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RINGS OF DIFFERENTIAL OPERATORS ON TORIC VARIETIES

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Let $\mathscr{X}$ be a finite dimensional toric variety over an algebraically closed field of characteristic zero, k. Let $\mathscr{D}_{\mathscr{X}}$ be the sheaf of differential operators on $\mathscr{X}$. We show that the ring of global sections, $\Gamma(\mathscr{X},\mathscr{D}_{\mathscr{X}})$ is a finitely generated Noetherian k-algebra and that its generators can be explicitly found. We prove a similar result for the sheaf of differential operators with coefficients in a line bundle.

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1. Introduction

Throughout this paper let k be an algebraically closed field of characteristic zero. For a commutative k-algebra A we define $D(A):=\bigcup_{i=0}^{\infty} D^i(A)$ where $D^0(A)=\mathrm{End}_A A$ and $D^i(A)=\{\theta\in\mathrm{End}_k A\,|\,[\theta,a]\in D^{i-1}(A)$ for all $a\in A\}$. Then $D(A)$ is a subring of $\mathrm{End}_k A$ called the ring of differential operators on A. For an irreducible affine variety $\mathscr{X}$, we define $\mathscr{D}(\mathscr{X}):=D(\mathscr{O}(\mathscr{X}))$ where $\mathscr{O}(\mathscr{X})$ is the ring of regular functions on $\mathscr{X}$. For a general irreducible variety we define the sheaf of differential operators on $\mathscr{X}$ to be the unique quasi-coherent $\mathscr{O}_{\mathscr{X}}$-module for which $\Gamma(U,\mathscr{D}_{\mathscr{X}})=D(\Gamma(U,\mathscr{O}_{\mathscr{X}}))$ for U an open affine subset of $\mathscr{X}$. Now if $\{U_i\}$ is an open affine cover of $\mathscr{X}$ then $\mathscr{D}_{\mathscr{X}}(U_i):=\Gamma(U_i,\mathscr{D}_{\mathscr{X}})=D(\mathscr{O}_{\mathscr{X}}(U_i))$. Thus the global sections of the sheaf of differential operators is

$$\mathscr{D}(\mathscr{X}):=\Gamma(\mathscr{X},\mathscr{D}_{\mathscr{X}})=\cap\,\mathscr{D}_{\mathscr{X}}(U_i)$$

and is called the ring of differential operators on $\mathscr{X}$. In order to study the differential operators on $\mathscr{X}$ we will calculate the above intersection.

Differential operators on smooth affine varieties have been well studied and also for certain classes of non-affine varieties such as projective spaces. We study a collection which contains examples of smooth, singular, affine and non-affine varieties. The collection in question consists of toric varieties, which are defined via cones and a lattice in a finite dimensional $\mathbb{Q}$-vector space. We shall use the geometry of the cones to prove results about the differential operators. The main result we prove is:

Theorem. *Let $\mathscr{X}$ be a finite dimensional toric variety over an algebraically closed field k. Then the ring of global sections of the sheaf of differential operators on $\mathscr{X}$ is a finitely generated Noetherian k-algebra. An explicit method to find this ring is given.*

The paper is split into four main sections. In Section 2 we recall the definition of a toric variety and state some of its properties. In Section 3 we establish some abstract properties of certain rings which enable us to calculate the intersection. In Section 4 we show that these properties hold for differential operators on a toric variety and finally in Section 5 we give some examples.

2. Toric varieties

The aim of this paper is to investigate the global sections of the sheaf of differential operators on a toric variety. First we need to define the notion of such a variety. We adopt the approach of [1] for simplicity.

Let V be a finite dimensional vector space over $\mathbb{Q}$. A subset $\{v\in V \mid \lambda(v)\geqq 0\}$ where $\lambda\colon V\to\mathbb{Q}$ is a non-zero linear functional is called a halfspace of V. A cone, σ in V is the intersection of a finite number of halfspaces. A subset of σ of the form $\{v\in V \mid \lambda(v)=0\}$ where $\lambda\colon V\to\mathbb{Q}$ is a linear functional positive on σ is called a face of σ. The dimension of the cone σ is the dimension of the vector space $\sigma-\sigma$ over $\mathbb{Q}$.

Let σ be a n-dimensional cone and M an n-dimensional lattice both embedded in the space $M_{\mathbb{Q}}=M\otimes_{\mathbb{Z}}\mathbb{Q}$. Without loss of generality we can consider the lattice to be $\mathbb{Z}^n$ and our cone therefore embedded in $\mathbb{Q}^n$. We can consider $\Lambda=\sigma\cap M$ as an additive semigroup. Let $k\Lambda$ be the semigroup algebra associated to Λ. The varieties of the form $\mathscr{X}=\operatorname{Spec} k\Lambda$ are affine toric varieties.

Examples. (i) $\mathscr{X}=\operatorname{Spec} k[x^2,xy,y^2]$. Here $k[x^2,xy,y^2]\cong k[x,xy,xy^2]$. Then $\sigma=(1,0)\mathbb{Q}_{\geqq 0}+(1,2)\mathbb{Q}_{\geqq 0}$ and $M=\mathbb{Z}^2$.

(ii) $\mathscr{X}=\operatorname{Spec} k[x,y,xz,yz]$. Here $\sigma=(1,0,0)\mathbb{Q}_{\geqq 0}+(0,1,0)\mathbb{Q}_{\geqq 0}+(0,0,1)\mathbb{Q}_{\geqq 0}+(1,1,-1)\mathbb{Q}_{\geqq 0}$ and $M=\mathbb{Z}^3$.

The following results give some idea of what sort of variety we are dealing with.

Proposition 2.1. *Let $\mathscr{X}$ be an affine toric variety. Then*

(1) *$\mathcal{O}(\mathscr{X})$ is a finitely generated k-algebra;*
(2) *$\mathcal{O}(\mathscr{X})$ is normal;*
(3) *$\mathcal{O}(\mathscr{X})$ is Cohen–Macaulay.*

Proof. See [1, 3].

General toric varieties are obtained by gluing together affine ones.

Definition 2.2. *A fan in V is a collection Σ of cones satisfying the following conditions*

(1) *Every cone has a vertex;*
(2) *If τ is a face of a cone $\sigma\in\Sigma$ then $\tau\in\Sigma$;*
(3) *If $\sigma,\sigma'\in\Sigma$ then $\sigma\cap\sigma'$ is a face of both σ and σ'.*

Let M, N be lattices dual to one another and let Σ be a fan in $N_{\mathbb{Q}}$. Again we can assume that $M=\mathbb{Z}^n$ and $V=\mathbb{Q}^n$. Define an affine toric variety for each $\sigma\in\Sigma$,

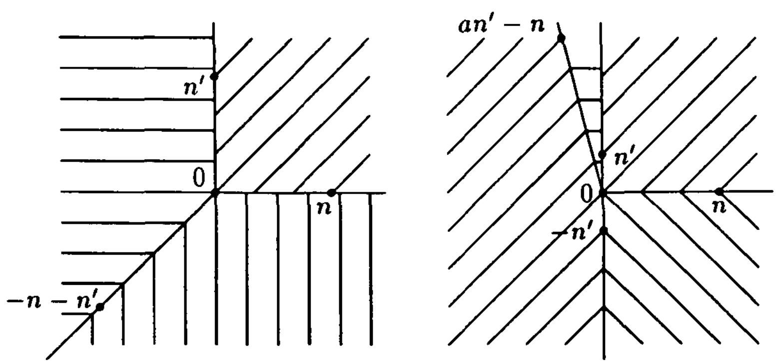
$$\mathscr{X}_\sigma=\operatorname{Spec} k[\sigma^\vee\cap\mathbb{Z}^n]$$

where $\sigma^\vee=\{\lambda\in N_{\mathbb{Q}}\mid\lambda(\sigma)\geqq 0\}$ is the dual cone of σ. These varieties can be glued together to give a variety over k denoted by $\mathscr{X}_\Sigma$. We consider toric varieties where Σ consists of a finite number of cones. Now if $\{\sigma_1,\dots,\sigma_r\}$ are cones such that $\dim_{\mathbb{Q}}(\sigma_i\cap\sigma_j)<\min(\dim_{\mathbb{Q}}\sigma_i,\ \dim_{\mathbb{Q}}\sigma_j)$ for all $i\neq j$ which cover $\bigcup_{\sigma\in\Sigma}\sigma$ then $\{U_1,\dots,U_r\}$, where $U_i=\operatorname{Spec} k[\sigma_i^\vee\cap M]$, is an open affine cover of $\mathscr{X}_\Sigma$. We call this cover *the standard open affine cover* of $\mathscr{X}$.

