Uniqueness of the Fréchet space topology on certain topological algebras

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It is well known that the complete norm topology on a Banach
algebra is not unique in general, though semisimplicity is
sufficient (but not necessary) for uniqueness. In this note we
consider a class of topological algebras of formal power series
which have unique Fréchet space topology. The structure of
these algebras in the Banach algebra case will be considered in
a later paper.

Let A be an associative algebra over a field K, either the real
or complex field, having an identity e and carrying a Fréchet space
topology. $\underline{A}$ will denote an algebra of formal power series over A in a
commutative indeterminate t, which we suppose to contain t. The
elements of $\underline{A}$ will be denoted by expressions of the form $\sum a_i t^i$, where
$t^0 = e$, $et = t$. $\underline{A}$ will also be supposed to carry a Fréchet space
topology, such that the projections $p_j : \sum a_i t^i \mapsto a_j$ of $\underline{A}$ into A are
continuous for each $j \geq 0$.

Note that if $\underline{A}$ has an identity and is a unitary left A-module
under the natural multiplication: $a \sum a_i t^i = \sum a a_i t^i$, we may embed A
in $\underline{A}$ by the injection $a \mapsto a + 0.t + \dots$. Then A can be given the
relative topology induced from $\underline{A}$, and if the projections are continuous

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with this topology, $A = \bigcap\limits_{j \geq 1} \ker p_j$ is closed, and hence is a Fréchet space under the relative topology. Also, we see that if multiplication in $\underline{A}$ is continuous then $\underline{A}$ is a topological left A-module with the relative topology on A. It is perhaps of interest to note that this is true more generally. Indeed, in the general case suppose that multiplication in A and $\underline{A}$ is continuous and that $\underline{A}$ is a left A-module under the natural multiplication. Then for $x \in \underline{A}$ the map $R_x : A \to \underline{A} : a \mapsto ax$ is continuous. For suppose $\{y_n\} \subseteq A$, $y_n \to 0$, $R_x(y_n) \to z$. Then $y_n x \to z$ in $\underline{A}$ so that $p_j(y_n x) \to p_j(z)$ in A for each $j \geq 0$. But $p_j(y_n x) = y_n p_j(x) \to 0$ for each j , whence $p_j(z) = 0$, $j \geq 0$, so that $z = 0$. The continuity of R_x follows by the closed graph theorem. By using the fact that t is not a zero divisor a similar argument shows that for $a \in A$ the map $L_a : \underline{A} \to \underline{A} : x \mapsto ax$ is continuous. A category argument $(cf.$ [1], Theorem 5) now shows that $\underline{A}$ is a topological left A-module.

In [3] I showed, in the case A the complex field and multiplication in $\underline{A}$ continuous, that under certain conditions on the projections $\{p_j\}$ derivations on $\underline{A}$ are necessarily continuous (the same argument works for more general $A)$, and in [4] that $\underline{A}$ has unique complete norm topology if it is a Banach algebra and A is commutative semisimple. By modifying the argument of [3] we now extend this second result. We will say that $\underline{A}$ satisfies condition (E) if there is a sequence $\{\gamma_n\}$ of positive reals such that $\left\{\gamma_n^{-1} p_n\right\}$ is equicontinuous. The first result is the following which, surprisingly, does not require continuity of multiplication in $\underline{A}$.

THEOREM 1. *Let B be an algebra over K which is a topological algebra under a Fréchet space topology, $\varphi : B \to \underline{A}$ a homomorphism such that $t \in \varphi(B)$, and suppose $\underline{A}$ satisfies (E) . Then φ is continuous.*

Proof. Since $\{p_j\}$ is a separating family of continuous linear mappings on $\underline{A}$ the closed graph theorem shows it is sufficient to prove that $p_j\varphi$ is continuous for each j . Thus suppose that some $p_k\varphi$ is

discontinuous, and let k be the least integer with this property. If $\{\|\cdot\|_n\}$ is a sequence of seminorms defining the topology on A there is thus some integer N such that $p_k\varphi : B \to \left(A, \|\cdot\|_N\right)$ is discontinuous. Let $\left\{\gamma_n^{-1}p_n\right\}$ be equicontinuous, and take a neighbourhood V of zero in $\underline{A}$ such that if $x \in V$ then $\|p_n(x)\|_N \leq \gamma_n$ for each n. Also, since $p_0\varphi, \ldots, p_{k-1}\varphi$ are continuous (possibly vacuously so) there is a neighbourhood U of zero in B such that if $x \in U$ then $\|p_i\varphi(x)\|_N \leq 1$, $0 \leq i \leq k-1$.

