Lambda)} \tau(x)\chi_{Q_1}(x)\chi_{R_1}(u_{s_j}x)|\eta(x)^{-1}| \tag{3.21}$$

and

$$\pi(W(B_2)) \ge \liminf_{j\to\infty} \frac{2\delta}{s_j} \sum_{x \in \mathcal{P}(\Lambda)} \tau(x)\chi_{Q_2}(x)\chi_{R_2}(u_{s_j}x)|\eta(x)^{-1}|. \tag{3.22}$$

Clearly $Q_1 \triangle Q_2$ and $R_1 \triangle R_2$ are contained in B_0. Hence for $x \in \mathcal{P}(\Lambda)$ and $j \in \mathbb{N}$ we have

$$|\chi_{Q_1}(x)\chi_{R_1}(u_{s_j}x) - \chi_{Q_2}(x)\chi_{R_2}(u_{s_j}x)| \le \chi_{B_0}(x) + \chi_{B_0}(u_{s_j}x).$$

Since $\delta \le \varepsilon(I_0)$, as in the proofs of propositions 3.9 and 3.15 using lemma 3.4 we can deduce from the above data that as $j \to \infty$

$$\frac{1}{s_j}\left| \sum_{x \in \mathcal{P}(\Lambda)} \tau(x)|\eta(x)^{-1}|(\chi_{Q_1}(x)\chi_{R_1}(u_{s_j}x) - \chi_{Q_2}(x)\chi_{R_2}(u_{s_j}x))\right| \to 0. \tag{3.23}$$

In view of (3.23) the relations (3.21) and (3.22) imply that

$$\pi(W(B_1)) \le \pi(W(B_2)).$$

That is,

$$\pi(W(B(I_1, \delta))) \le \pi(W(B(I_2, \delta)))$$

for any $\delta \in D$. Applying this to a sequence $\{\delta_k\}$ in D such that $\delta_k \to 0$ and using lemma 3.2 we deduce that

$$\pi(\{\infty\}) \le \pi(W(I_2)).$$

But since $W(I_2) \subset \mathcal{L}_0$, by corollary 3.19 $\pi(W(I_2)) = 0$. Hence $\pi(\{\infty\}) = 0$. □

Proof of theorem 3.1. In view of corollary 3.19, proposition 3.20 and theorem 2.5, no measure other than the measure μ as in the statement of theorem 3.1 belongs to L. Since L is non-empty we get $L = \{\mu\}$. Thus for any sequence $\{s_j\}$ such that $s_j \to \infty$ the measures π_{s_j} defined by (3.3) converge to μ in the weak* topology. Therefore for any continuous function on X the contention of the theorem holds. □

4. *Invariant measures of horocycle transformations*

As before let $G = SL(2, \mathbb{R})$, $\Gamma = SL(2, \mathbb{Z})$ and P be the subgroup consisting of all upper triangular matrices in G. Let $u \in G$ be the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The aim of this section is to prove the following analogue of theorem 2.4 for the cyclic subgroup generated by u.

(4.1) THEOREM. *Let σ be a measure on G/Γ which is invariant under the action (on the left) of u on G/Γ. Suppose that $\sigma(P\Gamma/\Gamma)=0$. Then π is G-invariant.*

There is a well-known duality, introduced by H. Furstenberg [**5**], which (for the case at hand) gives a natural 1–1 correspondence between H-invariant measures on G/Γ and Γ-invariant measures on G/H, where H is any closed unimodular subgroup of G (cf. [**3**, § 1] for details regarding the correspondence). Because of the duality, to prove theorem 4.1 it is enough to prove the following.

(4.2) THEOREM. *Let σ be a Γ-invariant measure on G/U where U is the cyclic subgroup generated by u. Suppose that $\sigma(\Gamma P/U)=0$. Then σ is G-invariant.*

Proof. To begin with we note that in view of the duality as mentioned above, now for the subgroup

$$N=\left\{\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\middle|\, t\in\mathbb{R}\right\}$$

for H, the latter part of theorem 2.4 implies the following: If ρ is a Γ-invariant measure on G/N and $\rho(\Gamma P/N)=0$ then ρ is G-invariant. We shall now deduce theorem 4.2 from this.

