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ON ISOMETRIC REPRESENTATION SUBSETS OF BANACH SPACES

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Abstract

Let X, Y be two Banach spaces and B_X the closed unit ball of X. We prove that if there is an isometry $f : B_X \to Y$ with $f(0) = 0$, then there exists an isometry $F : X \to Y^{**}$. If, in addition, Y is weakly nearly strictly convex, then there is an isometry $F : X \to Y$. Making use of these results, we show that if Y is weakly nearly strictly convex and there is an isometry $f : B_X \to Y$ with $f(0) = 0$, then there exists a linear isometry $S : X \to Y$.

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1. Introduction

Let (M_1, d_1) and (M_2, d_2) be two metric spaces. A mapping $f : M_1 \to M_2$ is called an isometry if, for any $u, v \in M_1$, $d_2(f(u), f(v)) = d_1(u, v)$. Suppose that X and Y are two Banach spaces with $0 \in M_1 \subset X$ and $0 \in M_2 \subset Y$. We call $f : M_1 \to M_2$ a standard isometry if f is an isometry with $f(0) = 0$. We investigate the following two problems.

Problem 1.1. Does the existence of a standard isometry $f : B_X \to Y$ imply the existence of an isometry $F : X \to Y$?

Problem 1.2. Does the existence of a standard isometry $f : B_X \to Y$ imply the existence of a linear isometry $S : X \to Y$?

Mazur and Ulam [9] proved that a surjective standard isometry between two Banach spaces is a linear isometry. Benyamini and Lindenstrauss [1] pointed out that this celebrated result demonstrates that the linear structure of a Banach space is completely determined by its structure as a metric space. There are many remarkable results concerning the properties of isometries and perturbations of isometries between

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Banach spaces. Figiel [6] showed the following remarkable result, which guarantees the existence and uniqueness of a continuous linear left inverse for any standard isometric embedding.

THEOREM 1.3 [6]. *If X, Y are two Banach spaces and $F : X \to Y$ is a standard isometry, then there exists a unique bounded linear operator $T : \overline{\operatorname{span}}(F(X)) \to X$ with $\|T\| = 1$ such that*

$$T \circ F = \mathrm{Id}_X. \tag{1.1}$$

Whether there exists a linear isometric right inverse of the operator T mentioned in Theorem 1.3 has also attracted attention. Godefroy and Kalton [8] showed the following deep theorem, which resolved a long-standing open problem of whether the existence of an isometry implies the existence of a linear isometry.

THEOREM 1.4 (Godefroy–Kalton [8]). *Let X, Y be two Banach spaces and $F : X \to Y$ a standard isometry. Suppose that T is the linear left inverse operator of F with $\|T\| = 1$.*

(I) [8, Proposition 2.9 and Theorem 3.1] *If X is a separable Banach space, then there is a linear isometry $S : X \to \overline{\operatorname{span}}(F(X))$ such that $T \circ S = \mathrm{Id}_X$.*
(II) [8, page 133] *If X is a nonseparable weakly compact generated space, then there exist a Banach space Y and a nonlinear isometric embedding $F : X \to Y$. However, X is not linear isomorphic to any subspace of Y.*

By making use of these results of Godefroy and Kalton, Dutrieux and Lancien [5] investigated the *compact representation subset of isometric embedding* between Banach spaces. Let $K_0 = \{f \in C([0, 1], |\cdot|) : \|f\|_\infty \le 1 \text{ and } \|f\|_{Lip} \le 1\}$. From [5], if a Banach space Y contains an isometric copy of K_0, then Y contains an isometric copy of any separable metric space and any separable Banach space is linearly isometric to a subspace of Y.

Dutrieux and Lancien introduced the notion of an isometrically representing subset of a Banach space X. A nonempty subset $M \subset X$ is called an *isometrically representing subset* of the Banach space X if any Banach space Y containing an isometric copy of M contains a subset which is isometric to X. Dutrieux and Lancien proved that the unit ball of a Banach space X is an isometrically representing subset of X if X is a finite-dimensional polyhedral Banach space (or $X = c_0$). Consequently, in view of Theorem 1.4, if X is a finite-dimensional polyhedral Banach space (or $X = c_0$) and there exists a standard isometry $f : B_X \to Y$, then X is linearly isometric to a subspace of Y. Dutrieux and Lancien [5, page 500] also proposed the following problem: is a Banach space always isometrically represented by its unit ball? We refer also to [3, 4, 11] and references therein for recent contributions to the study of perturbations of isometries.

