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NEARLY EXTREMAL COHEN–MACAULAY AND GORENSTEIN ALGEBRAS

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This paper study nearly extremal Cohen–Macaulay and Gorenstein algebras and characterise them in terms of their minimal free resolutions. Explicit bounds on their graded Betti numbers and their multiplicities are obtained.

1. INTRODUCTION

The concept of an extremal Cohen–Macaulay or Gorenstein algebra appeared in the work of Sally [**6**] and Schenzel [**7**]. These algebras have the smallest possible reduction number and their minimal resolutions are pure. Thus using Herzog and Kühl [**2**] and Huneke and Miller [**4**] formulae, Betti numbers and multiplicity of these algebras are obtained.

Let $R = k[x_1, x_2, \dots, x_n]$ be a standard polynomial ring over a field k. Let I be a graded ideal in R of height g and initial degree p. Let $(h_0, h_1, \dots, h_s)$ be the h-vector of R/I.

(1) Suppose R/I is Cohen–Macaulay. Then $s \geqslant p-1$. If $s = p-1$, the algebra R/I is called an *extremal Cohen–Macaulay algebra.*

(2) Suppose R/I is Gorenstein. Then $s \geqslant 2(p-1)$. If $s = 2(p-1)$, the algebra R/I is called an *extremal Gorenstein algebra.*

The extremal Cohen–Macaulay (or extremal Gorenstein) algebra R/I is interesting because it has linear (respectively, almost linear) minimal resolution. Algebras with linear resolutions have been widely studied. It was shown by Eagon and Reiner [**1**] that a Stanley-Reisner ring of a simplicial complex has linear minimal resolution if and only if the Alexander dual of the simplicial complex is Cohen–Macaulay. This result has been extended by Herzog and Hibi [**3**].

We introduce a notion of *nearly extremal algebra* and showed that many properties of extremal algebras extend analogously to them. For a graded Cohen–Macaulay ideal I in R of height g and initial degree p, we say that the algebra R/I is *nearly extremal*

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Cohen–Macaulay if $s = p$. Similarly, for a Gorenstein ideal I, we say that the algebra R/I is *nearly extremal Gorenstein* if $s = 2p-1$. In other words, nearly extremal algebras have penultimate reduction numbers.

2. Nearly Extremal Cohen–Macaulay Algebra

For a nearly extremal Cohen–Macaulay algebra, we have the following characterisation theorem.

Theorem 2.1. *Let $R = k[x_1, x_2, \dots, x_n]$ be a standard polynomial ring over a field k and I be a Cohen–Macaulay graded ideal in R of height g and initial degree p. Set $\delta = \binom{g+p-1}{p} - \nu(I_p)$, where $\nu(I_p)$ is the minimal number of generators of I in degree p. Then the following conditions are equivalent:*

(1) *R/I is a nearly extremal Cohen–Macaulay-algebra.*

(2) *$\delta > 0$ and $\nu(I_{p+1}) = \binom{g+p}{p+1}$.*

(3) *The minimal resolution of R/I is of the form*

$$0 \to F_g \to F_{g-1} \to \cdots \to F_2 \to F_1 \to R \to R/I \to 0,$$

where $F_i = R[-(p+i-1)]^{b'_i} \oplus R[-(p+i)]^{b''_i}$ and, $\beta_{i,p+i-1} = b'_i$ and $\beta_{i,p+i} = b''_i$ are graded Betti numbers with $b''_g \neq 0$.

(4) *Hilbert series of R/I is of the form*

$$\mathbf{F}(R/I, t) = \frac{\sum_{i=0}^{p-1} \binom{i+g-1}{g-1} t^i + \delta\, t^p}{(1-t)^d},$$

where $d = \dim(R/I)$ and $\delta > 0$.

Proof: It can be assumed that k is an infinite field. Thus there is a regular system of parameters $\mathbf{y} = \{y_1, \dots, y_d\}$ of R/I such that each y_i is of degree 1 in R. Then $\overline{R} = R/\mathbf{y}R$ is a polynomial ring in g variables and $\overline{R}/\overline{I} = R/(\mathbf{y}, I)$ is Artinian. The h-vector of R/I satisfies $h_i = H(\overline{R}/\overline{I}, i) = \dim(\overline{R}_i/\overline{I}_i)$. Clearly, $h_i = \binom{g+i-1}{i}$ for $0 \leqslant i < p$, and $h_p = \binom{g+p-1}{p} - \nu(I_p)$.