Let C_c^+ be the space of non-negative continuous functions on N having compact support. For $\phi\in C_c^+$ let σ_ϕ be the measure on G/U defined by

$$\sigma_\phi(E)=\int_N \sigma(\psi_n^{-1}E)\phi(n)\,dn$$

for any Borel set E, where dn is a fixed Haar measure on N and for $n\in\mathbb{N}$,

$$\psi_n: G/U\to G/U$$

is the homeomorphism defined by

$$\psi_n(gU)=gnU$$

for all $g\in G$. (Since U is normal in N this is well defined.)

It is well known (cf. [**7**, theorem 7]) that the Γ-action on G/U is ergodic with respect to the G-invariant measure λ (the latter is unique up to a scalar multiple). Under this condition and with the above notation proposition 2.5 in [**10**] asserts the following: If σ_ϕ is absolutely continuous with respect to λ for all $\phi\in C_c^+$ then σ is a multiple of λ, that is σ is G-invariant. Thus we only need to check that each σ_ϕ, $\phi\in C_c^+$ is absolutely continuous with respect to λ.

Let $\phi\in C_c^+$ and consider σ_ϕ. Let $\eta: G/U\to G/N$ be the map defined by $\eta(gU)=gN$ for all $g\in G$. Since N/U is compact η is a proper map. Therefore σ_ϕ projects under η to a (locally finite) measure $\eta(\sigma_\phi)$; we have

$$\eta(\sigma_\phi)(E)=\sigma_\phi(\eta^{-1}E)$$

for any Borel set E. It is easy to verify that if $\eta(\sigma_\phi)$ be absolutely continuous with respect to the (unique up to scalar) G-invariant measure on G/N then σ_ϕ is absolutely continuous with respect to λ. But $\eta(\sigma_\phi)$ is evidently a Γ-invariant measure on G/N and

$$\begin{aligned}\eta(\sigma_\phi)(\Gamma P/N) &= \sigma_\phi(\eta^{-1}(\Gamma P/N)) \\ &= \sigma_\phi(\Gamma P/U) \\ &= \int \sigma(\psi_n^{-1}\Gamma P/U)\phi(n)\,dn \\ &= \int \sigma(\Gamma P/U)\phi(n)\,dn = 0.\end{aligned}$$

Hence by the observation made in the beginning of the proof, $\eta(\sigma_\phi)$ is indeed a G-invariant measure itself. Hence σ_ϕ is absolutely continuous for any $\phi \in C_c^+$ and consequently σ is G-invariant, thus proving theorem 4.2 (and therefore theorem 4.1 also). □

In terms of the identification of G/Γ and $\mathscr{L}$, as in § 2, theorem 4.1 may be restated as follows:

(4.3) THEOREM. *Let π be a measure on $\mathscr{L}$ which is invariant under the action of u. Suppose that $\pi(\mathscr{L}_0) = 0$, where $\mathscr{L}_0$ is the subset of $\mathscr{L}$ as defined in* § 2. *Then π is G-invariant.*

It may be noted that the same method as above can be applied to extend H. Furstenberg's result on the unique egodicity of the horocycle flow (corresponding to a compact surface of constant negative curvature) to the following.

(4.4) THEOREM. *Let D be a discrete subgroup of* SL (2, ℝ) *such that* SL (2, ℝ)/*D is compact. Let*

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then the action of u on SL (2, ℝ)/*D is uniquely ergodic; that is the* SL (2, ℝ)*-invariant probability measure is the only invariant probability measure.*

Similarly the results in [**10**] and [**3**] can be extended to invariant measures (on appropriate homogeneous spaces) of those subgroups U of a maximal horospherical subgroup N such that U is normal in N and N/U is compact.

5. *Time averages of functions* (*discrete time*)

Let the notation be as in § 3. We now prove the analogue of theorem 3.1 for the action of (iterates of) the single element

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(5.1) THEOREM. *Let X be the one-point compactification of $\mathscr{L}$ and let ϕ be the homeomorphism of X extending the action of u on $\mathscr{L}$. Let*

$$\Lambda \in \mathscr{L} - \mathscr{L}_0 \subset X.$$

Let f be any continuous function on X. Then as $n \to \infty$

$$\frac{1}{n}\sum_{j=0}^{n-1} f(\phi^j\Lambda) \to \int_X f\,d\mu$$

where μ is the probability measure on X such that $\mu(\mathscr{L}) = 1$ and the restriction to $\mathscr{L}$ is G-invariant.