In this paper, we first show that if there is a standard isometry $f : B_X \to Y$, then there is a standard isometry $F : X \to Y^{**}$ for all Banach spaces X and Y. Moreover, if Y is weakly nearly strictly convex (see Definition 2.9) and $f : B_X \to Y$ is a standard isometry, then there is a standard isometry $F : X \to Y$. Finally, if Y is weakly nearly

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strictly convex and $f : B_X \to Y$ is a standard isometry, then there is a linear isometry $S : X \to Y$.

All symbols and notation are standard. All Banach spaces are real and we use X to denote a real Banach space and X^* its dual. The closed unit ball (respectively sphere) in X is B_X (respectively S_X) and $B(x, \lambda)$ (respectively $S(x, \lambda)$) is the closed ball (respectively sphere) with centre x and radius λ. For any set G, $\ell_\infty(G, \mathbb{R})$ denotes the Banach space comprising all uniformly bounded functions $m : G \to \mathbb{R}$, endowed with the supnorm. Given a bounded linear operator $T : X \to Y$, $T^* : Y^* \to X^*$ is its conjugate operator. For a subset $A \subset X$, $\mathrm{co}(A)$ (respectively $\overline{\mathrm{co}}(A)$, $\overline{\mathrm{span}}(A)$) stands for the convex hull of A (respectively closed convex hull of A, closed subspace linearly generated by A).

2. Isometric embedding

In this section, we study Problem 1.1. We first show that there is an isometry $F : X \to Y^{**}$ if there exists a standard isometry $f : B_X \to Y$. If, in addition, Y is weakly nearly strictly convex, then the isometry $F : X \to Y^{**}$ is actually from X into Y. Before describing our main results, we first recall some preliminaries (see, for example, [10] and [3, page 718]).

Recall that a Banach space X is said to be a Gâteaux differentiability space (GDS) provided every continuous convex function on X is densely Gâteaux differentiable. Typical Gâteaux differentiability spaces are separable Banach spaces [10, Theorem 1.20]. A point x^* in a w^*-closed convex set C of a dual space X^* is said to be a w^*-exposed point of C provided there exists a point $x \in X$ such that $\langle x^*, x\rangle > \langle y^*, x\rangle$ for all $y^* \in C$ with $y^* \neq x^*$. In this case, the point x is called a w^*-exposing functional of C exposing C at x^*. We denote by $w^*\text{-}\exp C$ the set of all w^*-exposed points of C.

The following proposition is classical and easy to prove (see [3, 10]).

Proposition 2.1 [3, Proposition 2.2]. *Suppose that X is a Banach space and $C \subset X^*$ is a nonempty w^*-compact convex set. Then $x^* \in C$ is w^*-exposed by $x \in X$ if and only if $\sigma_C : X \to \mathbb{R}$, $\sigma_C(\cdot) \equiv \sup_{x^* \in C}\langle x^*, \cdot\rangle$ is Gâteaux differentiable at x and the Gâteaux derivative $d\sigma_C(x) = x^*$.*

Lemma 2.2. *Let X, Y be two Banach spaces and $f : B_X \to Y$ a standard isometry. Then, for any $x^* \in w^*\text{-}\exp B_{X^*}$, there exists a $\phi_{x^*} \in Y^*$ with $\|\phi_{x^*}\| = \|x^*\| = 1$ such that*

$$\langle x^*, x\rangle = \lim_{n\to\infty}\langle \phi_{x^*}, nf(n^{-1}x)\rangle \tag{2.1}$$

for any $x \in X$ and $n \in \mathbb{N}$ when n is so large that $n^{-1}x \in B_X$.

Proof. Given any $x^* \in w^*\text{-}\exp B_{X^*}$, by Proposition 2.1, there exists a Gâteaux differentiability point $x_0 \in S_X$ such that $x^* = d\|\cdot\|(x_0)$, and this shows that $\langle x^*, \pm x_0\rangle = \pm 1$. Since $f : B_X \to Y$ is a standard isometry, by the Hahn–Banach theorem, we can choose a $\phi_{x^*} \in S_{Y^*}$ with

$$\phi_{x^*}(f(x_0) - f(-x_0)) = \|f(x_0) - f(-x_0)\| = \|x_0 - (-x_0)\| = 2. \tag{2.2}$$