Let $\{|\cdot|_n\}$ be a sequence of seminorms defining the topology on B. Since multiplication in B is jointly continuous there is a sequence $\{M_n\}$ of positive integers, and a sequence $\{\delta_n\}$ of positive reals such that if $x, y \in B$, $|x|_i < \alpha$, $|y|_i < \beta$, $1 \leq i \leq M_n$, where $\alpha\beta < \delta_n$, then $|xy|_j < 1$, $1 \leq j \leq n$. Finally, take $s \in B$ such that $\varphi(s) = t$, and let $\{\mu_n\}$ be a sequence of positive reals such that $\mu_n^{-1}\gamma_n \to 0$.

Define inductively a sequence $\{x_n\} \subseteq B$ such that:

(i) $x_n \in U$;

(ii) $|x_n|_i \leq 2^{-n}\delta_n \min_{\substack{1 \leq j \leq n \\ 1 \leq m \leq n}} \left(1+|s^j|_m\right)^{-1}$ for $1 \leq i \leq M_n$;

(iii) $\|p_k\varphi(x_n)\|_N \geq \mu_{n+k} + k + \sum_{i=1}^{n-1} \|p_{i+k}\varphi(x_{n-i})\|_N$.

From (ii) it follows that if $m \geq j$ then $\left|s^{m-j}x_m\right|_i < 2^{-m}$ for $1 \leq i \leq m$, and so for each j the series $\sum_{m \geq j} s^{m-j}x_m$ converges in B .

Set $y = \sum_{m \geq 1} s^m x_m$. Then

$$p_n\varphi(y) = p_n\varphi\left\{ \sum_{m=1}^{n} s^m x_m + s^{n+1}.\sum_{m\geq n+1} s^{m-n-1}x_m \right\}$$

$$= p_n\left\{ \sum_{m=1}^{n} t^m\varphi(x_m) + t^{n+1}.\varphi\left(\sum_{m\geq n+1} s^{m-n-1}x_m \right) \right\}$$

$$= \sum_{m=1}^{n} \sum_{i=0}^{n} p_{n-i}(t^m)p_i\varphi(x_m) + 0$$

$$= \sum_{i=0}^{n-1} p_i\varphi(x_{n-i}) \ .$$

Now suppose $n \geq k+2$, so that

$$\|p_n\varphi(y)\|_N \geq \|p_k\varphi(x_{n-k})\|_N - \left\{ \sum_{i=0}^{k-1} + \sum_{i=k+1}^{n-1} \right\}\|p_i\varphi(x_{n-i})\|_N$$

$$\geq \|p_k\varphi(x_{n-k})\|_N - k - \sum_{i=1}^{n-k-1} \|p_{i+k}\varphi(x_{n-k-i})\|_N$$

$$\geq \mu_n \ .$$

Thus taking a non-zero scalar λ so small that $\lambda\varphi(y) \in V$ we have $|\lambda|\mu_n \leq \|p_n\varphi(\lambda y)\|_N \leq \gamma_n$, whence $|\lambda| \leq \mu_n^{-1}\gamma_n$, which is impossible for any sufficiently large n by definition of $\{\mu_n\}$. This gives the desired contradiction.

REMARK. The supposition that $t \in \varphi(B)$ can almost certainly be weakened somewhat, at the cost of complication. However some condition of surjectivity is required, since the result is clearly false in certain cases when $\varphi(B) \subseteq A$.

COROLLARY 1. *If* $\underline{\underline{A}}$ *is itself a topological algebra then every automorphism of* $\underline{\underline{A}}$ *is continuous.*

COROLLARY 2. $\underline{\underline{A}}$ *has unique Fréchet space topology as a topological algebra.*

Proof. Suppose that $\underline{\underline{A}}$ is a topological algebra under a Fréchet space topology determined by seminorms $\{|\cdot|_n\}$. By Theorem 1 the identity map $(\underline{\underline{A}}, \{|\cdot|_n\}) \to (\underline{\underline{A}}, \{\|\cdot\|_n\})$ is continuous; and hence bicontinuous.

