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ON THE CONVERGENCE VECTOR SPACE $\mathscr{L}_c(E, F)$ AND ITS DUAL SPACE

BY
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Introduction. Let E be a locally convex tvs, F a normed space and $\mathscr{L}(E, F)$ the space of continuous linear mappings from E into F. In this paper, we investigate the continuous convergence structure (c-structure) on $\mathscr{L}(E, F)$. $\mathscr{L}_c(E, F)$ denotes the resulting convergence vector space (cvs).

The c-structure is by definition the coarsest cvs structure on $\mathscr{L}(E, F)$ making evaluation a continuous mapping. If $F = \mathbf{R}$, it is well-known that the c-structure is also the finest agreeing with the topology of pointwise convergence on every equicontinuous subset of $\mathscr{L}(E, \mathbf{R})$. In §2, this characterization of $\mathscr{L}_c(E, \mathbf{R})$ is extended to $\mathscr{L}_c(E, F)$, F any normed space.

Section 3 is devoted to a discussion of the dual space $\mathscr{L}\mathscr{L}_c(E, F)$ of $\mathscr{L}_c(E, F)$. If F is reflexive, $\mathscr{L}\mathscr{L}_c(E, F)$ can be computed explicitly as the completed projective tensor product $E\tilde{\otimes}F'$ of E and the norm dual F' of F. In fact, $\mathscr{L}_c(E, F)$ and $E\tilde{\otimes}F'$ then form a dual pair of cvs, each of which is the dual space of the other.

§1. **Preliminaries.** In this section, we present some of the notation and define many of the terms which will be used in the following sections. For the basic theory of convergence spaces and convergence vector spaces, we refer the reader to [1] and [3].

Throughout, E denotes a topological vector space, F a normed space and F' the norm dual space of F. $\mathscr{U}$, $\mathscr{V}$ and $\mathscr{V}'$ denote the zero neighbourhood filter in E, F and F' respectively.

We define two evaluation mappings, ω and e, as follows:

$$\omega : \mathscr{L}(E, F) \times E \to F \text{ by } \omega(t, x) = t(x) \quad \text{for} \quad t \text{ in } \mathscr{L}(E, F),\ x \text{ in } E,$$

$$e : \mathscr{L}(E, F) \times E \times F' \to \mathbf{R} \text{ by } e(t, x, y') = y'(t(x)) \quad \text{for} \quad t \text{ in } \mathscr{L}(E, F),\ x \text{ in } E,\ y' \text{ in } F'.$$

The continuous convergence structure (briefly c-structure) is defined to be the coarsest convergence structure on $\mathscr{L}(E, F)$ making ω continuous. A filter $\mathscr{T}$ on $\mathscr{L}(E, F)$ converges to t in the c-structure if, for every filter $\Theta \to x$ in E,

$$\omega(\mathscr{T} \times \Theta) \to t(x) \text{ in } F,$$

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where $\omega(\mathcal{T}\times\Theta)$ is the filter generated by sets of the form $\omega(T\times\vartheta)=\{t(x):t\in T, x\in\vartheta\}$ where $T\in\mathcal{T}$ and $\vartheta\in\Theta$.

We define analogously the weakly continuous convergence structure (briefly w-structure) to be the coarsest convergence structure on $\mathcal{L}(E,F)$ making the map e continuous. A filter $\mathcal{T}$ on $\mathcal{L}(E,F)$ converges to t in the w-structure if, for every filter $\Theta\to x$ in E and every filter $\Phi'\to y'$ in F',

$$e(\mathcal{T}\times\Theta\times\Phi')\to y'(t(x)) \text{ in } \mathbf{R}.$$

$\mathcal{L}_c(E,F)$ and $\mathcal{L}_w(E,F)$ denote the vector space $\mathcal{L}(E,F)$ equipped with the $\mathcal{C}$-structure and w-structure respectively. These are convergence vector spaces; the verification that the convergence structures are compatible with the vector space operations of $\mathcal{L}(E,F)$ is straightforward. Further, $\mathcal{L}_s(E,F)$ and $\mathcal{L}_{wo}(E,F)$ denote the vector space $\mathcal{L}(E,F)$ equipped with the topology of pointwise convergence (strong operator topology, topology of simple convergence) and the weak operator topology respectively. If $\dot{x}$ and $\dot{y}'$ denote the ultrafilters on E and F' associated to the points x in E and y' in F' (the filters of all sets containing the points), we have:

$\mathcal{T}\to t$ in $\mathcal{L}_s(E,F)$ if, for x in E, $\omega(\mathcal{T}\times\dot{x})\to t(x)$, and

$\mathcal{T}\to t$ in $\mathcal{L}_{wo}(E,F)$ if, for x in E, y' in F', $e(\mathcal{T}\times\dot{x}\times\dot{y}')\to y'(t(x))$.