Equation (2.2) implies that

$$1 \geq \phi_{x^*}(f(x_0)) = \phi_{x^*}(f(x_0) - f(-x_0)) + \phi_{x^*}(f(-x_0)) \geq 2 - \|f(-x_0)\| \geq 1 \tag{2.3}$$

and (2.3) further implies that

$$-1 \leq \phi_{x^*}(f(-x_0)) = \phi_{x^*}(f(-x_0) - f(x_0)) + \phi_{x^*}(f(x_0)) = -1.$$

Therefore, $\phi_{x^*}(f(x_0)) = 1$ and $\phi_{x^*}(f(-x_0)) = -1$. For any $x \in X$ and $n \in \mathbb{N}$ when n is so large that $n^{-1}x \in B_X$,

$$\begin{aligned}\|x_0\| - \phi_{x^*}(f(n^{-1}x)) = 1 - \phi_{x^*}(f(n^{-1}x)) &= \phi_{x^*}(f(x_0)) - \phi_{x^*}(f(n^{-1}x)) \\ &\leq \|f(x_0) - f(n^{-1}x)\| = \|x_0 - n^{-1}x\|.\end{aligned}$$

Consequently,

$$-(\|x_0 - n^{-1}x\| - \|x_0\|) \leq \phi_{x^*}(f(n^{-1}x)).$$

Dividing this inequality by n^{-1},

$$-\Big(\frac{\|x_0 - n^{-1}x\| - \|x_0\|}{n^{-1}}\Big) \leq \phi_{x^*}(nf(n^{-1}x));$$

therefore, since $x^* = d\|\cdot\|(x_0)$,

$$-\langle x^*, -x\rangle \leq \liminf_{n\to\infty} \phi_{x^*}(nf(n^{-1}x)). \tag{2.4}$$

Conversely,

$$\begin{aligned}\|x_0\| + \phi_{x^*}(f(n^{-1}x)) = 1 + \phi_{x^*}(f(n^{-1}x)) &= -\phi_{x^*}(f(-x_0)) + \phi_{x^*}(f(n^{-1}x)) \\ &\leq \|f(n{-}1x) - f(-x_0)\| = \|n^{-1}x + x_0\|.\end{aligned}$$

Thus,

$$\phi_{x^*}(f(n^{-1}x)) \leq \|n^{-1}x + x_0\| - \|x_0\|.$$

Dividing by n^{-1},

$$\phi_{x^*}(nf(n^{-1}x)) \leq \frac{\|x_0 + n^{-1}x\| - \|x_0\|}{n^{-1}}.$$

Again, since $x^* = d\|\cdot\|(x_0)$,

$$\limsup_{n\to\infty} \phi_{x^*}(nf(n^{-1}x)) \leq \langle x^*, x\rangle. \tag{2.5}$$

Combining (2.4) and (2.5) yields

$$\langle x^*, x\rangle = \lim_{n\to\infty} \langle \phi_{x^*}, nf(n^{-1}x)\rangle. \qquad \square$$

For any $x \in X$, let $n_x = \inf\{n \in \mathbb{N}, n^{-1}x \in B_X\}$. We give the following definition.

Definition 2.3. For any $x \in X$, we define

$$\alpha_n(x) = n_x f(n_x^{-1}x) \quad \text{if } n < n_x; \text{otherwise}, \alpha_n(x) = nf(n^{-1}x).$$

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Since $f : B_X \to Y$ is a standard isometry, we choose $y^* \in S(Y^*)$ such that $y^*(f(n_x{}^{-1}x)) = \|f(n_x{}^{-1}x)\| = \|n_x{}^{-1}x\|$. Then, for all $n \geq n_x$,

$$\left\|\frac{x}{n}\right\| \geq y^*\Big(f\Big(\frac{x}{n}\Big)\Big) = y^*\Big(f\Big(\frac{x}{n_x}\Big)\Big) - \Big(y^*\Big(f\Big(\frac{x}{n_x}\Big) - f\Big(\frac{x}{n}\Big)\Big)\Big) \geq \left\|f\Big(\frac{x}{n_x}\Big)\right\| - \Big(\frac{1}{n_x} - \frac{1}{n}\Big)\|x\| = \left\|\frac{x}{n}\right\|.$$

This means that $y^*(nf(n^{-1}x)) = \|x\|$ for any $n \geq n_x$. Therefore,

$$\{\alpha_n(x)\}_{n=1}^{\infty} \subset \{y^{**} \in S_{Y^{**}}(0, \|x\|), y^*(y^{**}) = \|x\|\} \subset B_{Y^{**}}(0, \|x\|).$$