As far as our interests are concerned we have that the dual of any cone in Σ is of dimension n. Thus $\mathbb{Z}(\sigma^\vee\cap M)=M$ for all $\sigma\in\Sigma$.

Examples. (i) $\mathscr{X}=\mathbb{P}^2$.

(ii) $\mathscr{X}=\mathbb{P}(\mathscr{O}_{\mathbb{P}^1}\oplus\mathscr{O}_{\mathbb{P}^1}(a))$. This is a Hirzebruch surface. The construction of projective space bundles can be found in [**2**]. The fans of both of these examples are as given in the following diagram.



(i) (ii)

Another approach to toric varieties can be found in [**6**] where one embeds a torus into an affine toric variety (hence the name) as a dense open subset and then patch affines together to give the general case. In fact this idea of embedding a torus yields the following theorem.

Theorem 2.3. *Suppose an algebraic torus $T\cong(k^*)^n$ acts algebraically on an irreducible normal variety $\mathscr{X}$ locally of finite type over k. If $\mathscr{X}$ contains an open orbit isomorphic to T, then there exists a unique fan Σ such that $\mathscr{X}$ is equivariantly isomorphic to $\mathscr{X}_\Sigma$.*

Proof. See [**6**]. □

3. Calculating intersections

In this section we abstract the properties we shall use to perform our calculations.

Let $R=k[x_1^{\pm 1},\dots,x_n^{\pm 1},\partial_1,\dots,\partial_n]$ be the ring of differential operators on an n dimensional torus. Consider the $(k^*)^n$-action on R given by

$$(k^*)^n\times R\to R$$

$$(\lambda_1,\dots,\lambda_n)\cdot x_i\mapsto \lambda_i x_i$$

$$(\lambda_1,\dots,\lambda_n)\cdot \partial_i\mapsto \lambda_i^{-1}\partial_i.$$

Define

$$G_R(i_1,\dots,i_n)=\{\theta\in R\,|\,(\lambda_1,\dots,\lambda_n)\cdot\theta=\lambda_1^{i_1}\dots\lambda_n^{i_n}\theta \text{ for all } (\lambda_1,\dots,\lambda_n)\in(k^*)^n\}.$$

We call these the homogeneous elements of degree $(i_1,\dots,i_n)$. Let $W=G_R(0,\dots,0)$. Then $W=k[x_1\partial_1,\dots,x_n\partial_n]$. Let $gr\,R$ be the associated graded ring with respect to the differential operator filtration. We note that $gr\,R$ is a commutative domain. Let X_i be the image of x_i in the associated graded ring, and likewise, Δ_i the image of ∂_i. Consider the map

$$sym\colon R\to gr\,R=k[X_1^{\pm 1},\dots,X_n^{\pm 1},\Delta_1,\dots,\Delta_n]$$

$$\sum a_{i_1,\dots,i_n,j_1,\dots,j_n}x_1^{i_1}\dots x_n^{i_n}\partial_1^{j_1}\dots\partial_n^{j_n}\mapsto \sum\nolimits_{j_1+\dots+j_n=t}a_{i_1,\dots,i_n,j_1,\dots,j_n}X_1^{i_1}\dots X_n^{i_n}\Delta_1^{j_1}\dots\Delta_n^{j_n}$$

where t is the maximum integer such that $t=j_1+\dots+j_n$ and $a_{i_1,\dots,i_n,j_1,\dots,j_n}\neq 0$. There is an action induced on $gr\,R$ given by

$$(k^*)^n\times gr\,R\to gr\,R$$

$$(\lambda_1,\dots,\lambda_n)\cdot X_i\mapsto \lambda_i X_i$$

$$(\lambda_1,\dots,\lambda_n)\cdot \Delta_i\mapsto \lambda_i^{-1}\Delta_i.$$

Define

$$G_{gr\,R}(i_1,\dots,i_n)=\{\theta\in gr\,R\,|\,(\lambda_1,\dots,\lambda_n)\cdot\theta=\lambda_1^{i_n}\dots\lambda_n^{i_n}\theta \text{ for all } (\lambda_1,\dots,\lambda_n)\in(k^*)^n\}$$

to be the set of elements of $gr\,R$ of degree $(i_1,\dots,i_n)$. Now,

$$(\lambda_1,\dots,\lambda_n)\cdot sym(r)=sym((\lambda_1,\dots,\lambda_n)\cdot r)$$

for all $r\in R$ and so the $(k^*)^n$-action is equivalent with respect to *sym*. Thus we have a $\mathbb{Z}^n$-graded structure on $gr\,R$ compatible with the grading arising from the differential operator filtration. In the rest of this paper we shall write $\mathbf{x}^\lambda := x_1^{\lambda_1} \dots x_n^{\lambda_n}$ for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$. We now think about subalgebras of R. If A is a $\mathbb{Z}^n$-graded subalgebra of R then we can filter A by the differential operator filtration and form $gr\,A$, the associated graded ring. Note that $gr\,A$ will be a $\mathbb{Z}^n$-graded subalgebra of $gr\,R$. Then $gr\,A$ is a commutative domain since $gr\,A \subseteq gr\,R$. All our calculations involving $gr\,A$ take place in $gr\,R$. Thus for $r \in A$ we denote by $\bar{r}$, the image $sym(r)$. To illustrate the above, W is a graded subalgebra of R and $gr\,W = k[X_1\Delta_1, \dots, X_n\Delta_n]$ is a graded subalgebra of $gr\,R$.

We will be using the concept of gcd and lcm in both of the rings R and $gr\,R$. Consider $r_1, \dots, r_p \in R$ all of the same degree, say λ. Then $r_j = \mathbf{x}^\lambda f_j$ for some $f_j \in W$ for all $j = 1, \dots, p$. Then $(r_1, \dots, r_p) := \mathbf{x}^\lambda (f_1, \dots, f_p)$ where $(f_1, \dots, f_p)$ is the standard gcd in W. Similarly $lcm(r_1, \dots, r_p) := \mathbf{x}^\lambda lcm(f_1, \dots, f_p)$. The same process is used for $gr\,R$ with the gcd and lcm really being found in $gr\,W$. Thus these notions are well-defined.

Lemma 3.1. *Let $L = \{q_1, \dots, q_p\}$ be a finite collection of points in $\mathbb{Q}^n$ such that no two elements of L are linearly dependent. Suppose that the $\mathbb{Q}$-span of the collection L is $\mathbb{Q}^n$. Then there exists a collection of sets $\{I_\ell\}$ such that $\bigcup I_\ell = L$, $|I_\ell| = n$ and $q_i \in S_\ell := \sum_{h \in I_\ell} \mathbb{Q}_{\geqq 0} h \Rightarrow q_i \in I_\ell$.*

Proof. Choose $d_1, \dots, d_n \in L$ such that they form a $\mathbb{Q}$-basis of $\mathbb{Q}^n$. Suppose $d' \in L \setminus \{d_1, \dots, d_n\}$ such that $d' \in \sum_{i=1}^n \mathbb{Q}_{\geqq 0} d_i$. Then

$$d_r \notin \sum_{\substack{i=1 \\ i \neq r}}^{n} \mathbb{Q}_{\geqq 0} d_i + \mathbb{Q}_{\geqq 0} d'.$$

By repetition of this and the fact that $|L| < \infty$ we can construct a collection $\{I_\ell\}$ satisfying the condition. □

Definition 3.2. With the notation of the previous lemma, the collection of cones $\{S_\ell\}$ as constructed in the proof is called a partition of $\mathbb{Q}_{\geqq 0} L$. Let $(gr\,A)^\ell$ be the k-algebra generated by $\{a \in gr\,A \mid \deg a \in S_\ell\}$.