Proof. For any n let ρ_n be the probability measure on X such that for any continuous function f

$$\int_X f\,d\rho_n = \frac{1}{n}\sum_{j=0}^{n-1} f(\phi^j\Lambda).$$

Let L' be the set of limit points of the sequence $\{\rho_n\}$. It is well known that any element of L' is a ϕ-invariant measure. As in the case of continuous time averages in § 3 we shall be through if we show that $L' = \{\mu\}$.

Let $\rho \in L'$ be arbitrary. There exists a sequence $\{n_k\}$ in $\mathbb{N}$ such that $n_k \to \infty$ and $\rho_{n_k} \to \rho$ in the weak* topology. Now let θ be the measure defined by

$$\theta(E) = \int_0^1 \rho(u_t E)\,dt$$

for any Borel subset E of X. Since ρ is invariant under $u_1 = u$ it follows that θ is a (u_t)-invariant measure. For any continuous function f on X we have

$$\begin{aligned}\int_X f\,d\theta &= \int_0^1 \int_X f(u_t x)\,d\rho(x)\,dt\\ &= \int_0^1 \lim_{k\to\infty} \int_X f(u_t x)\,d\rho_{n_k}(x)\,dt\\ &= \lim_{k\to\infty} \int_0^1 \int_X f(u_t x)\,d\rho_{n_k}(x)\,dt\\ &= \lim_{k\to\infty} \int_0^1 \left(\frac{1}{n_k}\sum_{j=0}^{n_k-1} f(u_t u^j\Lambda)\right) dt\\ &= \lim_{k\to\infty} \frac{1}{n_k}\int_0^{n_k} f(u_t\Lambda)\,dt\\ &= \int_X f\,d\mu\end{aligned}$$

where the last step follows from theorem 3.1. This being true for all continuous functions we get that $\theta = \mu$.

It is evident from the definition of θ that for any Borel subset E of X which is invariant under the action of the flow $\{\phi_t\}$ (extending the (u_t)-action on $\mathscr{L}$ to X) we have $\theta(E) = \rho(E)$. Since $\mathscr{L}_0$ and $\{\infty\}$ are clearly $\{\phi_t\}$-invariant we have

$$\rho(\mathscr{L}_0) = \theta(\mathscr{L}_0) = \mu(\mathscr{L}_0) = 0$$

and

$$\rho(\{\infty\}) = \theta(\{\infty\}) = \mu(\{\infty\}) = 0.$$

Therefore by theorem 4.1 $\rho = \mu$. Since $\rho \in L'$ was arbitrary we get that $L' = \{\mu\}$.

Since the space of probability measures is compact with respect to the weak* topology this means that ρ_n converges to μ in the weak* topology, which is precisely the contention of the theorem. □

6. *Conclusions and questions*

I. *Uniform distribution*

Theorems 3.1 and 5.1 mean that the orbits of elements $g\Gamma \in G/\Gamma$ where $g \notin P\Gamma$ under (u_t) or u respectively are 'uniformly distributed' in G/Γ. To illustrate this and bring it closer in form to what is more widely understood as uniform distribution we note the following consequence of theorems 3.1 and 5.1.

(6.1) THEOREM. *Let* $G = \mathrm{SL}(2, \mathbb{R})$, $\Gamma =$ *a subgroup of finite index in* $\mathrm{SL}(2, \mathbb{Z})$ *and* P *be the subgroup consisting of all upper triangular matrices in* G. *Let*

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and for $t \in \mathbb{R}$,

$$u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Let μ *be the* G*-invariant probability measure on* G/Γ. *Let* Ω *be any open subset of* G/Γ *such that* $\mu(\partial\Omega) = 0$ *where* $\partial\Omega$ *is the topological boundary of* Ω. *Let* χ_Ω *denote the characteristic function of* Ω. *Then for any* $x = g\Gamma \in G/\Gamma$ *where* $g \notin P\Gamma$

$$\frac{1}{T}\int_0^T \chi_\Omega(u_t x)\, dt \to \mu(\Omega) \quad \text{as } T \to \infty$$

and

$$\frac{1}{n}\sum_{j=0}^{n-1} \chi_\Omega(u^j x) \to \mu(\Omega) \quad \text{as } n \to \infty.$$

Proof. For $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ this follows from theorems 3.1 and 5.1 and lemma 3.2. The general case may be deduced from the fact that any u-invariant measure on G/Γ which projects to the G-invariant measure on $G/\mathrm{SL}(2, \mathbb{Z})$ is itself G-invariant. □

II. *Recurrent and generic points*

The class of dynamical systems for which all the points are recurrent/generic has attracted some attention in the literature (cf. [**6**] and other references therein). The homeomorphism ϕ of X as in § 5 (extending the u-action on G/Γ to its one-point compactification) provides a natural example of a topologically transitive homeomorphism for which these properties hold.