Let $\mathcal{U}$ be a free ultrafilter of $\mathbb{N}$. By the w^*-compactness of $B_{Y^{**}}(0, \|x\|)$, the limit $\lim_{\mathcal{U}} \alpha_n(x)$ with respect to the ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and with respect to the w^*-topology on Y^{**} exists for any $x \in X$, and $\lim_{\mathcal{U}} \alpha_n(x) \in \{y^{**} \in S_{Y^{**}}(0, \|x\|), y^*(y^{**}) = \|x\|\}$. Therefore, $F : X \to Y^{**}, F(x) = \lim_{\mathcal{U}} \alpha_n(x)$ is a well-defined mapping for all $x \in X$. Obviously, we have the following proposition.

Proposition 2.4. *For any $x \in X$, $\|F(x)\| = \|x\|$, where $F : X \to Y^{**}, F(x) = \lim_{\mathcal{U}} \alpha_n(x)$ for any $x \in X$.*

Proposition 2.5 [10, Theorem 6.2]. *A Banach space X is a Gâteaux differentiability space if and only if every nonempty w^*-compact convex set of its dual is the w^*-closed convex hull of its w^*-exposed points.*

Now we are ready to state and prove our main results in this section.

Lemma 2.6. *Let X, Y be two Banach spaces and $f : B_X \to Y$ a standard isometry. Suppose that $F : X \to Y^{**}, F(x) = \lim_{\mathcal{U}} \alpha_n(x)$. Then, for any $x^* \in X^*$, there exists a $\phi_{x^*} \in Y^{***}$ with $\|\phi_{x^*}\| = \|x^*\| \equiv \gamma$ such that*

$$\langle x^*, x\rangle = \langle \phi_{x^*}, F(x)\rangle \quad \textit{for any } x \in X. \tag{2.6}$$

Proof. The proof is inspired by [3, Lemma 2.4]. We will complete the proof in several steps. Firstly, we show that (2.6) is true if X is finite dimensional. Then we prove that (2.6) holds for any Banach space X and any norm-attaining functional $x^* \in X^*$. Finally, making use of these results and the Bishop–Phelps theorem, we prove that (2.6) holds for any Banach space X and any bounded linear functional $x^* \in X^*$.

Without loss of generality, we assume that $\gamma = 1$. Let X be finite dimensional, so that X is a GDS [10, Theorem 1.20]. We first show that (2.6) is true if x^* is a w^*-exposed functional of B_{X^*}. Indeed, for any $x^* \in w^*\text{-exp}B_{X^*}$, by (2.1) of Lemma 2.2, there exists a $\phi_{x^*} \in Y^*$ with $\|\phi_{x^*}\| = \|x^*\| = 1$ such that

$$\langle x^*, x\rangle = \lim_{n\to\infty} \langle \phi_{x^*}, nf(n^{-1}x)\rangle$$

for all $x \in X$ and $n^{-1}x \in B_X$. Consequently, Definition 2.3 implies that

$$\langle x^*, x\rangle = \lim_{n\to\infty} \langle \phi_{x^*}, nf(n^{-1}x)\rangle = \lim_{n\to\infty} \langle \phi_{x^*}, \alpha_n(x)\rangle.$$

Then, since $F(x) = \lim_{\mathcal{U}} \alpha_n(x)$ and $\mathcal{U}$ is free,

$$\langle x^*, x\rangle = \lim_{n\to\infty} \langle \phi_{x^*}, \alpha_n(x)\rangle = \langle \phi_{x^*}, F(x)\rangle. \tag{2.7}$$

This means that (2.6) is true.

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Next, we show that for arbitrary $x^* \in S(X^*)$, there exists a linear functional $\phi_{x^*} \in S(Y^{***})$ with $\|\phi_{x^*}\| = \|x^*\| = 1$ such that (2.6) holds. By Proposition 2.5, since $\dim X < \infty$, we see that $\mathrm{co}(w^*\text{-exp}B_{X^*})$ is dense in B_{X^*}. Therefore, there is a sequence $\{x_n^*\}_{n=1}^\infty \subset \mathrm{co}(w^*\text{-exp}B_{X^*})$ such that $x_n^* \to x^*$. Equivalently, for any $n \in \mathbb{N}$, there are $m(n)$ elements $\{x_{n,i}^*\}_{i=1}^{m(n)} \subset w^*\text{-exp}B_{X^*}$ and $m(n)$ positive real numbers $\{\lambda_{n,i}\}_{i=1}^{m(n)} \subset \mathbb{R}$ with $\sum_{i=1}^{m(n)} \lambda_{n,i} = 1$ for some $m(n) \in \mathbb{N}$ such that