To proceed with any calculations we need to consider a particular type of graded subalgebra of R.

Definition 3.3. Let $L \subseteq \mathbb{Z}^n$ such that $|L| < \infty$ and no two elements of L are linearly dependent. A graded k-subalgebra A of R is said to be L-faithful if

(1) $W \subseteq A$;

(2) There exist $a_l \in A$ for $l \in L$ such that $\deg a_l = l$ for all $l \in L$, $A = k[a_l \mid l \in L][W]$ and $gr\,A = k[\overline{a_l} \mid l \in L][gr\,W]$;

(3) $G_{gr\,A}(s_1, \dots, s_n)$ is a cyclic $gr\,W$-module for all $(s_1, \dots, s_n) \in \mathbb{Z}_{\geqq 0} L$. Here $G_{gr\,A}(s_1, \dots, s_n) = \{\theta \in gr\,A \mid \deg \theta = (s_1, \dots, s_n)\}$;

(4) $(gr\,A)^{\ell}=k[\bar{a}_l \mid l\in I_\ell][gr\,W]$.

We say that $\{a_l \mid l\in L\}$ is the set of L-generators of A. A is said to be faithful if it is L-faithful for some L.

We will work with a collection of graded subalgebras of R but the faithful condition is not enough to proceed. We need some information about the algebras relative to each other.

Definition 3.4. A collection $A_1,\dots,A_r$ of L-faithful subalgebras of R where A_i has L-generators $\{a_{i,l} \mid l\in L\}$ is said to be L-compatible if

(1) $\Theta((gr\,A_j)^{\ell},(gr\,A_k)^{\ell})=1$ for all $j,k=1,\dots,r$ and ℓ, where

$$\Theta((gr\,A_j)^{\ell},(gr\,A_k)^{\ell})=\left(\prod_{l\in I_\ell}\frac{\bar{a}_{j,l}}{(\bar{a}_{j,l},\bar{a}_{k,l})},\ \prod_{l\in I_\ell}\frac{\bar{a}_{k,l}}{(\bar{a}_{j,l},\bar{a}_{k,l})}\right)$$

(2) $lcm(\bar{a}_{1,l},\dots,\bar{a}_{r,l})=\overline{lcm(a_{1,l},\dots,a_{r,l})}$ for all $l\in L$.

where $\{a_{i,l} \mid l\in L\}$ are the L-generators of A_i.

Lemma 3.5. *Let A,B be L-compatible and faithful subalgebras of R with L-generators $\{a_l \mid l\in L\}$, and $\{b_l \mid l\in L\}$ respectively. Then, $gr\,A\cap gr\,B=k[\bar{\theta}_l \mid l\in L][gr\,W]$ where $\bar{\theta}_l=lcm(\bar{a}_l,\bar{b}_l)$.*

Proof. Write $\bar{a}_l=X_1^{r_{l,1}}\dots X_n^{r_{l,n}}m_{a_l}$ and $\bar{b}_l=X_1^{r_{l,1}}\dots X_n^{r_{l,n}}m_{b_l}$ where $m_{a_l},m_{b_l}\in gr\,W$. Consider $(s_1,\dots,s_n)\in\mathbb{Z}^n$. Then $(s_1,\dots,s_n)\in S_\ell$ for some ℓ and so there exist $q_1,\dots,q_p\in\mathbb{N}_0$ such that at most n are non-zero and

$$G_{gr\,A}(s_1,\dots,s_n)=\prod_{l\in I_\ell}\bar{a}_l^{q_l}gr\,W$$

$$G_{gr\,B}(s_1,\dots,s_n)=\prod_{l\in I_\ell}\bar{b}_l^{q_l}gr\,W.$$

This uses condition 4 of the definition of L-faithful. Now

$$\begin{aligned}G_{gr\,A\cap gr\,B}(s_1,\dots,s_n)&=G_{gr\,A}(s_1,\dots,s_n)\cap G_{gr\,B}(s_1,\dots,s_n)\\&=G_{(gr\,A)^{\ell}}(s_1,\dots,s_n)\cap G_{(gr\,B)^{\ell}}(s_1,\dots,s_n)\\&=X_1^{s_1}\dots X_n^{s_n}\left(\prod_{l\in I_\ell}m_{a_l}^{q_l}gr\,W\right)\cap X_1^{s_1}\dots X_n^{s_n}\left(\prod_{l\in I_\ell}m_{b_l}^{q_l}gr\,W\right)\end{aligned}$$

$$=X_1^{s_1}\dots X_n^{s_n}\prod_{l\in I_\ell} m_{\theta_l}^{q_l}\left(\prod_{l\in I_\ell} w_{a_l}^{q_l} gr\,W\cap\prod_{l\in I_\ell} w_{b_l}^{q_l} gr\,W\right)$$

where $m_{\theta_l}=(m_{a_l},m_{b_l})$, $w_{a_l}=m_{a_l}/m_{\theta_l}$ and $w_{b_l}=m_{b_l}/m_{\theta_l}$. The second equality follows by point 4 of the definition of faithful. Since A and B are L-compatible,

$$\left(\prod_{l\in I_\ell} w_{a_l}^{q_l},\prod_{l\in I_\ell} w_{b_l}^{q_l}\right)=1.$$

Thus $\prod_{l\in I_\ell} w_{a_l}^{q_l} gr\,W\cap\prod_{l\in I_l} w_{b_l}^{q_l} gr\,W=\prod_{l\in I_\ell}(w_{a_l}w_{b_l})^{q_l} gr\,W$. This gives

$$G_{gr\,A\cap gr\,B}(s_1,\dots,s_n)=\prod_{l\in I_\ell}(X_1^{r_{l,1}}\dots X_n^{r_{l,n}}w_{a_l}w_{b_l}m_{\theta_l})^{q_l} gr\,W$$

$$=\prod_{l\in I_\ell}\bar{\theta}_l^{q_l} gr\,W$$

where the $\bar{\theta}_l$ are as in the statement. Clearly $\bar{\theta}_l\in gr\,A\cap gr\,B$ for all $l\in L$ and $gr\,W\subseteq gr\,A\cap gr\,B$ by assumption and so $gr\,A\cap gr\,B=k[\bar{\theta}_l\,|\,l\in L][gr\,W]$. □

Remark. Without compatibility we can still use the techniques of the proof of the theorem to find out the monomial which generates elements of any particular degree (as a module over $gr\,W$).