COROLLARY 3. *Any topological algebra A with Fréchet space topology can be isometrically embedded in a topological algebra with unique Fréchet space topology.*

Proof. Adjoin an identity e to A if necessary, and if $\{\|\cdot\|_n\}$ is a sequence of seminorms defining the topology in A extend them by setting $\|x+\lambda e\|_n = \|x\|_n + |\lambda|$. Then take $\underline{\underline{A}}$ as the algebra of all formal power series over A such that $\left\|\sum a_i t^i\right\|_n = \sum \|a_i\|_n < \infty$ for each n , with these seminorms defining the topology. It is easily seen that $\underline{\underline{A}}$ is a topological algebra with this topology, satisfying (E) , and so has unique Fréchet space topology as a topological algebra by Corollary 2. The natural injection of $A \to \underline{\underline{A}}$ is the required embedding.

The fact that $\underline{\underline{A}}$ satisfied (E) played an essential role in the proof of Theorem 1, and it is of interest to obtain a characterization of this condition. In [4, Lemma 3.1], where the Banach algebra case was considered, the notion of a growth sequence was introduced, and it is precisely this concept which we require here: a sequence $\{\sigma_n\}$ of positive reals is a growth sequence for $\underline{\underline{A}}$ if $\sigma_n p_n(x) \to 0$ for each $x \in \underline{\underline{A}}$, convergence in the topology of A .

THEOREM 2. *Let A be as above, $\underline{\underline{A}}$ an algebra of formal power series over A which is a topological algebra under a Fréchet space topology. Then $\underline{\underline{A}}$ satisfies (E) if and only if $\underline{\underline{A}}$ admits a growth sequence.*

Proof. Suppose that $\left\{\gamma_n^{-1} p_n\right\}$ is equicontinuous. Let U be a neighbourhood of zero in A , and V a neighbourhood of zero in $\underline{\underline{A}}$ such that $\gamma_n^{-1} p_n(V) \subseteq U$ for all n . Then if $x \in \underline{\underline{A}}$ there is N such that $n^{-1}x \in V$ if $n > N$. Thus if $n > N$, $n^{-1}\gamma_n^{-1} p_n(x) = \gamma_n^- p_n(n^{-1}x) \in U$. It follows that $\left\{n^{-1}\gamma_n^{-1}\right\}$ is a growth sequence for $\underline{\underline{A}}$.

Conversely, suppose that $\{\sigma_n\}$ is a growth sequence for $\underline{\underline{A}}$. By an argument analogous to [4, Lemma 3.1] and Theorem 1 above it follows

that the projections are continuous. The Banach-Steinhaus theorem now shows that $\{\sigma_n p_n\}$ is equicontinuous.

   EXAMPLE. Let $\underline{A}$ be the algebra of all formal power series over the complex field, with the coordinate topology. Then $\underline{A}$ is a topological algebra under this Fréchet space topology, with continuous projections, but Theorem 2 shows that $\underline{A}$ does not satisfy (E). An argument of Scheinberg [5, §2] shows that automorphisms on $\underline{A}$ are continuous. (Note, however, that $t^n \underline{A} \neq \underline{A}^{n+1}$, so a little care is needed. Indeed, Scheinberg's argument is not applicable to derivations on $\underline{A}$; these are continuous by the remark at the end of [2], which is valid in this case.) The same argument shows that $\underline{A}$ has unique Fréchet space topology under the condition that projections are continuous, though this is easily seen to be true in the general case by use of the closed graph theorem. The present author has been unable to determine whether $\underline{A}$ has unique Fréchet space topology as a topological algebra, however the topology as a Q-algebra is unique. For suppose $\underline{A}$ is a Q-algebra under a Fréchet space topology τ, and consider the map $\Gamma : x \mapsto xt$. Then Γ is a τ-continuous bijection of $\underline{A}$ onto $t\underline{A}$ and the latter is τ-closed since it is a (in fact the) maximal ideal in $\underline{A}$ and $\underline{A}$ is a Q-algebra under τ. The open mapping theorem shows Γ^{-1} is τ-continuous. Since $\ker p_0 = t\underline{A}$ is τ-closed, p_0 is τ-continuous, and the relation

$$p_j(x) = p_{j-1}\left[\Gamma^{-1}\left(x - p_0(x)e\right)\right] , \quad x \in \underline{A} , \quad j \geq 1 ,$$ gives the τ-continuity of each p_j by induction. As remarked above an application of the closed graph theorem now shows that τ equals the coordinate topology.

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