Now R/I is nearly extremal Cohen–Macaulay-algebra if and only if $h_p = \delta > 0$ and $h_j = 0$ for $j \geqslant p+1$. In particular,

$$\nu(I_{p+1}) = \dim(\overline{R}_{p+1}) = \binom{g+p}{p+1}.$$

This proves that (1) and (2) are equivalent. On substituting the value of h-vector in the Hilbert series $\mathbf{F}(R/I, t) = (h_0 + h_1 t + \cdots + h_s t^s)/(1-t)^d$, we see that (1) and (4) are equivalent. Since R/I is Cohen–Macaulay, the projective dimension $pd(R/I) = ht(I) = g$. Thus R/I has a minimal resolution of the form

$$0 \to F_g \to F_{g-1} \to \cdots \to F_2 \to F_1 \to R \to R/I \to 0,$$

where $F_i = \bigoplus_{j=1}^{b_i} R[-d_{ij}]$ with $d_{i1} \leqslant d_{i2} \leqslant \cdots \leqslant d_{ib_i}$ for all i. By the minimality of resolution, we have $d_{11} < d_{21} < \cdots < d_{g1}$ and $p + g - 1 \leqslant d_{gj}$ for all j. On tensoring the minimal resolution of R/I by $R/(\mathbf{y})$, we get a minimal resolution of the Artinian ring $\overline{R}/\overline{I} = R/(\mathbf{y}, I)$. In this case, the socle of $\overline{R}/\overline{I}$ is isomorphic to $\bigoplus_{j=1}^{b_g} k\big[-(d_{gj} - g)\big]$. If R/I is nearly extremal, then the socle of $\overline{R}/\overline{I}$ can live in degrees $\leqslant p$. Thus $d_{gj} - g \leqslant p$. Combining the two inequalities, we get $p + (g-1) \leqslant d_{gj} \leqslant p + g$. As R/I is Cohen–Macaulay, we have $p \leqslant d_{1b_1} < d_{2b_2} < \cdots < d_{gb_g} \leqslant p + g$ (see [**9**, Proposition 4.2.3]). Therefore, either $d_{ij} = p + i - 1$ or $d_{ij} = p + i$. Since R/I is nearly extremal Cohen–Macaulay but not an extremal algebra, $d_{gb_g} = p + g$. This proves that (1) implies (3).

Finally, we shall show that (3) implies (1). If the minimal resolution of R/I is given as in (3), then its Hilbert series $\mathbf{F}(R/I, t)$ is of the form

$$\mathbf{F}(R/I, t) = \frac{1 + \sum_{i=1}^{g} (-1)^i (b_i' t^{p+i-1} + b_i'' t^{p+i})}{(1-t)^n}.$$

Since $d = \dim(R/I) = n - g$, the Hilbert series of R/I is of the form

$$\mathbf{F}(R/I, t) = \frac{\sum_{i=0}^{p} h_i t^i}{(1-t)^d}.$$

Thus the h-vector of R/I has length exactly p. □

Remarks.

1. If $\delta = 0$, then R/I becomes an extremal CM-algebra. In this case, R/I has a p-linear resolution and $\nu(I_p) = \binom{p+g-1}{p}$.
2. The Betti numbers $\beta_{i,p+i-1} = b_i'$ (for $i = 1, \ldots, g$) are called the *initial Betti numbers* of R/I.

In the next result we obtain some specific bounds on the Betti numbers of a nearly extremal Cohen–Macaulay-algebra.

Theorem 2.2. *With notation as in 2.1, let R/I be a nearly extremal Cohen–Macaulay-algebra. The initial Betti numbers $\beta_{i,p+i-1}$ (for $1 \leqslant i \leqslant g$) of R/I satisfy*

$$\beta_{i,p+i-1} = \binom{p+g-1}{g-i}\binom{p+i-2}{i-1} - \binom{g}{i-1}\delta + \beta_{i-1,p+i-1}.$$

Further,

$$0 \leqslant \beta_{i,p+i-1} \leqslant \binom{p+g-1}{g-i}\binom{p+i-2}{i-1}$$

and

$$0 \leqslant \beta_{i-1,p+i-1} \leqslant \binom{g}{i-1}\delta.$$

PROOF: Since R/I is a nearly extremal Cohen–Macaulay-algebra, its minimal resolution is given as in Theorem 2.1. Thus its Hilbert series $\mathbf{F}(R/I,t)$ is given by

$$\mathbf{F}(R/I,t) = \frac{1+\sum_{i=1}^{g}(-1)^i(b_i' t^{p+i-1} + b_i'' t^{p+i})}{(1-t)^n}.$$