$$x_n^* = \sum_{i=1}^{m(n)} \lambda_{n,i} x_{n,i}^* \to x^*.$$

By (2.7), there exists a linear functional $\phi_{x_{n,i}^*} \in S(Y^*)$ such that

$$\langle x_{n,i}^*, x\rangle = \langle \phi_{x_{n,i}^*}, F(x)\rangle$$

for all $x \in X$ and $x_{n,i}^* \subset w^*\text{-exp}B_{X^*}$. Let $\psi_n = \sum_{i=1}^{m(n)} \lambda_{n,i}\, \phi_{x_{n,i}^*}$. Then $\|\psi_n\| \le 1$ and

$$\langle x_n^*, x\rangle = \langle \psi_n, F(x)\rangle \quad \text{for all } x \in X. \tag{2.8}$$

Since $\{\psi_n\}_{n=1}^\infty \subset B_{Y^*}(0,1) \subset B_{Y^{***}}(0,1)$, $\{\psi_n\}_{n=1}^\infty$ is a relatively w^*-compact subset of $B_{Y^{***}}(0,1)$. Therefore, there exists a subnet of $\{\psi_n\}_{n=1}^\infty$ w^*-converging to some ϕ_{x^*} in $B_{Y^{***}}(0,1)$. This and (2.8) together imply that

$$\langle x^*, x\rangle = \langle \phi_{x^*}, F(x)\rangle \quad \text{for all } x \in X. \tag{2.9}$$

Further, by Proposition 2.4, $\|F(x)\| = \|x\|$ for any $x \in X$ and (2.9) shows that $\|\phi_{x^*}\| = 1$. Therefore, we have shown (2.6) for any finite-dimensional space. (For a simpler proof, see Remark 2.7.)

Now we will show that (2.6) is true for any Banach space X and any norm-attaining functional $x^* \in S_{X^*}$. Given any norm-attaining functional $x^* \in S_{X^*}$, choose $x_0 \in S_X$ such that $\langle x^*, x_0\rangle = 1$, and let $\mathcal{E}$ be the collection of all finite-dimensional subspaces of X containing x_0. Since any such $E \in \mathcal{E}$ is a GDS, by (2.9) there is a $\phi_E \in S_{Y^{***}}$ such that

$$\langle x^*, x\rangle = \langle \phi_E, F(x)\rangle \quad \text{for all } x \in E.$$

Let

$$\Phi_E = \{\phi \in B_{Y^{***}}, \langle \phi, F(x)\rangle = \langle x^*, x\rangle \text{ for all } x \in E\}$$

and $\Phi = \{\Phi_E, E \in \mathcal{E}\}$. It is clear that for every $E \in \mathcal{E}$, Φ_E is a nonempty w^*-compact convex subset of Y^*. Since, for any $G, H \in \mathcal{E}$, $\Phi_G \cap \Phi_H \supset \Phi_K$, where $K = \mathrm{span}\{G, H\}$, $\Phi = \{\Phi_E, E \in \mathcal{E}\}$ has the finite intersection property. This, together with the fact that any element Φ_E of $\Phi = \{\Phi_E, E \in \mathcal{E}\}$ is a w^*-compact subset, shows that

$$\bigcap_{E\in\mathcal{E}} \Phi_E \neq \emptyset.$$

Now any element of $\bigcap_{E\in\mathcal{E}} \Phi_E$ satisfies (2.6).

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Finally, by using the Bishop–Phelps theorem, we will show that (2.6) holds for a general Banach space X and any functional $u^* \in S_{X^*}$. For any $u^* \in S_{X^*}$, by the Bishop–Phelps theorem, we can choose a sequence of norm-attaining functionals $\{x_n^*\}_{n=1}^{\infty} \subset S_{X^*}$ such that $x_n^* \to u^*$. For any norm-attaining functional $x_n^* \in S_{X^*}$, there exists a linear functional $\phi_{x_n^*} \in Y^{***}$ with $\|\phi_{x_n^*}\| = 1$ such that $\langle x_n^*, x\rangle = \langle \phi_{x_n^*}, F(x)\rangle$ for any $x \in X$ and $n \in \mathbb{N}$. By the relative w^*-compactness of $\{\phi_{x_n^*}\}$, there is a w^*-cluster point $\phi_{u^*} \in B_{Y^{***}}$. Therefore,

$$\langle u^*, x\rangle = \langle \phi_{u^*}, F(x)\rangle \tag{2.10}$$

for any $x \in X$. Note that $\|F(x)\| = \|x\|$ for all $x \in X$ by Proposition 2.4. Therefore, (2.10) further implies that $\|\phi_{u^*}\| = \|u^*\|$, that is, (2.6) holds. □