Lemma 3.6. *Let $P\subseteq Q$ be an inclusion of $\mathbb{N}$-filtered rings such that $gr\,P=gr\,Q$. Then $P=Q$.*

Proof. Clearly $P_0=Q_0$. Suppose that $P_{i-1}=Q_{i-1}$ for $i\in\mathbb{N}$. Then let $q\in Q_i$. Thus by hypothesis, there is a $p\in P_i$ such that $q+Q_{i-1}=p+P_{i-1}$. Hence $q-p\in P_{i-1}$ and thus $q\in P_i$. Thus by induction, $P=Q$. □

Proposition 3.7. *Let A, B be compatible L-faithful subalgebras of R with L-generators $\{a_l\,|\,l\in L\}$, and $\{b_l\,|\,l\in L\}$ respectively. Then, $gr\,A\cap gr\,B=gr(A\cap B)$ and $A\cap B=k[\theta_l\,|\,l\in L,x_1\partial_1,\dots,x_n\partial_n]$ where $\theta_l=lcm(a_l,b_l)$. Also $A\cap B$ is an L-faithful Noetherian subalgebra of R.*

Proof. From the previous lemma we have $gr\,A\cap gr\,B=k[\bar{\theta}_l\,|\,l\in L][gr\,W]$. Define $\theta_l=lcm(a_l,b_l)$. Then $\theta_l\in A\cap B$. Also by the compatibility,

$$\overline{lcm(a_l,b_l)}=lcm(\bar{a}_l,\bar{b}_l)$$

for all $l\in L$ and so $\bar{\theta}_l$ is the image of θ_l under the symbol map. Thus $\bar{\theta}_l\in gr(A\cap B)$. Since $gr\,W\subseteq gr(A\cap B)$ then $gr\,A\cap gr\,B\subseteq gr(A\cap B)$. The opposite inclusion is obvious and so

$$gr\,A \cap gr\,B = gr(A \cap B).$$

Using the inclusions

$$gr\,A \cap gr\,B = k[\bar{\theta}_l \mid l \in L][gr\,W] \subseteq gr\,k[\theta_l \mid l \in L, x_1\partial_1, \ldots, x_n\partial_n] \subseteq gr(A \cap B)$$

we have equality and since $k[\theta_l \mid l \in L; x_1\partial_1, \ldots, x_n\partial_n] \subseteq A \cap B$, then Lemma 3.6 gives the required equality since the associated graded rings have been taken with respect to the differential operator filtration which is an $\mathbb{N}$-filtration. The Noetherian condition follows from the fact that by the lemma, $gr(A \cap B)$ is finitely generated and [4, 1.6.9]. Now

(1) $W \subseteq A \cap B$;
(2) $A \cap B = k[\theta_l \mid l \in L]W$ and $gr(A \cap B) = k[\bar{\theta}_l \mid l \in L][gr\,W]$ with $\deg \theta_l = l$;
(3) $G_{gr(A \cap B)}(s_1, \ldots, s_n)$ is a cyclic $gr\,W$-module for all $(s_1, \ldots, s_n) \in \mathbb{Z}^n$;
(4) If $(s_1, \ldots, s_n) \in I_\ell$ then $G_{gr(A \cap B)}(s_1, \ldots, s_n) \subseteq k[\bar{\theta}_l \mid l \in I_\ell][gr\,W]$.

Thus $A \cap B$ is an L-faithful subalgebra of R. □

We now have the required result to intersect two faithful and compatible subalgebras of R. We will use induction to prove the general result. To do this we need to show that compatibility and faithfulness are preserved by taking intersections.

Lemma 3.8. *Let A, B, C be L-compatible and faithful subalgebras of R. Then $A \cap B, C$ are L-compatible and faithful subalgebras of R.*

Proof. The important point is that $A \cap B$ and C are L-compatible. Let the L-generators of A, B, C be $\{a_l \mid l \in L\}$, $\{b_l \mid l \in L\}$, $\{c_l \mid l \in L\}$ respectively. By the previous proposition $A \cap B$ and C are L-faithful subalgebras of R. Also the compatibility of A, B and C gives $\Theta((gr(A \cap B))^\ell, (gr\,C)^\ell) = 1$. Finally for any l,

$$\begin{aligned} lcm(\overline{lcm(a_l, b_l)}, \bar{c}_l) &= lcm(lcm(\bar{a}_l, \bar{b}_l), \bar{c}_l) \\ &= lcm(\bar{a}_l, \bar{b}_l, \bar{c}_l) \\ &= \overline{lcm(a_l, b_l, c_l)} \\ &= \overline{lcm(lcm(a_l, b_l), c_l)} \end{aligned}$$

which gives the compatibility. □

Theorem 3.9. *Let $A_1, \ldots, A_r$ be a collection of L-compatible and faithful subalgebras of R. Let $\{a_{i,l} \mid l \in L\}$ be the L-generators of A_i. Then*

$$gr\,A_1 \cap \cdots \cap gr\,A_r = gr(A_1 \cap \cdots \cap A_r) = k[\bar{\theta}_l \mid l \in L][gr\,W];$$

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$$A_1 \cap \cdots \cap A_r = k[\theta_l \mid l \in L]W$$

where $\theta_l = lcm(a_{1,l}, \ldots, a_{r,l})$ *and* $\bar{\theta}_l = lcm(\bar{a}_{1,l}, \ldots, \bar{a}_{r,l})$. *Also* $A_1 \cap \cdots \cap A_r$ *is Noetherian.*

Proof. We proceed by induction on r. The result is trivial for $r=1$. Let $r \geqq 2$ and suppose the result holds for $r-1$. Then

$$gr\, A_1 \cap \cdots \cap gr\, A_{r-1} = gr(A_1 \cap \cdots \cap A_{r-1}) = k[\bar{\varphi}_l \mid l \in L]\,[gr\, W];$$

$$A_1 \cap \cdots \cap A_{r-1} = k[\varphi_l \mid l \in L]W$$

where $\varphi_l = lcm(a_{1,l}, \ldots, a_{r-1,l})$ and $\bar{\varphi}_l = lcm(\bar{a}_{1,l}, \ldots, \bar{a}_{r-1,l})$. By the previous proposition and induction we see that $A_1 \cap \cdots \cap A_{r-1}$ and A_r are compatible L-faithful subalgebras of R. Now by Proposition 3.7 and the induction hypothesis,

$$gr\, A_1 \cap \cdots \cap gr\, A_r = gr(A_1 \cap \cdots \cap A_{r-1}) \cap gr\, A_r = gr(A_1 \cap \cdots \cap A_r) = k[\bar{\theta}_l \mid l \in L]\,[gr\, W]$$

where

$$\theta_l = lcm(lcm(a_{1,l}, \ldots, a_{r-1,l}), a_{r,l}) = lcm(a_{1,l}, \ldots, a_{r,l})$$

$$\bar{\theta}_l = lcm(lcm(\bar{a}_{1,l}, \ldots, \bar{a}_{r-1,l}), \bar{a}_{r,l}) = lcm(\bar{a}_{1,l}, \ldots, \bar{a}_{r,l}).$$

Again $gr(A_1 \cap \cdots \cap A_r)$ is finitely generated and so $A_1 \cap \cdots \cap A_r$ is Noetherian. This completes the induction. □

We will be applying these ideas to calculating the global sections of sheaves on a toric variety. We need to define the notions of faithfulness and compatibility for these.

Definition 3.10. Let $\mathscr{F}$ be a quasi-coherent sheaf on a toric variety $\mathscr{X}$ such that $\{\mathscr{F}(U_i)\}$ is an L-faithful and compatible collection of subalgebras of R for some L where $\{U_i\}$ is the standard open affine cover of $\mathscr{X}$. Then $\mathscr{F}$ is said to be an *L-faithful and compatible sheaf* on $\mathscr{X}$.

4. Applications to toric varieties

We now apply the results of the previous section to toric varieties. We must first establish that our results are applicable to the rings in question. From now on until stated otherwise we let $\mathscr{X} = \operatorname{Spec} k[\sigma \cap \mathbb{Z}^n]$ be an n-dimensional affine toric variety for some rational n-dimensional cone σ given as the irredundant intersection of a finite collection of halfspaces $H_1, \ldots, H_r$ with boundaries $\partial H_1, \ldots, \partial H_r$ respectively. Recall that $\mathbb{Z}(\sigma \cap \mathbb{Z}^n) = \mathbb{Z}^n$. Let $h_1, \ldots, h_r$ be the linear polynomials defining $\partial H_1, \ldots, \partial H_r$ respectively. Define $H_i^c = \mathbb{Q}^n \setminus H_i$. Recall that all affine toric varieties can be written in this way.

In order to describe $\mathscr{D}(\mathscr{X})$ and how it relates to the underlying cone σ we use a

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correspondence between functionals on $\mathbb{Q}^n$ and the degree zero elements, $\mathbb{Q}[x_1\partial_1,\ldots,x_n\partial_n]$.