We recall that if ψ is a homeomorphism of a compact metric space Y it is said to be *topologically transitive* if there exists $y_0 \in Y$ such that

$$\{\psi^j y_0 \mid j \in \mathbb{Z}\}$$

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is dense in Y; further, if $y \in Y$ then (i) y is said to be *recurrent* if there exists a sequence $\{n_k\}$ such that $n_k \to \infty$ and $\psi^{n_k} y \to y$ and (ii) y is said to be *generic* if there exists a measure μ_y on Y such that for all continuous functions f on Y

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(\psi^j y) = \int f \, d\mu_y.$$

(6.2) THEOREM. *Let X be the one-point compactification of G/Γ (where $G = \mathrm{SL}(2, \mathbb{R})$ and $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, and let ϕ be the homeomorphism extending the action of*

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

to X. Then ϕ is topologically transitive and every point of X is both recurrent and generic with respect to ϕ.

Proof. By theorem 6.1 every $x = g\Gamma \in G/\Gamma$ where $g \notin P\Gamma$ is generic with respect to the G-invariant measure on G/Γ. Since the G-invariant measure assigns positive value to any open set, in particular we can deduce from the theorem that such an x is also recurrent. Similar argument also shows that ϕ is topologically transitive.

On the other hand if $x = g\Gamma$, where $g \in P\Gamma$, then by lemma 2.1 the (u_t)-orbit of x is periodic. The latter is therefore a ϕ-invariant circle and the restriction of ϕ is equivalent to a rotation of the circle in the usual sense. Hence every point on the circle including x is both generic and recurrent.

Finally, the point at infinity is evidently generic as well as recurrent, which completes the proof. □

In the light of various known results including those in [**2**] and the present paper it seems reasonable to conjecture the following:

CONJECTURE. *Let X be the one point compactification of*

$$\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$$

where $n \geq 2$ and let ϕ be the homeomorphism extending the action of a unipotent element u (i.e. $(u - I)^m = 0$ for some $m \geq 2$, I being the identity matrix) on the homogeneous space. Then every element of X is both generic and recurrent.

III. *An application to number theory*

For any $t \in \mathbb{R}$ let $[t]$ denote the largest integer not exceeding t and let

$$\{t\} = t - [t].$$

For any two positive integers m and n let (m, n) denote the g.c.d. of m and n.

(6.3) THEOREM. *For any irrational number θ*

$$\lim_{T\to\infty} \frac{1}{T} \sum_{\substack{0<m\leq T\{m\theta\} \\ (m,[m\theta])=1}} \{m\theta\}^{-1} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$$

where ζ stands for the Riemann zeta function.

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Proof. Let the notation be as in § 1. Further put $f_1=-c^{-1}e_1$ and $f_2=ce_2$ where $c>0$ is such that $c^2<\frac{1}{2}$. Let Λ be the lattice generated by $f_1+\theta f_2$ and f_2. Clearly $\Lambda\in\mathcal{L}-\mathcal{L}_0$. Put

$$\begin{aligned} S&=\{\alpha e_1+\beta e_2|-c<\alpha<0 \text{ and } 0<\beta<c\} \\ &=\{\rho f_1+\sigma f_2|0<\rho<c^2 \text{ and } 0<\sigma<1\}\end{aligned}\tag{6.4}$$

and let $\Omega=W(S)$. Since

$$m(C(S))=c^2<\tfrac{1}{2},$$

by proposition 1.4 $\partial\Omega=W(\partial S)$. Using a formula of Siegel, namely (25) in **[9]**, it is easy to see that

$$\mu(\Omega)=c^2/\zeta(2) \quad \text{and} \quad \mu(\partial\Omega)=\mu(W(\partial S))=0,$$

μ being the G-invariant probability measure on $\mathcal{L}$. Recall that we are identifying $\mathcal{L}$ with G/Γ and under the identification $\mathcal{L}_0$ corresponds to $P\Gamma$ as in the statement of theorem 6.1. Thus by theorem 6.1 we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \chi_\Omega(u_t\Lambda)\,dt=\mu(\Omega)=c^2/\zeta(2).\tag{6.5}$$