Remark 2.7. We thank the referee for providing a simpler proof of Equation (2.6) in the case of a finite-dimensional space. This alternative approach only uses the simple fact that convex functions defined on a finite-dimensional space are differentiable on a dense subset. The notion of Gâteaux differentiability space and Proposition 2.5 are not needed. Details can be found in [7]. Also, as the referee points out, it is not necessary to apply the Bishop–Phelps theorem in the proof of Lemma 2.6. Instead, we can use the easy fact that the norm-attaining functionals are w^*-dense.

Theorem 2.8. *Let X, Y be two Banach spaces and $f : B_X \to Y$ a standard isometry. If $F : X \to Y^{**}$, $F(x) = \lim_{\mathcal{U}} \alpha_n(x)$, then F is a standard isometric embedding.*

Proof. First, note that $F(0) = 0$. From the w^*-lower semi-continuity of the norm on Y^{**},

$$\|F(u) - F(v)\| \le \|u - v\|. \tag{2.11}$$

Conversely, let $x^* \in S_{X^*}$ be such that $\langle x^*, u - v\rangle = \|u - v\|$. By (2.6), there is a linear functional $\phi_{x^*} \in S_{Y^{***}}$ such that $\langle x^*, x\rangle = \langle \phi_{x^*}, F(x)\rangle$ for any $x \in X$. Thus,

$$\|F(u) - F(v)\| \ge \langle \phi_{x^*}, F(u) - F(v)\rangle = \langle x^*, u - v\rangle = \|u - v\|. \tag{2.12}$$

Combining (2.11) and (2.12) gives $\|F(u) - F(v)\| = \|u - v\|$. □

The following definition of weakly nearly strictly convex space is taken from [2].

Definition 2.9. A Banach space Y is called weakly nearly strictly convex (for short, WNSC) if $\{y \in S_Y, y^*(y) = 1\}$ is weakly compact for any norm-attaining functional $y^* \in S_{Y^*}$.

Clearly, strictly convex spaces are WNSC. Typical examples of WNSC spaces are the reflexive Banach spaces. The next Corollary 2.10 shows that if the range space Y of the isometric embedding $f : B_X \to Y$ is weakly nearly strictly convex, then the isometry $F : X \to Y^{**}$ derived in Theorem 2.8 is actually an isometric embedding from X into Y.

Corollary 2.10. *Let X, Y be Banach spaces and Y be weakly nearly strictly convex. Let $f : B_X \to Y$ be a standard isometry. Then there is a standard isometry $F : X \to Y$.*

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Proof. It is trivial that the desired standard isometry $F : X \to Y$ is provided, with the same notation, by Theorem 2.8. □

Remark 2.11. Let X, Y be Banach spaces. It follows from Corollary 2.10 that there is a standard isometry $F : X \to Y$ if there exists a standard isometry $f : B_X \to Y$ and Y is weakly nearly strictly convex. Furthermore, Theorem 1.4 implies that there is a linear isometry $S : X \to Y$ if X is separable. Therefore, if $f : B_X \to Y$ is a standard isometry from the unit ball of a separable Banach space X into another weakly nearly strictly convex Banach space Y, then there is a linear isometry $S : X \to Y$. In the next section, we will show that the separability assumption of X can be dropped (Theorem 3.4).

3. Linear isometric embedding

Suppose that $f : B_X \to Y$ is a standard isometric embedding. In this section, we consider Problem 1.2. Note that, for a general Banach space Y, we cannot derive a linear isometry $S : X \to Y$ from the existence of a standard isometry $f : B_X \to Y$, even though X is a nonseparable Hilbert space. For example, let H be a nonseparable Hilbert space; then H is a nonseparable weakly compact generated space. By the Godefroy–Kalton theorem 1.4, there exist a Banach space Y and a standard nonlinear isometry $f : H \to Y$. However, H is not linear isomorphic to any subspace of Y. Therefore, there is no linear isometry $S : X \to Y$.