Let $k\Lambda$ be the associated semigroup algebra of Λ and $D(k\Lambda)$ the ring of differential operators. We view $D(k\Lambda)$ as a subring of R as defined above. Now for $\mu\in\mathbb{Z}^n$, $x_i\partial_i * \mathbf{x}^\mu=\mu_i\mathbf{x}^\mu$ where $\mu=(\mu_1,\ldots,\mu_n)$. Now $W=\mathbb{Q}[x_1\partial_1,\ldots,x_n\partial_n]\otimes_{\mathbb{Q}} k$. Thus elements of W define polynomial functions from $\mathbb{Z}^n\subseteq\mathbb{Q}^n$ to k by the rule $x_i\partial_i*(\mu)=\mu_i$ for $\mu\in\mathbb{Z}\Lambda$. Thus for $f\in W$ and $\mu\in\mathbb{Z}^n$,

$$f(x_1\partial_1,\ldots,x_n\partial_n)(\mathbf{x}^\mu)=f(\mu)\mathbf{x}^\mu.$$

Hence we can now regard an element $f\in(\mathbb{Q}^n)^*$ as an element of $\mathbb{Q}[x_1\partial_1,\ldots,x_n\partial_n]$.

Let $\Lambda=\sigma\cap\mathbb{Z}^n$, $\Omega(\lambda)=\{\mu\in\Lambda\mid\lambda+\mu\notin\Lambda\}$ and $\Omega_{H_i}(\lambda)=\{\mu\in\Lambda\mid\lambda+\mu\notin H_i\cap\mathbb{Z}^n\}$ for $\lambda\in\mathbb{Z}^n$. Observe that $\Omega(\lambda)=\bigcup\Omega_{H_i}(\lambda)$. If $\Omega\subseteq\mathbb{Z}^n$, let $\bar{\Omega}$ be the Zariski closure of Ω in $\mathbb{Q}^n$ and $I(\Omega)=\{f\in W\mid f(\Omega)=0\}$.

We now give some results describing the differential operators on an affine toric variety. These results are taken from [5] with a slight change in the setting in which they work.

Lemma 4.1. *Let Λ be a semigroup of $\mathbb{Q}^n$ with $\mathbb{Q}\Lambda=\mathbb{Q}^n$. For $g\in(\mathbb{Q}^n)^*$ and $b\in\mathbb{Q}$ set $\Lambda_b=\{\lambda\in\Lambda\mid g(\lambda)=b\}$. Suppose that*

(1) $\Lambda_b\neq\emptyset$;

(2) $\dim_{\mathbb{Q}}\mathbb{Q}\Lambda_0=n-1$.

Then $\bar{\Lambda}_b=\{\lambda\in\mathbb{Q}^n\mid g(\lambda)=b\}$.

Proof. See [5, 2.5]. □

Lemma 4.2. *For $i=1,\ldots,r$, if $h_i(\mathbb{Z}^n)=a\mathbb{Z}$ for some $a>0$ then there exists $\lambda\in\Lambda$ such that $h_i(\lambda)=a$.*

Proof. By assumption $h_i(\lambda')=a$ for some $\lambda'\in\mathbb{Z}^n$. Now ∂H_i and ∂H_j are distinct subspaces of codimension one in $\mathbb{Q}^n$ and so $\partial H_i\nsubseteq\partial H_j$ and since $\partial H_i\cap\Lambda$ spans ∂H_i there exists $u_j\in\partial H_i\cap\Lambda$ with $u_j\notin\partial H_j$. Therefore $h_i(u_j)=0$ and $h_j(u_j)>0$.

Let

$$u=\sum_{\substack{j=1\\ j\neq i}}^{r}u_j\in\Lambda.$$

Then $h_i(u)=0$ and $h_j(u)>0$. Set $\lambda=\lambda'+Nu$ for $N\gg0$ to get the result. □

Lemma 4.3. *With the notation as above,*

$$\overline{\Omega_{H_i}(\lambda)} = \{\mu \in \mathbb{Q}^n \mid h_i(\mu) \in h_i(\mathbb{Z}^n) \text{ and } 0 \leqq h_i(\mu) < -h_i(\lambda)\}$$

for $i = 1, \ldots, r$. Hence $\overline{\Omega_{H_i}(\lambda)}$ is a finite union of hyperplanes parallel to ∂H_i.

Proof. Since h_i is linear and the right hand side is Zariski closed then we have "$\subseteq$". Let μ lie in the right hand side. Suppose $h_i(\mu) = b \in h_i(\mathbb{Z}^n)$ with $0 \leqq h_i(\mu) < -h_i(\lambda)$. Let $\Lambda_b = \{\lambda \in \Lambda \mid h_i(\mu) = b\}$. By the previous lemma we have that $\Lambda_b \neq \emptyset$. Also since ∂H_i is a face of σ then $\dim_{\mathbb{Q}} \bar{\Lambda}_0 = n-1$. Hence by Lemma 4.1, $\bar{\Lambda}_b = \{\lambda \in \mathbb{Q}^n \mid h_i(\lambda) = b\}$. Thus $\mu \in \Omega_{H_i}(\lambda)$. □

Proposition 4.4. *With the above notation,*

(1) $W \subseteq \mathscr{D}(\mathscr{X})$;

(2) $\mathscr{D}(\mathscr{X}) = \bigoplus_{\lambda \in \mathbb{Z}^n} \mathbf{x}^\lambda I(\overline{\Omega(\lambda)})$;

(3) $\overline{\Omega(\lambda)}$ *is a finite union of hyperplanes each parallel to an $(n-1)$-dimensional face of σ;*

(4) $\Omega(\lambda) = \overline{\Omega(\lambda)} \cap \Lambda$;

(5) *$gr\,\mathscr{D}(\mathscr{X})$ is finitely generated as a k-algebra.*

Proof. (1) This is obvious.

(2) Clearly $G_{\mathscr{D}(\mathscr{X})}(\lambda) \subseteq \mathbf{x}^\lambda W$. If $f \in W$ then $\mathbf{x}^\lambda f * \mathbf{x}^\mu = f(\mu)\mathbf{x}^{\lambda+\mu}$. Now since differential operators extend uniquely to localisations and

$$\mathscr{D}(\mathscr{X}) = \{\theta \in D(k\mathbb{Z}^n) \mid \theta * \mathscr{O}(\mathscr{X}) \subseteq \mathscr{O}(\mathscr{X})\}$$

then $\mathbf{x}^\lambda f \in \mathscr{D}(\mathscr{X}) \Leftrightarrow f(\mu) = 0$ for $\mu \in \Omega(\lambda)$. This gives the equality.

(3) We have $\Omega(\lambda) = \bigcup_i \Omega_{H_i}(\lambda)$. By Lemma 4.3, $\overline{\Omega_{H_i}(\lambda)}$ is a finite union of hyperplanes parallel to ∂H_i. Then

$$\overline{\Omega(\lambda)} = \overline{\bigcup_i \Omega_{H_i}(\lambda)} = \bigcup_i \overline{\Omega_{H_i}(\lambda)}$$

and so $\overline{\Omega(\lambda)}$ is a finite union of hyperplanes each parallel to some ∂H_i.

(4) Clearly we have $\Omega(\lambda) \subseteq \overline{\Omega(\lambda)} \cap \Lambda$. Conversely, suppose $F \subseteq \overline{\Omega(\lambda)}$ is a hyperplane parallel to some ∂H_i. Then by the following lemma, we have $\lambda + F \subseteq H_i^c$ and so $\lambda + (F \cap \Lambda) \subseteq H_i^c$ and hence $F \cap \Lambda \subseteq \Omega(\lambda)$. By 3, $\overline{\Omega(\lambda)}$ is a union of such hyperplanes and so

$$\overline{\Omega(\lambda)} \cap \Lambda = (\cup F) \cap \Lambda = \cup (F \cap \Lambda) \subseteq \Omega(\lambda)$$

which gives the equality.