Now let $x\in\mathcal{P}(\Lambda)$ and suppose that there exists $t>0$ such that $u_tx\in S$. Since $x\in\mathcal{P}(\Lambda)$ there exist coprime integers m and n such that

$$x=m(f_1+\theta f_2)+nf_2=mf_1+(m\theta+n)f_2.$$

Then

$$u_tx=\{m-c^2t(m\theta+n)\}f_1+(m\theta+n)f_2.$$

Since $u_tx\in S$ for some $t>0$ from (6.4) we have

$$0<m\theta+n<1 \quad \text{and} \quad m>c^2t(m\theta+n).$$

The first inequality implies that

$$n=-[m\theta] \quad \text{and} \quad m\theta+n=\{m\theta\}$$

and the second, in particular, implies that $m\geq 1$. Thus

$$x=mf_1+\{m\theta\}f_2$$

for some $m\geq 1$ such that $(m,[m\theta])=1$. Conversely for any $m\geq 1$ such that $(m,[m\theta])=1$

$$x=mf_1+\{m\theta\}f_2$$

is a primitive element of Λ for which there exists $t>0$ such that $u_tx\in S$.

As in the proof of proposition 3.9 we see that for any $T>0$

$$\begin{aligned}\int_0^T \chi_\Omega(u_t\Lambda)\,dt&=\sum_{x\in\mathcal{P}(\Lambda),\eta(x)>0}\int_0^T \chi_S(u_tx)\,dt \\ &=\sum_{m\geq 1,(m,[m\theta])=1} l(E_T^m)\end{aligned}\tag{6.6}$$

where

$$E_T^m=\{t|0\leq t\leq T \text{ and } u_t(mf_1+\{m\theta\}f_2)\in S\}$$

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and l is the Lebesgue measure. A straightforward computation shows that

$$(6.7)\qquad \begin{aligned} l(E_T^m) &= \{m\theta\}^{-1} && \text{if } c^2T\{m\theta\} \geq m \\ &= 0 && \text{if } c^2T\{m\theta\} \leq m - c^2 \end{aligned}$$

and

$$(6.8)\qquad 0 \leq l(E_T^m) \leq \{m\theta\}^{-1} \quad \text{if } m - c^2 < c^2T\{m\theta\} < m.$$

Again, as before, since $m(C(S)) < \frac{1}{2}$, for any $T > 0$ there exists at most one $m \geq 1$, say m_T, such that $(m, [m\theta]) = 1$ and

$$m - c^2 < c^2T\{m\theta\} < m;$$

the latter is equivalent to

$$u_T(mf_1 + \{m\theta\}f_2) \in S.$$

Further, by lemma 3.4 along any sequence of T's tending to ∞, for which m_T exists $T\{m_T\theta\} \to \infty$. This together with (6.6), (6.7) and (6.8) implies that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \chi_\Omega(u_t\Lambda)\, dt = \lim_{T\to\infty} \frac{1}{T} \sum_{\substack{c^2T\{m\theta\}\geq m \\ (m,[m\theta])=1}} \{m\theta\}^{-1}.$$

Therefore by (6.5)

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{T} \sum_{\substack{1\leq m\leq T\{m\theta\} \\ (m,[m\theta])=1}} \{m\theta\}^{-1} &= \frac{1}{c^2} \lim_{T\to\infty} \frac{1}{T} \sum_{\substack{c^2T\{m\theta\}\geq m \\ (m,[m\theta])=1}} \{m\theta\}^{-1} \\ &= \mu(\Omega)/c^2 \\ &= 1/\zeta(2) = 6/\pi^2 \end{aligned}$$

which proves the theorem. □

While above we have deduced theorem 6.3 from theorem 6.1, conversely it turns out that the contention of theorem 6.3 together with theorem 2.4 implies theorem 6.1. Initially I attempted to prove theorem 6.3 directly and then deduce theorem 6.1. The question was discussed with number theorists. M. Ram Murty showed me a proof of theorem 6.3 under a certain additional condition on θ, involving the growth of the denominators of convergents of θ (in its continued fraction development). Using Roth's theorem the condition was shown to be true for all algebraic numbers. However, it was not possible to get a proof for all irrational θ. It would be of interest to know whether the theorem could indeed be proved directly.

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