Our main result, Theorem 3.4, shows that if Y is weakly nearly strictly convex, then Problem 1.2 admits an affirmative answer. The following Lemma 3.3 is established in our paper ('Linearization of isometric embedding on Banach spaces', submitted for publication), but, since it plays an essential role in the proof of our main result, we give its proof here. To prove Lemma 3.3, we need the following definition of an invariant mean on a semigroup and some related results from Benyamini and Lindenstrauss [1, pages 417–418].

Definition 3.1. Let G be a semigroup. A left-invariant mean on G is a linear functional μ on $\ell_\infty(G, \mathbb{R})$ such that:

(I) $\mu(1) = 1$;
(II) $\mu(f) \geq 0$ for every $f \geq 0$;
(III) for all $f \in \ell_\infty(G, \mathbb{R})$ and $g \in G, \mu(f_g) = \mu(f)$, where f_g is the left translation of f by g, that is, $f_g(h) = f(gh)$, for all $h \in G$.

Analogously, we can define a right-invariant mean of G. An invariant mean is a linear functional on $\ell_\infty(G, \mathbb{R})$ which is both left-invariant and right-invariant. Note that (I) and (II) are equivalent to (I) and $\|\mu\| = 1$.

Proposition 3.2. *Every Abelian semigroup G (in particular, every linear space) has an invariant mean.*

Lemma 3.3. *Let X, Y be two Banach spaces, $F : X \to Y$ be a standard isometry and $T : \overline{\mathrm{span}}(F(X)) \to X$ be the operator defined in (1.1) such that $\|T\| = 1$ and $T \circ F = \mathrm{Id}_X$. If $\overline{\mathrm{span}}(F(X))$ is a WNSC space, then there exists a linear isometry $S : X \to \overline{\mathrm{span}}(F(X))$ such that $T \circ S = \mathrm{Id}_X$.*

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Proof. Note that X is an Abelian group with respect to the vector addition of X. By Proposition 3.2, there exists an invariant mean μ on X. We also denote the invariant mean by μ_z or $\mu_z(\cdot)$, to emphasise that the mean is taken with respect to the variable z. Since $F : X \to Y$ is an isometry, for any fixed $x \in X$ and $y^* \in \overline{\text{span}}(F(X))^*$ and for any $z \in X$,

$$|\langle F(x+z) - F(z), y^* \rangle| \le \|F(x+z) - F(z)\| \cdot \|y^*\| = \|x\| \cdot \|y^*\|.$$

Therefore, $(\langle F(x+z) - F(z), y^* \rangle)_{z \in X} \in \ell_\infty(X, \mathbb{R})$. For the sake of simplicity, we denote $(\langle F(x+z) - F(z), y^* \rangle)_{z \in X}$ by $\langle F(x+z) - F(z), y^* \rangle_{z \in X}$. Making use of the invariant mean $\mu_z \in \ell_\infty(X, \mathbb{R})^*$, we define the desired linear isometry $S : X \to \overline{\text{span}}(F(X))$ as follows: for any $z^* \in \overline{\text{span}}(F(X))^*$ and $x \in X$,

$$\langle z^*, S(x) \rangle = \langle \mu_z, \langle F(x+z) - F(z), z^* \rangle_{z \in X} \rangle. \tag{3.1}$$

Indeed, it is obvious that $S(x)$ is a bounded linear functional on $\overline{\text{span}}(F(X))^*$ with $\|S(x)\| \le \|x\|$. We assert that $S(x) \in \overline{\text{span}}(F(X))$. Without loss of generality, we assume that $\|x\| = 1$. Since $\|x\| = 1$, there exists an $x^* \in S(X^*)$ such that $\langle x^*, x \rangle = 1$. Since T^* is a w^*-to-w^* continuous linear isometry, $\|T^*(x^*)\| = 1$. For any $z \in X$,

$$T^*(x^*)(F(x+z) - F(z)) = \langle x^*, x + z - z \rangle = \langle x^*, x \rangle = 1. \tag{3.2}$$

Equation (3.2) together with the fact that $\|F(x+z) - F(z)\| = \|x + z - z\| = 1$ imply that

$$\{F(x+z) - F(z) : z \in X\} \subseteq \{y \in \overline{\text{span}}(F(X)) : \|y\| = 1, T^*(x^*)(y) = 1\}. \tag{3.3}$$