On comparing this Hilbert series with the Hilbert series in Theorem 2.1, we see that the polynomial $f(t) = 1 + \sum_{i=1}^{g}(-1)^i(b_i' t^{p+i-1} + b_i'' t^{p+i})$ has exactly g-zeros at $t = 1$ and $b_g'' = \delta$. Thus on differentiating $f(t)$ successively $g-1$ times and putting $t = 1$, we obtain a system of g equations in $\lambda_i = (-1)^i(b_i' - b_{i-1}'')$ of the form

$$\sum_{i=1}^{g} \lambda_i = -1 - (-1)^g\delta,$$

$$\sum_{i=1}^{g} \lambda_i(p+i-1)\dots(p+i-j) = -(-1)^g(p+g)\dots(p+g-j)\delta$$

for $j = 1, 2, \dots, g-1$. By applying elementary row operations, we see that this system of linear equations in λ_i is equivalent to the system of linear equations

$$\sum_{i=1}^{g} \lambda_i = -1 - (-)^g\delta, \quad \& \quad \sum_{i=1}^{g} \lambda_i(p+i-1)^j = -(-1)^g(p+g)^j\delta$$

for $j = 1, 2, \dots, g-1$. On solving this system of linear equations by Cramer's rule, we obtain

$$D = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ p & p+1 & \cdots & p+g-1 \\ p^2 & (p+1)^2 & \cdots & (p+g-1)^2 \\ \vdots & \vdots & \ddots & \vdots \\ p^{g-1} & (p+1)^{g-1} & \cdots & (p+g-1)^{g-1} \end{vmatrix}$$

and

$$D_i = \begin{vmatrix} 1 & \cdots & 1 & -1-(-1)^g\delta & 1 & \cdots & 1 \\ p & \cdots & p+i-2 & -(-1)^g(p+g)\delta & p+i & \cdots & p+g-1 \\ p^2 & \cdots & (p+i-2)^2 & -(-1)^g(p+g)^2\delta & (p+i)^2 & \cdots & (p+g-1)^2 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p^{g-1} & \cdots & (p+i-2)^{g-1} & -(-1)^g(p+g)^{g-1}\delta & (p+i)^{g-1} & \cdots & (p+g-1)^{g-1} \end{vmatrix}.$$

Thus $D = (g-1)!(g-2)!\dots 1!$ and

$$D_i = \frac{(-1)^i}{(i-1)!}\Big(\prod_{j=p, j\neq p+i-1}^{p+i} j \prod_{r=1, r\neq g-i}^{g-1} r! - \delta \prod_{m=1, m\neq g-i+1}^{g} m!\Big).$$

But $\lambda_i = D_i/D$. Therefore,

$$b_i' - b_{i-1}'' = (-1)^i \frac{D_i}{D} = \binom{p+g-1}{g-i}\binom{p+i-2}{i-1} - \binom{g}{i-1}\delta \dots (*).$$

Since

$$b_i' = \beta_{i,p+i-1}(R/I) \leqslant \beta_{i,p+i-1}(R/in_>(I))$$

(see [**5**]), where $in_>(I)$ is the intial ideal of I with respect to any monomial order $>$. Thus without loss of generality, we may assume that I is a monomial ideal. Further, on going modulo a regular sequence $\mathbf{y} = y_1, \dots, y_{n-g}$, we can assume that R is a polynomial ring in g-variables and R/I is an Artinian ring. We are given that

$$\nu(I_p) = \binom{p+g-1}{p} - \delta < \binom{p+g-1}{p}.$$

Let J be a monomial ideal such that $I \subseteq J$ and $\nu(J_p) = \binom{p+g-1}{p}$. Then R/J is an extremal Cohen–Macaulay algebra and it has linear resolution with Betti numbers

$$\beta_{i,p+i-1}(R/J) = \binom{p+g-1}{g-i}\binom{p+i-2}{i-1}.$$

Since $\nu(I_p) < \nu(J_p)$ and all the minimal generators of the i-th syzygy of R/I in degree $p+i-1$ also appears in the minimal generators of the i-th syzygy of R/J, we clearly have

$$b_i' = \beta_{i,p+i-1}(R/I) \leqslant \beta_{i,p+i-1}(R/J) = \binom{p+g-1}{g-i}\binom{p+i-2}{i-1}.$$

Using this inequality in (*), we obtain $b_{i-1}'' \leqslant \binom{g}{i-1}\delta$. ∎

REMARKS.