(5) This is proved in [5]. □

The next lemma shows exactly which hyperplanes lie in $\overline{\Omega(\lambda)}$.

Lemma 4.5. *Let F be a hyperplane in $\mathbb{Q}^n$ parallel to some ∂H_i with $F=\overline{F\cap\Lambda}$. Then for $\lambda\in\mathbb{Z}^n$, $\lambda+F\subseteq H_i^c$ if and only if $F\subseteq\Omega(\lambda)$.*

Proof. $\Rightarrow$ is obvious.

For the converse, note that $h_i(F)=b$ for some $b\in\mathbb{Q}_{\geq 0}$ and using Lemma 4.3 we have $h_i(\lambda+F)<0$ which implies that $\lambda+F\subseteq H_i^c$. □

Now that we have described $\mathscr{D}(\mathscr{X})$, we wish to check that it is of the correct form to apply the ideas of the previous section.

Proposition 4.6. *With $\mathscr{X}$ as above, $\mathscr{D}(\mathscr{X})$ is L-faithful for some L. Also if $L'\supseteq L$ with $|L'|<\infty$ such that no two elements of L' are linearly dependent then $\mathscr{D}(\mathscr{X})$ is L'-faithful for some set of L'-generators.*

Proof. We only need to check 2 and 4 of the definition for a valid L as the rest is done. Consider the partition of $\mathbb{Q}^n$ given by

$$\{S_\ell\}:=\{(\textstyle\bigcap_{i\in I}H_i)\cap(\bigcap_{i\notin I}-H_i)\}_{I\subseteq\{1,\ldots,r\}}.$$

For each S_ℓ choose elements of $\mathbb{Z}^n$ such that they generate $S_\ell\cap\mathbb{Z}^n$ over $\mathbb{Z}_{\geq 0}$. Consider the set of all such elements $\{d_1,\ldots,d_t\}$ and the elements $\{a_1,\ldots,a_t\}$ of $\mathscr{D}(\mathscr{X})$ such that $G_{\mathscr{D}(\mathscr{X})}(d_i)=a_iW$. Consider the set of all such elements arising from all of the S_ℓ and add in any elements of non-zero degree needed to generate $\mathscr{D}(\mathscr{X})$ along with W, such that their images in $gr\,\mathscr{D}(\mathscr{X})$ along with $gr\,W$ generate $gr\,\mathscr{D}(\mathscr{X})$. This will be the set of L-generators where L is the set of all their degrees. For the moment we will refer to this set as simply the generators. We further divide the partition as in Lemma 3.1 to give a new partition which we still call $\{S_\ell\}$ such that the degrees in S_ℓ of the set of generators form a basis of $\mathbb{Q}^n$ for each ℓ. It is clear that we have condition 2 with these generators.

Now each $S_\ell\subseteq(\bigcap_{i\in I}H_i)\cap(\bigcap_{i\notin I}-H_i)$ for some I. Consider any $\lambda\in S_\ell$. Then $\overline{\Omega(\lambda)}$ consists of hyperplanes parallel to some ∂H_i for $i\notin I$. This follows from the previous lemma and the fact that $\lambda+\Lambda\subseteq H_i$ for all $i\in I$. For each ∂H_i where $i\notin I$, $\overline{\Omega(\lambda)}$ contains at least one hyperplane parallel to ∂H_i unless $\lambda\in\partial H_i$.

Consider λ, $\mu\in S_\ell$ for some ℓ. Then

$$\begin{aligned}\overline{\Omega(\lambda+\mu)}&=\overline{\{p\in\Lambda\mid p+\lambda+\mu\notin\Lambda\}}\\&\subseteq\overline{\{p\in\Lambda\mid p+\mu\notin\Lambda\}}\cup\overline{\{p\in\Lambda\mid p+\mu\in\Lambda,p+\lambda+\mu\notin\Lambda\}}\\&=\overline{\Omega(\mu)}\cup\overline{(\Omega(\lambda)-\mu)\cap\Lambda}\\&\subseteq\overline{\Omega(\mu)}\cup(\overline{\Omega(\lambda)}-\mu).\end{aligned}$$

Now $\overline{\Omega(\mu)}$ is a union of hyperplanes, each parallel to some ∂H_i for $i\notin I$. By Lemma 4.3

such a hyperplane, say F, is defined by an equation $h_i - a = 0$ where $0 \leqq a < -h_i(\mu)$ and $a \in h_i(\mathbb{Z}^n)$. Now since $\lambda \in S_\ell$, then $h_i(\lambda) \leqq 0$ and so $h_i(\lambda+\mu) \leqq h_i(\mu)$. Thus by another application of Lemma 4.3 we have $F \subseteq \overline{\Omega(\lambda+\mu)}$. A similar argument shows that $\overline{\Omega(\lambda)} - \mu \subseteq \overline{\Omega(\lambda+\mu)}$. Thus the above inclusion becomes equality.

Hence

$$\overline{\Omega(\lambda+\mu)} = \overline{\Omega(\mu)} \cup \overline{\Omega(\lambda)} - \mu)$$

and since the right hand side is a disjoint union of hyperplanes then $f_{\lambda+\mu} = f_\lambda^* f_\mu$ where

$$I(\Omega(\lambda)) = f_\lambda W$$

$$I(\Omega(\mu)) = f_\mu W$$

$$I(\Omega(\lambda+\mu)) = f_{\lambda+\mu} W$$

and $f_\lambda^*(p) := f_\lambda(p+\mu)$. Hence $\mathbf{x}^{\lambda+\mu} f_{\lambda+\mu} = \mathbf{x}^\lambda f_\lambda \mathbf{x}^\mu f_\mu$. Thus $(gr\,\mathcal{D}(\mathcal{X}))^\ell$ is generated by generators with degree in S_ℓ. This gives condition 4. Also by the above we can reduce our L until we have that every two elements are linearly independent. Thus $\mathcal{D}(\mathcal{X})$ is L-faithful.

Consider $\lambda \in \mathbb{Z}^n \backslash L$. Suppose $G_{\mathcal{D}(\mathcal{X})}(\lambda) = \alpha_\lambda W$. Then by adding a_λ to the L-generators and setting $L' = L \cup \{\lambda\}$ does not affect any part of the above proof. Hence $\mathcal{D}(\mathcal{X})$ is L'-faithful. This completes the proof. □

We now consider a general toric variety. When we take an open affine cover of $\mathcal{X}$ it is of the form as described in Section 2. Let $\{U_i\}$ be the cover. Then each U_i is an affine toric variety. Thus without loss of generality, $\mathcal{O}_{\mathcal{X}}(U_i) = k[\sigma_i \cap \mathbb{Z}^n]$ for an n-dimensional cone σ_i in $\mathbb{Q}^n$.

Proposition 4.7. *Let $\mathcal{X}$ be a toric variety. Then there exists an L such that each $\mathcal{D}_{\mathcal{X}}(U_i)$ is L-faithful and that if σ_i is an intersection of halfspaces $H_1, \ldots, H_r$, then either $S_\ell \subseteq H_i$ or $S_\ell \subseteq -H_i$ for all ℓ.*

Proof. By the above proposition, $\mathcal{D}_{\mathcal{X}}(U_i)$ is L_i-faithful for some L_i. Let $L = \cup L_i$. Then again by the above $\mathcal{D}_{\mathcal{X}}(U_i)$ is L-faithful as we can subdivide the partition until the statement is true. □

Thus we have that $\mathcal{D}_{\mathcal{X}}$ is L-faithful for some carefully chosen L with the property of the above proposition. We shall use this property to show compatibility.