Since $\overline{\text{span}}(F(X))$ is weakly nearly strictly convex,

$$C = \{y \in \overline{\text{span}}(F(X)) : \|y\| = 1, T^*(x^*)(y) = 1\}$$

is a weakly compact set. Let $(\overline{\text{span}}(F(X))^*, m)$ be the locally convex space $\overline{\text{span}}(F(X))^*$ endowed with the Mackey topology (that is, the topology of uniform convergence on weakly compact subsets of $\overline{\text{span}}(F(X))$). Then $(\overline{\text{span}}(F(X))^*, m)^* = \overline{\text{span}}(F(X))$. Suppose that $\{z^*_\alpha\}_{\alpha \in D} \subset \overline{\text{span}}(F(X))^*$ is a Mackey convergent net with $z^*_\alpha \xrightarrow{m} z^*_0$ for some $z^*_0 \in \overline{\text{span}}(F(X))^*$. Since C is a weakly compact set, $\{z^*_\alpha\}_{\alpha \in D}$ is uniformly convergent to z^*_0 on C and so $\{z^*_\alpha\}_{\alpha \in D}$ is uniformly convergent to z^*_0 on $\{F(x+z) - F(z) : z \in X\}$ by (3.3). According to (3.1), $\langle z^*_\alpha, S(x) \rangle \to \langle z^*_0, S(x) \rangle$. This means that $S(x)$ is a Mackey continuous bounded linear functional on $\overline{\text{span}}(F(X))^*$. Consequently, we see that $S(X) \in \overline{\text{span}}(F(X))$.

By (III) in Definition 3.1, for any $u, v \in X$,

$$\begin{aligned}
\langle z^*, S(u+v) \rangle &= \langle \mu_z, \langle F(u+v+z) - F(z), z^* \rangle_{z \in X} \rangle \\
&= \langle \mu_z, \langle F(u+v+z) - F(u+z), z^* \rangle_{z \in X} \rangle + \langle \mu_z, \langle F(u+z) - F(z), z^* \rangle_{z \in X} \rangle \\
&= \langle \mu_z, \langle F(v+z) - F(z), z^* \rangle_{z \in X} \rangle + \langle \mu_z, \langle F(u+z) - F(z), z^* \rangle_{z \in X} \rangle \\
&= \langle z^*, S(v) \rangle + \langle z^*, S(u) \rangle.
\end{aligned} \tag{3.4}$$

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This establishes the additivity of S. Furthermore,

$$\begin{aligned}
|\langle z^*, S(u)\rangle - \langle z^*, S(v)\rangle| &\\
&= \langle \mu_z, \langle F(u+z) - F(z), z^*\rangle_{z\in X}\rangle - \langle \mu_z, \langle F(v+z) - F(z), z^*\rangle_{z\in X}\rangle \\
&= \langle \mu_z, \langle F(u+z) - F(v+z), z^*\rangle_{z\in X}\rangle \leq \|u-v\| \cdot \|z^*\|, \qquad (3.5)
\end{aligned}$$

which shows that S is 1-Lipschitz. Together, (3.4) and (3.5) imply that S is a bounded linear operator on X with $\|S\| \leq 1$.

Finally, for any $x^* \in X^*, x \in X$,

$$\begin{aligned}
\langle x^*, T \circ S(x)\rangle &= \langle T^*(x^*), S(x)\rangle \\
&= \langle \mu_z, \langle F(x+z) - F(z), T^*(x^*)\rangle_{z\in X}\rangle \\
&= \langle \mu_z, \langle T(F(x+z) - F(z)), x^*\rangle_{z\in X}\rangle = \langle \mu_z, \langle x, x^*\rangle_{z\in X}\rangle = \langle x, x^*\rangle,
\end{aligned}$$

that is, $T \circ S = \mathrm{Id}_X$. Therefore, S is a linear isometry. This completes the proof. □

Theorem 3.4. *Let X, Y be two Banach spaces and $f : B_X \to Y$ a standard isometry. If Y is WNSC, then there is a linear isometry $S : X \to Y$.*

Proof. It follows from Corollary 2.10 that there is a standard isometry $F : X \to Y$ if there is a standard isometry $f : B_X \to Y$ and Y is WNSC. Note that $\overline{\mathrm{span}}(F(X)) \subset Y$ is WNSC since Y is WNSC. Therefore, by Lemma 3.3, we obtain a linear isometry $S : X \to \overline{\mathrm{span}}(F(X)) \subset Y$. □

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