1. If $\delta = 0$, then R/I is an extremal Cohen–Macaulay algebra and in this case, we have $b_{i-1}'' = 0$ for all i, as desired.
2. The multiplicity $e(R/I)$ of R/I can easily be calculated from the Hilbert series given in Theorem 2.1 and it is equal to $e(R/I) = \binom{p+g-1}{g} + \delta$.

We shall now deduce a result on Stanley-Reisner rings, which can be thought of as an extension of a result of Eagon and Reiner [**1**]. Let Δ be a simplicial complex. Let I_Δ be the Stanley-Reisner ideal and $k[\Delta] = R/I_\Delta$ be the Stanley-Reisner ring of Δ. If Δ^* is the Alexander dual of Δ, then Eagon-Reiner proved that Δ^* is CM if and only if $k[\Delta]$ has a linear resolution. Since $(\Delta^*)^* = \Delta$, we can interchange Δ and Δ^*. Now we prove the following result on nearly extremal Cohen–Macaulay–Stanley Reisner rings.

Proposition 2.3. *Let Δ be a CM-simplicial complex on the vextex set $V = \{x_1, \dots, x_n\}$. Suppose that the Stanley-Reisner ideal I_Δ is of initial degree p and height g. Then $k[\Delta]$ is a nearly extremal Cohen–Macaulay-algebra if and only if* $\dim\big(k[\Delta^*]\big) - \operatorname{depth}\big(k[\Delta^*]\big) = 1$.

Proof: The Alexander dual Δ^* is defined by $\Delta^* = \{F \subseteq V : V - F \notin \Delta\}$. Since I_Δ has initial degree p, it follows that $\dim\big(k[\Delta^*]\big) = n - p$. Now assume that $k[\Delta]$ is a nearly extremal Cohen–Macaulay-algebra. Then $s = p$. Since Δ is CM, it follows from Eagon and Reiner [**1**, Theorem 3, 4], that $k[\Delta^*]$ has a linear resolution and its Betti numbers satisfy the identity

$$\sum_{i\geqslant 1} \beta_i\big(k[\Delta^*]\big)t^{i-1} = \sum_{i=0}^{p} h_i(\Delta)(t+1)^i.$$

Clearly, $\beta_{p+1}\big(k[\Delta^*]\big) = h_p(\Delta)$ and all the higher Betti numbers are zero. Thus the projective dimension $pd\big(k[\Delta^*]\big) = p + 1$. By Auslander–Buchsbaum formula,

$$pd(k[\Delta^*]) + \operatorname{depth}\big(k[\Delta^*]\big) = n.$$

Thus the $\operatorname{depth}\big(k[\Delta^*]\big) = n - p - 1$. This shows that

$$\dim\big(k[\Delta^*]\big) - \operatorname{depth}\big(k[\Delta^*]\big) = 1.$$

Conversely, if the condition holds, then going backward, we see that $pd\big(k[\Delta^*]\big) = p + 1$. Now the above identity becomes

$$\sum_{i=1}^{p+1} \beta_i\big(k[\Delta^*]\big)t^{i-1} = \sum_{i=0}^{s} h_i(\Delta)(t+1)^i,$$

from which we conclude that $s = p$. ∎

3. Nearly Extremal Gorenstein Algebra

For a nearly extremal Gorenstein algebra, we have the following characterisation theorem.

Theorem 3.1. *Let $R = k[x_1, x_2, \dots, x_n]$ be a standard polynomial ring over a field k and I be a Gorenstein graded ideal in R of height g and initial degree p. Then the following conditions are equivalent:*

(1) *R/I is a nearly extremal Gorenstein-algebra.*

(2) *The minimal resolution of R/I is of the form*

$$0 \to F_g \to F_{g-1} \to \cdots \to F_2 \to F_1 \to R \to R/I \to 0,$$

where $F_i = R\big[-(p+i-1)\big]^{b_i'} \oplus R\big[-(p+i)\big]^{b_i''}$; for $1 \leqslant i \leqslant g-1$ and $F_g = R\big[-(2p+g-1)\big]$.

