THE LIMITING BEHAVIOR OF TRANSIENT BIRTH AND DEATH PROCESSES CONDITIONED ON SURVIVAL

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1. Introduction

The development of a population over time can often be simulated by the behavior of a birth and death process, whose transition probability matrix $P(t) = (P_{ij}(t))$,

$$P_{ij}(t) = \Pr\{X(t+s) = j | X(s) = i\} \qquad i, j = 0, 1, \cdots : t, s \geqq 0$$

where $X(t)$ denotes the number of individuals at time t, satisfies the differential equations

$$P'(t) = AP(t) \qquad t \geqq 0 \tag{1.1}$$

$$P'(t) = P(t)A \tag{1.2}$$

and the initial condition

$$P(0) = I. \tag{1.3}$$

The matrix $A = (a_{ij})$ in (1.1) is of the form

$$\begin{aligned} a_{i\,i+1} &= \lambda_i \\ a_{i\,i} &= -(\lambda_i+\mu_i) \\ a_{i\,i-1} &= \mu_i \\ a_{i\,j} &= 0 \quad \text{if} \quad |i-j| > 1, \end{aligned}$$

with

$$\begin{aligned} \mu_i &> 0 \quad \text{for} \quad i > 0 \\ \mu_0 &= 0 \\ \lambda_i &> 0 \quad \text{for} \quad i > 0 \end{aligned}$$

and

$$\lambda_0 = 0.$$

The condition of mortality, $\mu_i > 0$ for all i, implies that state zero is accessible from every state. A process absorbed at zero becomes extinct,

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because of the condition $\lambda_0 = 0$. Consequently, every non-zero finite state, j, is transient, that is

$$\lim P_{ij}(t) = 0 \qquad j = 1, 2, \cdots. \tag{1.4}$$

A series of computer simulations of such models were conducted by the author. In almost every realization, the population was not absorbed during the period of observation but instead appeared to grow to and oscillate about an equilibrium level. This level was reproducible and was a function of the parameters of the system. The purpose of the present note is to characterize this pseudo-equilibrium.

Consider the event that a population survives up to and including the time t

$$\{X(s) \neq 0,\ 0 \leqq s \leqq t\} = \{X(t) \neq 0\}$$

and the probability of transitions conditioned with respect to it

$$\begin{aligned} P^*_{ij}(t) &= \Pr\{X(t) = j | X(0) = i,\ X(s) \neq 0,\ 0 \leqq s \leqq t\} \\ &= P_{ij}(t)/(1-P_{i0}(t)). \end{aligned} \tag{1.5}$$

It is with the limits of these conditional transition probabilities that the present note is concerned. Theorem 3.1 of section 3 provides existence criteria and an explicit form for the limit. Theorem 3.2 is concerned with the existence of higher moments of the conditional distribution. Section 4 discusses the relation of the existence criteria to the moments of the distributions of absorption times and conditional recurrence times.

The proofs of these results are facilitated by the use of an integral representation discussed in the next section.

2. Representation formula

Associated with the matrix A is a system of polynomials satisfying the recurrence relations

$$\begin{aligned} &Q_0(x) \equiv 0,\ Q_1(x) \equiv 1, \\ &-xQ_n(x) = \mu_n Q_{n-1}(x) - (\lambda_n + \mu_n)Q_n(x) + \lambda_n Q_{n+1}(x) n \geqq 1. \end{aligned} \tag{2.1}$$

Karlin and McGregor (1957a) show that under certain conditions there exists a unique regular measure ψ on $0 \leqq x < \infty$ such that

$$\int Q_i(x)Q_j(x)d\psi = \delta_{ij}/\pi_j \qquad i, j = 0, 1, 2, \cdots \tag{2.2}$$

where

$$\pi_1 = 1,\ \pi_n = \lambda_1\lambda_2 \cdots \lambda_{n-1}/\mu_2\mu_3 \cdots \mu_n \text{ for } n > 1. \tag{2.3}$$

In particular, existence and uniqueness of the measure are assured if

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$$\sum_{n=1}^{\infty} \pi_n \sum_{k=1}^{n-1} 1/\lambda_k \pi_k = \infty. \tag{2.4}$$

A sufficient condition that there exist a unique Markov process with transition matrix $P(x)$ satisfying the relations (1.1)—(1.3) is

$$\sum 1/\lambda_n \pi_n = \infty. \tag{2.5}$$

Both these conditions will be assumed to hold in the sequel.

Accordingly, the transition matrix $P(t)$ may be represented by the formula

$$P_{ij}(t) = \pi_j \int \exp(-xt) Q_i(x) Q_j(x) d\psi(x) \qquad i, j \geqq 1. \tag{2.6}$$

As Karlin and McGregor (1957a) note, the usefulness of this integral representation (2.6) derives from the simple monotonic properties of exp $(-xt)$ and from the fact that the dependence on the time, the initial and the final states are separated in the integral.

3. Results

Transient Markov chains with a finite-state space were considered in Good (1965). An eigenvalue representation is possible and was used to demonstrate the explicit form of the limiting conditional distribution under very general conditions. When the transition matrix P is infinite, there may be no solutions of the characteristic equation and it is necessary to use the alternative representation 2.6 given in the preceding section.

