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ON VON NEUMANN REGULAR RINGS

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Recently, in the Research Problems of Canadian Mathematical Bulletin, Vol. 14, No. 4, 1971, there appeared a problem which asks "Is a prime Von Neumann regular ring pimitive?" While we are not able to settle this question one way or the other, we prove that in a Von Neumann regular ring, there is a maximal annihilator right ideal if and only if there is a minimal right ideal. Hence a prime Von Neumann regular ring is a primitive ring with the non-zero socle if and only if it has a maximal annihilator right ideal. Recall that a ring R is called "Von Neumann regular" if for every $a \in R$, there is $x \in R$ such that $axa=a$. A right ideal I of a ring R is called "an annihilator right ideal" if and only if there is a non-empty subset S of R such that $I=\{x \in R \mid sx=0 \text{ for every } s \in S\}$. A right ideal I is "a maximal annihilator right ideal" provided that I is an annihilator right ideal such that $I\neq R$ and if J is an annihilator right ideal such that $J \geq J$ then either $J=R$ or $J=I$.

THEOREM. *Let R be a Von Neumann regular ring. Then a maximal annihilator right ideal I of R (if it exists) is a maximal right ideal which is a direct summand of R.*

Proof. Suppose that I is a maximal annihilator right ideal of R. Then there exists a non-empty subset S in R such that $I=\{x \in R \mid sx=0 \text{ for every } s \in S\}$. Since $I\neq R$, there is $a \in S$ and $a\neq 0$ such that $I=\{x \in R \mid ax=0\}$. Let b be an element of R such that $a=aba$. Let $e=ba$. Then $e^2=e$ and the set $A(e)=\{r-er \mid r \in r\}$ is a subset of I. If $ax=0$ for some $x \in R$, then $ex=0$ since $e=ba$ and $x=x-ex \in A(e)$. Hence $I=A(e)$. Let $exe\neq 0$, $eye\neq 0$ be two non-zero elements of eRe. Then $exeI=0$ since $I=A(e)$. Since $exe\neq 0$ and I is a maximal annihilator right ideal, $\{r \in R \mid exer=0\}=I$. Hence if $exeeye=exeye=0$, then $eye=r-er$ for some $r \in R$ and $eeye=eye=e(r-er)=0$. So the ring eRe has no zero divisors. Since eRe is a Von Neumann regular ring without zero divisor, it must be a division ring. Hence by [1: Proposition 1, p. 65], eR is a minimal right ideal. Since $R/A(e)\cong eR$, I is a maximal right ideal and $R=I\oplus R$.

COROLLARY. *If R is a Von Neumann regular ring then there is a minimal right ideal if and only if there is a maximal annihilator right ideal.*

Proof. If M is a minimal right ideal of R then $M=eR$ for some $e \in R$ such that $e^2=e$ by [1: Proposition 1, p. 57]. Since $M \cong R/A(e)$, $A(e)$ is a maximal right ideal which is also an annihilator right ideal. Conversely, if there is a maximal annihilator right ideal, then there is a minimal right ideal by Theorem.

Reference

1. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloquim publication, Vol. **37** (1964).