Proposition 4.8. *Let $\mathcal{X}$ be a toric variety with open affine cover $\{U_i\}$. Then $\mathcal{D}_{\mathcal{X}}$ is compatible. Moreover:*

(1) *with the partition as given in the above lemma,* $\Theta((gr\,\mathscr{D}_{\mathfrak{X}}(U_j))^{\ell}, (gr\,\mathscr{D}_{\mathfrak{X}}(U_k))^{\ell})=1$ *for all* j, k *and* ℓ;

(2) *if* $a_1,\ldots,a_r$ *are elements of distinct* $\mathscr{D}_{\mathfrak{X}}(U_i)$ *of the same degree such that* $\mathscr{D}_{\mathfrak{X}}(U_i)=a_iW$ *then*

$$lcm(\overline{a_1},\ldots,\overline{a_r})=\overline{lcm(a_1,\ldots,a_r)}.$$

Proof. We already have that $\mathscr{D}_{\mathfrak{X}}$ is faithful and so proving 1 and 2 gives the compatibility condition.

(1) We have seen that for an affine toric variety, $\overline{\Omega(\lambda)}$ is a finite union of hyperplanes each parallel to a face of the underlying cone. Hence for any partition, $\Theta(gr\,\mathscr{D}_{\mathfrak{X}}(U_j)^{\ell}, gr\,\mathscr{D}_{\mathfrak{X}}(U_k)^{\ell})$ is a monomial in terms of the linear equations of the faces of the underlying cones. In fact it is a monomial in terms of the equations of the hyperplanes which form $(n-1)$-dimensional faces to both of the underlying cones. Thus the only way statement 2 could fail is if some cones σ_j and σ_k have parallel faces. Say H is the hyperplane with linear equation h passing through two such parallel faces. Then either σ_j and σ_k lie on the same side of H or on opposite sides. We consider these two possibilities.

(A) *Same side.* Consider $\lambda\in S^{\ell}$. Let

$$\Omega^j(\lambda)=\{\mu\in\sigma_j\cap\mathbb{Z}^n \mid \lambda+\mu\notin\sigma_j\cap\mathbb{Z}^n\}$$

$$\Omega^k(\lambda)=\{\mu\in\sigma_k\cap\mathbb{Z}_n \mid \lambda+\mu\notin\sigma_k\cap\mathbb{Z}^n\}.$$

From above both closures consist of a finite union of hyperplanes parallel to faces of the underlying cones. By Lemma 4.3 any hyperplane parallel to H belongs to the union in both cases or neither. Thus $h\nmid(\overline{a_{j,l}}/(\overline{a_{j,l}},\overline{a_{k,l}}))$ for all $l\in L$ in the notation of Section 3. Hence $h\nmid\Theta(gr\,\mathscr{D}_{\mathfrak{X}}(U_j)^{\ell}, gr\,\mathscr{D}_{\mathfrak{X}}(U_k)^{\ell})$.

(B) *Opposite sides.* The partition is such that either $h(S_\ell)\geqq 0$ or $h(S_\ell)\leqq 0$ for all S_ℓ. Then without loss of generality we can suppose that σ_j and S_ℓ lie on the same side of the hyperplane H. In this instance, $h\nmid a$ for any L-generator $a\in gr\,\mathscr{D}_{\mathfrak{X}}(U_j)^{\ell}$. Hence $h\nmid\Theta(gr\,\mathscr{D}_{\mathfrak{X}}(U_j)^{\ell}, gr\,\mathscr{D}_{\mathfrak{X}}(U_k)^{\ell})$.

Hence $\Theta(gr\,\mathscr{D}_{\mathfrak{X}}(U_j)^{\ell}, gr\,\mathscr{D}_{\mathfrak{X}}(U_k)^{\ell})=1$ for all j, k and ℓ.

(2) Let $p=\deg a_i$. Then from Proposition 4.4 we have

$$a_i=\mathrm{x}^p\prod_j f_{i,j}$$

for some degree one polynomials $f_{i,j}$. Consider the following statement,

There exists $\lambda\in\mathbb{Q}$ and linear f such that $f\mid a_i, f+\lambda\mid a_j$ and $f+\lambda\nmid a_i$ for some $i\neq j$. (†)

The linearity of the $f_{i,j}$ allows the following statement.

$$lcm(\overline{a_1},\ldots,\overline{a_r})=\overline{lcm(a_1,\ldots,a_r)}\Leftrightarrow(\dagger)\text{ does not hold for any } i,j.$$

Suppose that (†) holds for some i, j. Since the $\{f_{i,j}\}_j$ are completely determined by the underlying cone of $\mathcal{O}_{\mathcal{X}}(U_i)$, say σ_i, then (†) $\Rightarrow \sigma_i$ and σ_j both have a face parallel to $V(f)$. There is only one such hyperplane through the origin parallel to $V(f)$. Now by (†) and Proposition 4.4 σ_i and σ_j lie on the same side of this hyperplane. Let H be the halfspace, defined by this hyperplane, containing σ_i and σ_j. Let $\Omega_H^i(\lambda) = \{\mu \in \sigma_i \cap \mathbb{Z}^n \mid \lambda + \mu \notin H\}$ for all i. Then by (†) there exists $q \in \overline{\Omega_H^i(p)}$ such that $q + \lambda \in \overline{\Omega_H^j(p)}$ and $q + \lambda \notin \overline{\Omega_H^i(p)}$. Since $\overline{\Omega_H^i(p)} = \overline{\Omega_H^j(p)}$ we have a contradiction. □

We can now restate Proposition 3.9 in terms of differential operators on toric varieties.

Theorem 4.9. *Let $\mathcal{X}$ be a toric variety. Then*

$$gr\,\mathcal{D}(\mathcal{X}) = gr\,\Gamma(\mathcal{X}, \mathcal{D}_{\mathcal{X}}) = \Gamma(\mathcal{X}, gr\,\mathcal{D}_{\mathcal{X}}).$$

Thus $\mathcal{D}(\mathcal{X})$ is Noetherian and

$$\mathcal{D}(\mathcal{X}) = k[lcm(a_{1,l}, \ldots, a_{r,l}) \mid l \in L; x_1\partial_1, \ldots, x_n\partial_n]$$

where $\mathcal{X}$ has open affine cover $\{U_i\}_{i=1}^r$ and in the notation of Section 3, $A_i = \mathcal{D}_{\mathcal{X}}(U_i)$ with the $a_{i,j}$ defined appropriately.

A similar result holds for the global sections of the sheaf of differential operators on a toric variety with coefficients in some invertible sheaf.

Corollary 4.10. *Let $\mathcal{X}$ be a toric variety and $\mathcal{E}_{\mathcal{X}}$ an invertible sheaf on $\mathcal{X}$. Then $\mathcal{M}_{\mathcal{X}} := \mathcal{E}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{D}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{E}_{\mathcal{X}}^{-1}$ is the sheaf of differential operators on $\mathcal{X}$ with coefficients in $\mathcal{E}_{\mathcal{X}}$. Then*

$$gr\,\Gamma(\mathcal{X}, \mathcal{M}_{\mathcal{X}}) = \Gamma(\mathcal{X}, gr\,\mathcal{M}_{\mathcal{X}}).$$

Thus $\Gamma(\mathcal{X}, \mathcal{M}_{\mathcal{X}})$ is Noetherian and

$$\Gamma(\mathcal{X}, \mathcal{M}_{\mathcal{X}}) = k[lcm(a_{1,l}, \ldots, a_{r,l}) \mid l \in L; x_1\partial_1, \ldots, x_n\partial_n]$$

where $\mathcal{X}$ has open affine cover $\{U_i\}_{i=1}^r$ and in the notation of Section 3, $A_i = \Gamma(U_i, \mathcal{M}_{\mathcal{X}})$ with the $a_{i,j}$ defined appropriately.