(3) *Hilbert series of R/I is of the form*

$$\mathbf{F}(R/I,t) = \frac{\sum_{i=0}^{p-1}\binom{i+g-1}{g-1}(t^i + t^{2p-1-i})}{(1-t)^d},$$

where $d = \dim(R/I)$.

PROOF: As in the proof of Theorem 2.1, we may assume that k is an infinite field, $\mathbf{y} = \{y_1, \dots, y_d\}$ is a regular system of parameters of R/I such that each y_i is of degree 1 in R and $\overline{R} = R/\mathbf{y}R$ is a polynomial ring in g-variables such that $\overline{R}/\overline{I} = R/(\mathbf{y}, I)$ is Artinian. Thus going modulo $\mathbf{y}$, we may assume that R is a standard polynomial ring in g-variables and R/I is Artinian. Since R/I is Gorenstein, the minimal resolution of R/I is of the form

$$0 \to R[-d_g] \to \bigoplus_{i=1}^{b_{g-1}} R[-d_{(g-1)i}] \to \cdots \to \bigoplus_{i=1}^{b_1} R[-d_{1i}] \to R \to R/I \to 0,$$

with $d_{k1} \leqslant \cdots \leqslant d_{kb_k}$ and $\bigoplus_{i=1}^{b_{g-k}} R\big[-(d_g - d_{(g-k)i})\big] \simeq \bigoplus_{i=1}^{b_k} R[-d_{ki}]$; for $1 \leqslant k \leqslant g-1$ (see [9, Theorem 4.3.11]). Also the socle of R/I lives in degree $d_g - g$. Let R/I be a nearly extremal Gorenstein-algebra. Since R/I is also Artinian, its socle lies in the last non-zero graded component. Thus $d_g - g = s = 2p-1$ or $d_g = 2p+g-1$. From the duality isomorphisms $\bigoplus_{i=1}^{b_{g-k}} R\big[-(d_g - d_{(g-k)i})\big] \simeq \bigoplus_{i=1}^{b_k} R[-d_{ki}]$; for $1 \leqslant k \leqslant g-1$ and the minimal resolution of R/I, we obtain $d_{11} = p$, $d_{k1} \geqslant p+k-1$ and $d_g - d_{(g-k)b_{(g-k)}} = d_{k1}$; for $1 \leqslant k \leqslant g-1$. Therefore, $d_{(g-k)b_{(g-k)}} = d_g - d_{k1} \leqslant 2p+g-1-(p+k-1) = p+g-k$; for $1 \leqslant k \leqslant g-1$. Thus the minimal resolution of R/I has the required form. This proves that (1) implies (2).

Now assume that the minimal resolution of R/I is as in (2). Then the socle of R/I lives in degree $d_g - g = 2p-1$ as $d_g = 2p+g-1$. Since R/I is Artinian Gorenstein algebra, its socle lies in the last non-zero graded component. Thus we have $s = 2p-1$, which shows that (2) implies (1).

Now we shall show that (1) implies (3). As R is a polynomial ring in g-variables and I is a Gorenstein ideal of initial degree p with R/I Artinian, we have $h_i = H(R/I, i)$ $= \binom{g+i-1}{g-1}$ and $h_i = h_{2p-1-i}$; for $0 \leqslant i \leqslant p-1$. Thus the Hilbert series of R/I has the required form given in (3). Finally, (3) implies (1) is clear. ∎

THEOREM 3.2. *Let $R = k[x_1, x_2, \dots, x_n]$ be a standard polynomial ring over a field k and I be a Gorenstein graded ideal in R of height g and initial degree p. Suppose that the h-vector $h = (h_0, h_1, \dots, h_s)$ of R/I satisfies $s \geqslant 2p-1$.Then the multiplicity of R/I satisfies $e(R/I) \geqslant 2\binom{p+g-1}{g}$. Further, $e(R/I) = 2\binom{p+g-1}{g}$ if and only if R/I is a nearly extremal Gorenstein-algebra.*

PROOF: As in the proof of Theorem 3.1, we may assume that R is a polynomial ring in g-variables and R/I is Artinian. Further, $h_i = H(R/I, i) = \binom{g+i-1}{g-1}$; for $0 \leqslant i \leqslant p-1$, and R/I is Gorenstein implies that $h_i = h_{s-i}$; for $0 \leqslant i \leqslant [s/2]$. Thus the multiplicity of R/I is

$$\begin{aligned} e(R/I) &= \sum_{i=0}^{s} h_i = 2\sum_{i=0}^{p-1} h_i + h_p + \dots + h_{s-p} \\ &\geqslant 2\sum_{i=0}^{p-1} h_i = 2\sum_{i=0}^{p-1} \binom{g+i-1}{g-1} \\ &= 2\binom{g+p-1}{g}. \end{aligned}$$