An additional complication arises; a drift to infinity may occur in the general case. The divergence of $\sum 1/\lambda_n \pi_n$, assumed in the preceding section, is sufficient to assure that absorption at zero is certain.

If absorption at zero is certain, it can be shown (Theorem 1) that the conditional transition probabilities converge, $P^*(t) \to P^*$. There are still two possibilities:

either 1) $\sum P_j^* = 1$

or 2) $P_i^* = 0$ for all i.

A distinction between the two cases can be made on the basis of the value of a, the lower limit of the support of ψ. It is already known (Karlin and McGregor (1957b)), that a sufficient but not necessary condition for the expected time to absorption to be finite is that $a > 0$. In Theorem 1 it is shown that this is both a necessary and sufficient condition for P^* to be an honest distribution.

An initial, less intuitive result is required and is provided by the following lemma.

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LEMMA. *For a process satisfying* (2.4) *and* (2.5)

$$\cdot\, a > 0 \textit{ if and only if } \sum_n \pi_n Q_n(a) < \infty$$
$$\cdot\, \sum_n \pi_n Q_n(a) = \mu_1/a.$$

PROOF. Using the recurrence formula for $Q_n(x)$, (2.1), it is easy to show that

$$\begin{aligned} Q_n(x)-Q_{n-1}(x) &= xQ_{n-1}(x)/\lambda_{n-1}+[Q_{n-1}(x)-Q_{n-2}(x)]\mu_{n-1}\pi_{n-1} \\ &= -x\sum_{k=1}^{n-1} Q_k(x)\pi_k/\lambda_{n-1}\pi_{n-1}+\mu_1/\lambda_{n-1}\pi_{n-1}. \end{aligned}$$

Summing these differences

$$Q_n(x) = 1+\mu_1\sum_{k=1}^{n-1}(1/\lambda_k\pi_k)-x\sum_{j=1}^{n-1}(1/\lambda_j\pi_j)\sum_{k=1}^{j}\pi_k Q_k(x) \text{ for } n \geqq 1. \tag{3.2}$$

In particular, with $x = a$ and $n \geqq 1$, we have

$$Q_n(a) = 1+\sum_{j=1}^{n-1}(b_j/\lambda_j\pi_j)$$

where

$$b_j = \mu_1 - a\sum_{1}^{j}\pi_k Q_k(a) \qquad j = 1, \cdots.$$

It is known (Karlin and McGregor (1957a)) that $Q_n(a) > 0$ for all n; clearly the sequence b_j is monotone decreasing. Suppose, for some n, that $b_n < 0$. Since, by assumption, $\sum_j 1/\lambda_j\pi_j$ diverges $\sum_j b_j/\lambda_j\pi_j \to -\infty$. But $Q_n(a) > 0$; a contradiction. Hence $b_j \geqq 0$ for all j, so that from the definition of b_j, it is seen that

$$\sum \pi_k Q_k(a) \leqq \mu_1/a. \tag{3.3}$$

Now, suppose b_j converges to $b^* > 0$. Consider $\sum \pi_n Q_n(a)$. It is seen that from (3.2) that

$$\sum \pi_n Q_n(a) = \sum \pi_n + \sum \pi_n \sum_{j=1}^{n-1} b_j/\lambda_j\pi_j > b^* \sum \pi_n \sum_{j=1}^{n-1} 1/\lambda_j\pi_j.$$

By (2.4), $\sum \pi_n Q_n(a)$ diverges, a contradiction if $a > 0$; b_j must converge to zero, and

$$\sum_{n=1}^{\infty}\pi_n Q_n(a) = \mu_1/a \text{ if } a > 0.$$

If $a = 0$, then by (2.4) $\sum \pi_n Q_n(a)$ diverges.

Let a, as before, denote the lower limit of the support of ψ.

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THEOREM 1. *If* $a > 0$

$$\cdot\ P^*(t) \to P^*$$

that is

$$\lim_{t\to\infty} P^*_{ij}(t) = P^*_j = a\pi_j Q_j(a)/\mu_1$$

and

$$\cdot \sum P^*_j = 1.$$

COROLLARY. *If* $a = 0$, $P^*_j = 0$ *for all* j.

PROOF. Theorem 11 of Karlin and McGregor (1957b) states that $\lim_{t\to\infty} P_{ij}(t)/P_{mk}(t)$ exists and is finite and positive for $i, j, m, k > 0$. In particular,

$$\lim_{t\to\infty} P_{ij}(t)/P_{ik}(t) \to \pi_j Q_j(a)/\pi_k Q_k(a).$$

$$P^*_{ik}(t) = P_{ik}(t)/(1-P_{i0}(t))$$

$$= P_{ik}(t)/\sum_{j=1}^{\infty} P_{ij}(t);$$

$$\lim_{t\to\infty} \sum_{j=1}^{\infty} P_{ij}(t)/P_{ik}(t)$$

$$= \sum_{j=1}^{\infty} \lim_{t\to\infty} P_{ij}(t)/P_{ik}(t)$$

$$= \sum_{j=1}^{\infty} \pi_j Q_j(a)/\pi_k Q_k(a).$$

The interchange of the operations of limit and summation is permitted in the present case if $\sum_j \pi_j Q_j(a)$ converges. The conditions for convergence were established in the lemma.