Proof. By [1, 6.1], locally we have

$$\begin{aligned}\Gamma(U_i, \mathcal{M}_{\mathcal{X}}) &= \mathbf{x}^{\alpha}\mathcal{O}_{\mathcal{X}}(U_i) \otimes \left(\bigoplus\nolimits_{\lambda \in \mathbb{Z}^n} \mathbf{x}^{\lambda} I\overline{\Omega(\lambda)}\right) \otimes \mathbf{x}^{-\alpha}\mathcal{O}_{\mathcal{X}}(U_i) \\ &= \mathbf{x}^{\alpha}\mathcal{O}_{\mathcal{X}}(U_i) \otimes \left(\bigoplus\nolimits_{\lambda \in \mathbb{Z}^n} \mathbf{x}^{\lambda} f_{\lambda}(x_1\partial_1, \ldots, x_n\partial_n)\right) \otimes \mathbf{x}^{-\alpha}\mathcal{O}_{\mathcal{X}}(U_i) \\ &\cong \left(\bigoplus\nolimits_{\lambda \in \mathbb{Z}^n} \mathbf{x}^{\lambda} f_{\lambda}(x_1\partial_1 - \alpha_1, \ldots, x_n\partial_n - \alpha_n)\right).\end{aligned}$$

for some $\alpha \in \mathbb{Z}^n$. Since $\mathscr{D}_{\mathfrak{X}}$ is faithful and compatible then so is $\mathscr{M}_{\mathfrak{X}}$. Thus we have the results. □

5. Examples

We now give some examples to illustrate the process in operation.

(i) Consider $\mathbb{P}^n$ with $\{U_i\}_{i=0}^n$ as standard open affine cover. We know that

$$\mathscr{D}_{\mathbb{P}^n}(U_i) = k[x_i^{-1}, x_i^{-1}x_1, \ldots, x_i^{-1}x_n, x_i\pi, x_i\partial_1, \ldots, x_i\partial_n] \text{ for } i \neq 0$$

$$\mathscr{D}_{\mathbb{P}^n}(U_0) = k[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n].$$

Calculating the required *lcm*'s gives $\mathscr{D}(\mathbb{P}^n)$ Noetherian and

$$\mathscr{D}(\mathbb{P}^n) = k[x_i\pi, \partial_i, x_1\partial_i, x_i\partial_1, x_i\partial_i \mid i = 1, \ldots, n].$$

Also

$$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a) \otimes_{\mathbb{P}^n} \mathscr{D}_{\mathbb{P}^n} \otimes_{\mathbb{P}^n} \mathcal{O}_{\mathbb{P}^n}(-a)) = k[x_i(\pi - a), \partial_i, x_1\partial_i, x_i\partial_1, x_i\partial_i \mid i = 1, \ldots, n]$$

is Noetherian for all $a \in \mathbb{Z}$.

(ii) For the line bundles $\mathscr{X}_m = \mathbb{V}(\mathcal{O}_{\mathbb{P}^n}(m)) = \operatorname{Spec} \mathscr{S}(\mathcal{O}_{\mathbb{P}^n}(m))$ we have an affine morphism $\pi: \mathscr{X}_m \to \mathbb{P}^n$. Let $\{U_i\}$ be the standard open affine cover of $\mathbb{P}^n$. The affine toric varieties $\operatorname{Spec} \mathcal{O}_{\mathscr{X}_m}(U_i)$ glue to give $\mathscr{X}_m$. For the construction of vector bundles see [2]. Then

$$\mathscr{D}(\mathscr{X}_m) = \bigcap \pi_* \mathscr{D}_{\mathscr{X}_m}(U_i)$$

The calculation is similar to the above for projective space yielding

$$\mathscr{D}(\mathscr{X}_m) = k[\mathbf{x}^\lambda \pi, \partial^\lambda, x_1\partial_i, x_i\partial_1, x_i\partial_i \mid i = 1, \ldots, n; \lambda \in \mathbb{Z}^n_{\geqq 0} \text{ such that } \textstyle\sum \lambda_i = m]$$

as a Noetherian ring.

Consider the invertible sheaves on $\mathscr{X}_m$, $\mathcal{O}_{\mathscr{X}_m}(a)$ defined by

$$\mathcal{O}_{\mathscr{X}_m}(a)(\pi^{-1}(U_i)) := \mathcal{O}_{\mathbb{P}^n}(a)(U_i)$$

for $a \in \mathbb{Z}$. Then

$$\Gamma(\mathscr{X}_m, \mathcal{O}_{\mathscr{X}_m}(a) \otimes_{\mathcal{O}_{\mathscr{X}_m}} \mathscr{D}_{\mathscr{X}_m} \otimes_{\mathcal{O}_{\mathscr{X}_m}} \mathcal{O}_{\mathscr{X}_m}(-a)) = k[\mathbf{x}^\lambda(\pi - a), \partial^\lambda, x_1\partial_i, x_i\partial_1, x_i\partial_i \mid i = 1, \ldots, n; \lambda \in \mathbb{Z}^n_{\geqq 0} \text{ such that } \textstyle\sum \lambda_i = m]$$

is Noetherian for all $a \in \mathbb{Z}$.

(iii) Consider the Hirzebruch surfaces $\pi: F_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a)) \to \mathbb{P}^1$ for $a > 0$. Here we have $\mathscr{D}_{F_a}$ on the open affine cover $V_0^{\pm}$, $V_1^{\pm}$ as

$$\mathscr{D}_{F_a}(V_0^+)=k[y,x,\partial_y,\partial_x]$$

$$\mathscr{D}_{F_a}(V_0^-)=k[y,x^{-1},\partial_y,x^2\partial_x]$$

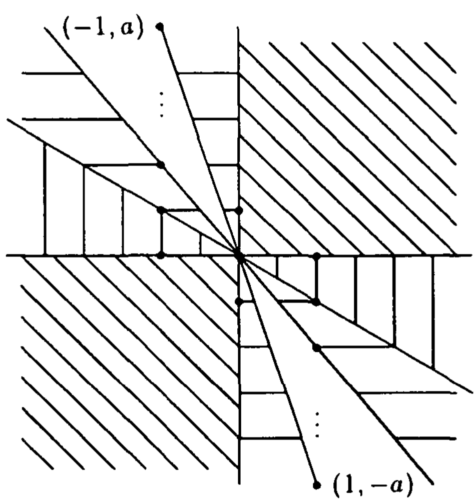
$$\mathscr{D}_{F_a}(V_1^+)=k[y^{-1},x^{-1}y^a,y\pi,y^{-a}x^2\partial_x]$$

$$\mathscr{D}_{F_a}(V_1^-)=k[y^{-1},xy^{-a},y\pi,y^a\partial_x]$$

where $\pi=y\partial_y+ax\partial_x$. Hence we have $\mathscr{D}(F_a)$ as a Noetherian ring. We have the L-generators of the pieces as in the following table

L	$\mathscr{D}_{F_a}(V_0^+)$	$\mathscr{D}_{F_a}(V_0^-)$	$\mathscr{D}_{F_a}(V_1^+)$	$\mathscr{D}_{F_a}(V_1^-)$	$\theta_i=lcm$
$(0,1)$	y	y	$y\pi$	$y\pi$	$y\pi$
$(0,-1)$	∂_y	∂_y	y^{-1}	y^{-1}	∂_y
$(-1,j)$	$y^i\partial_x$	$x^{-1}y^j$	$x^{-1}y^j$	$y^j\partial_x$	$y^j\partial_x$
$(1,-j)$	$x\partial_y^j$	$x\partial_y^j x\partial_x$	$x^2y^{-}\partial_x f_{a-j}(\pi)$	$xy^{-j}f_{a-j}(\pi)$	$x\partial_y^j x\partial_x f_{a-j}(\pi)$

where $f_{a-j}(\pi)=\pi(\pi+1)\cdots(\pi+a-j-1)$. The partition in this case is the following collection of cones



Hence $\mathscr{D}(F_a)$ generated by $y\partial_y, x\partial_x$ and the entries in the last column of the above table. That is,

$$\mathscr{D}(F_a)=k[\partial_y,y^j\partial_x,y\pi,\partial_y^j x^2\partial_x\pi(\pi+1)\cdots(\pi+a-j-1),x\partial_x,y\partial_y\,|\,j=0,1,\ldots,a].$$

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