The equality holds in the above inequality if and only if $s = 2p-1$. ∎

REMARK. For R/I, as in Theorem 3.2, the multiplicity $e(R/I)$ satisfies

$$e(R/I) \geqslant \binom{g+p-1}{g} + \binom{g+p-2}{g}$$

with equality if and only if R/I is an extremal Gorenstein algebra, that is, $s = 2p - 2$ (see [**9**]). Thus we conclude from Theorem 3.2 that multiplicity of nearly extremal Gorenstein-algebra is the second lowest. We further remark that nothing can be said about the minimum value of the multiplicity if $s > 2p-1$.

Now we proceed to derive some numerical identity satisfied by the graded Betti numbers of a nearly extremal Gorenstein-algebra.

THEOREM 3.3. *With notation as in 3.1, let R/I be a nearly extremal Gorenstein-algebra. Then the graded Betti numbers b_i' and b_i'' of R/I satisfy the identity*

$$b_i' - b_{i-1}'' = \binom{p+i-2}{i-1}\binom{p+g-1}{g-i} - \binom{p+g-1}{i-1}\binom{p+g-i-1}{g-i},$$

for $1 \leqslant i \leqslant g$, where $b_0'' = 0$. In particular, the minimal number of generators of I in degree p is given by

$$\nu(I_p) = b_1' = \binom{p+g-1}{g-1} - \binom{p+g-2}{g-1}.$$

PROOF: Since R/I is a nearly extremal Gorenstein-algebra, its minimal resolution is given as in Theorem 3.1. Thus its Hilbert series $\mathbf{F}(R/I, t)$ is given by

$$\mathbf{F}(R/I,t) = \frac{1 + \sum_{i=1}^{g-1}(-1)^i(b_i' - b_{i-1}'')t^{p+i-1} + (-1)^{g-1}(b_{g-1}''t^{p+g-1} - t^{2p+g-1})}{(1-t)^n}.$$

On comparing this Hilbert series with the Hilbert series in Theorem 3.1, we see that $f(t) = 1 + \sum_{i=1}^{g-1}(-1)^i(b_i' - b_{i-1}'')t^{p+i-1} + (-1)^{g-1}b_{g-1}''t^{p+g-1} + (-1)^g t^{2p+g-1}$ is a polynomial having exactly g-zeros at $t = 1$. Thus on differentiating $f(t)$ successively $g-1$ times and putting $t = 1$, we obtain a system of g equations in $\lambda_i = (-1)^i(b_i' - b_{i-1}'')$ of the form

$$\sum_{i=1}^{g} \lambda_i = -1 - (-1)^g,$$

$$\sum_{i=1}^{g} \lambda_i(p+i-1)\dots(p+i-j) = -(-1)^g(2p+g-1)\dots(2p+g-j)$$

for $j = 1, 2, \dots, g-1$, where $b_0'' = 0$ and $b_g' = 0$. Now, on proceeding as in Theorem 2.2, we obtain the required identity. Further,

$$\nu(I_p) = b_1' = (-1)\lambda_1 = \binom{p+g-1}{g-1} - \binom{p+g-2}{g-1}. \qquad \square$$

REMARKS.

1. Let R/I be the Stanley-Reisner ring of a Gorenstein simplicial polytope. Then by Stanley [8], the h-vector $h = (h_0, h_1, \dots, h_s)$ of R/I satisfies the condition that $h_0 \leqslant h_1 \leqslant \cdots \leqslant h_{[s/2]}$.......(*). Thus, if $s \geqslant 2p - 1$, then we see that $\nu(I_p) \leqslant \binom{p+g-1}{g-1} - \binom{p+g-2}{g-1}$ and equality holds if and only if R/I is a nearly extremal Gorenstein-algebra.
2. Since the h-vector of a Gorenstein algebra need not satisfy the condition (*), there may exist a Gorentein ideal I with initial degree p and height g such that $s > 2p - 1$ and

$$\binom{p+g-1}{g-1} - \binom{p+g-2}{g-1} \leqslant \nu(I_p) \leqslant \binom{p+g-1}{g-1} - \binom{p+g-3}{g-1}.$$

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