Let

$$P^*_{ikn}(t) = P_{ik}(t)/\sum_{j=1}^{n} P_{kj}(t).$$

Let

$$P^*_{ikn} = \lim_{t\to\infty} P^*_{ikn}(t).$$

$$P^*_{ikn}(t) \downarrow P^*_{ik}(t) \text{ for all } t \text{ and } P^*_{ikn} \downarrow P^*_{ik}.$$

If $a = 0$, $\sum_j \pi_j Q_j(a)$ diverges; for all ε, we may find an n_1 such that $\varepsilon > P^*_{i1n_1} > P^*_{i1} \geqq 0$, an $n_2 > n_1$ such that $\varepsilon > P^*_{i2n_2} > P^*_{i2} \geqq 0$, and so forth. The corollary follows.

A stronger result is possible under certain conditions.

THEOREM 2. *If* $|\lambda_n| < C_1$, $|\mu_n| < C$ *for all* n *then if* $a > 0$

$$\sum P^*_k = 1 \text{ and } \sum k^p P^*_k < \infty \text{ for all } p = 0, 1, \cdots.$$

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PROOF. Multiplying the recurrence relation (2.1), by $\pi_n n^p$

$$-xn^p\pi_n Q_n(x) = n^p[\pi_n\mu_n Q_{n-1}(x) - \pi_n(\lambda_n+\mu_n)Q_n(x) + \pi_n\lambda_n Q_{n+1}(x)]$$
$$= n^p[\lambda_{n-1}\pi_{n-1}Q_{n-1}(x) - \pi_n(\lambda_n+\mu_n)Q_n(x) + \mu_{n+1}\pi_{n+1}Q_{n+1}(x)]$$

and rearranging terms, it is seen that

$$\begin{aligned}-xn^p\pi_n Q_n(x) = {} & (n-1)^p\lambda_{n-1}\pi_{n-1}Q_{n-1}(x) - n^p(\lambda_n+\mu_n)\pi_n Q_n(x) \\ & + (n+1)^p\mu_{n+1}\pi_{n+1}Q_{n+1}(x) \\ & + \sum_{v=1}^{p-1}\binom{p}{v}(n-1)^v\lambda_{n-1}\pi_{n-1}Q_{n-1}(x) \\ & + \sum_{v=1}^{p-1}\binom{p}{v}(-1)^{p-v}(n+1)^v\mu_{n+1}\pi_{n+1}Q_{n+1}(x);\end{aligned}$$

or summing over n

$$-x\sum_{n=1}^{\infty} n^p\pi_n Q_n(x) = \sum_{v=1}^{p-1}\binom{p}{v}\left[\sum_{n=1}^{\infty} n^v\lambda_n\pi_n Q_n(x) + (-1)^{p-v}\sum_{n=2}^{\infty} n^v\mu_n\pi_n Q_n(x)\right] - x\mu_1.$$

Assuming that

$$\sum_{n=1}^{\infty} n^v\pi_n Q_n(x)$$

converges for $v = 0, 1, \cdots, p-1$,

$$\sum_{n=1}^{\infty} n^p\pi_n Q_n(x)$$

converges and the result proceeds by induction.

4. Relation to the distributions for absorption times and recurrence times

Denote by $I(x)$ the largest integer k, such that the k^{th} moment of the random variable x is finite.

Let R_{ij} be the random time required for first passage from the state i to the state j. The results of Hodges and Rosenblatt (1953) for the discrete time case may be readily extended to show that $I(R_{ij})$ has a constant value, say $I(R)$, for all $i > j$.

In particular, it suffices to examine the moments of R_{10}.

$$ER_{10}^p = \int t^p P_{11}(t)\mu_1 dt$$
$$= \iint \mu_1 t^p \exp(-xt) dt d\psi$$
$$= \mu_1 \int \frac{p!}{x^{p+1}} d\psi.$$

If $a > 0$, ER_{10}^p is finite. But then $ER_{ij}^p < \infty$ for all $i > j$, $p = 0, 1, 2, \cdots$. Hence $I(R_{10}) = I(R_{ij}) = I(R) = \infty$.

THEOREM 3. *$P^* > 0$ for some i, only if $I(R) = \infty$.*

By Theorem 1, the existence of the limiting conditional distribution P^* ensures that $a > 0$, hence that the moments of all orders of the absorption time are finite.

The existence of an arbitrary but finite number of moments of the absorption time distribution does not guarantee the existence of P^* as is shown by the following example:

Let

$$\lambda_n = nk \qquad n = 0, 1, \cdots,$$
$$\mu_n = (n+\alpha)k \qquad n = 0, 1, \cdots; \alpha > 0.$$

Then $\psi(x) = \exp(-x/k)x^\alpha$, so that $a = 0$, while $I(R) = [\alpha]$.

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