§2. **Properties of $\mathcal{L}_c(E,F)$ and $\mathcal{L}_w(E,F)$.** We characterize the notions of continuous convergence and weakly continuous convergence on $\mathcal{L}(E,F)$ in the following:

THEOREM 2.1. *Let $\mathcal{T}$ be a filter on $\mathcal{L}(E,F)$. Then*

(i) *$\mathcal{T}\to 0$ in $\mathcal{L}_c(E,F)$ iff $\mathcal{T}$ contains an equicontinuous set and $\mathcal{T}\to 0$ in $\mathcal{L}_s(E,F)$.*

(ii) *$\mathcal{T}\to 0$ in $\mathcal{L}_w(E,F)$ iff $\mathcal{T}$ contains an equicontinuous set and $\mathcal{T}\to 0$ in $\mathcal{L}_{wo}(E,F)$.*

Proof. (i) Let $\mathcal{U}_x=\mathcal{U}+\dot{x}$ be the neighbourhood filter of x in E. $\mathcal{T}\to 0$ in $\mathcal{L}_c(E,F)$ iff, for all x in E, $\omega(\mathcal{T}\times\mathcal{U}_x)\to 0$. But

$$\omega(\mathcal{T}\times\mathcal{U}_x)=\omega(\mathcal{T}\times(\mathcal{U}+\dot{x}))=\omega(\mathcal{T}\times\mathcal{U})+\omega(\mathcal{T}\times\dot{x}).$$

Hence $\mathcal{T}\to 0$ in $\mathcal{L}_c$ (E,F) iff for all x in E,

$$\omega(\mathcal{T}\times\mathcal{U})\to 0 \quad\text{and}\quad \omega(\mathcal{T}\times\dot{x})\to 0.$$

But $\omega(\mathcal{T}\times\dot{x})\to 0$ for every x in E iff $\mathcal{T}\to 0$ in $\mathcal{L}_s(E,F)$. So it remains to show that $\omega(\mathcal{T}\times\mathcal{U})\to 0$ iff $\mathcal{T}$ contains an equicontinuous set. Now $\omega(\mathcal{T}\times\mathcal{U})\to 0$ iff for every $\varepsilon>0$, there exists a T_ε in $\mathcal{T}$ and a U_ε in $\mathcal{U}$ such that

$$\{\|\omega(T_\varepsilon\times U_\varepsilon)\|\}=\{\|t(x)\|:t\in T_\varepsilon, x\in U_\varepsilon\}\subset[0,\varepsilon].$$

Since F is normed, for any $T\in\mathcal{T}$ and any $U_1\in\mathcal{U}$,

$$\{\|\omega(T\times U_1)\|\}\subset[0,1] \quad\text{iff for all } \delta>0,\quad \{\|\omega(T\times\delta U_1)\|\}\subset[0,\delta].$$

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Hence $\omega(\mathcal{T}\times\mathcal{U})\to 0$ iff there is $T\in\mathcal{T}$ and $U_1\in\mathcal{U}$ such that

$$\{\|\omega(T\times U_1)\|\}\subset[0,1].$$

But this is so, iff for all $\varepsilon>0$, there exists $U_\varepsilon=\varepsilon U_1\in\mathcal{U}$ such that

$$\{\|\omega(T\times U_\varepsilon)\|\}\subset[0,\varepsilon],$$

i.e. iff

$$\omega(T\times\mathcal{U})\to 0,$$

i.e. iff T is equicontinuous.

The proof of (ii) is completely analogous.

By Theorem 2.1, it follows that the c-structure (w-structure) is the finest structure on $\mathcal{L}(E,F)$ which agrees with the topology of pointwise convergence (weak operator topology) on every equicontinuous subset of $\mathcal{L}(E,F)$. If we denote by $\mathcal{H}$ the collection of equicontinuous subsets of $\mathcal{L}(E,F)$, we have shown:

Corollary 2.2. $\mathcal{L}_c(E,F)=\operatorname{ind}_{H\in\mathcal{H}}(H,s)$ *and* $\mathcal{L}_w(E,F)=\operatorname{ind}_{H\in\mathcal{H}}(H,wo)$ *where the inductive limit in the category of convergence spaces is taken over the equicontinuous subsets of* $\mathcal{L}(E,F)$ *equipped with the topology of pointwise convergence and the weak operator topology respectively.*

The above results are analogues of the well-known result (see eg. [1, Cor. 29]) that $\mathcal{L}_c(E)=\operatorname{ind}_{U\in\mathcal{U}}(U^0,s)$. (As usual, $\mathcal{L}_c(E,\mathbf{R})$ is abbreviated to $\mathcal{L}_c(E)$.) $\mathcal{L}_c(E)$ is always locally compact, i.e. every convergent filter contains a compact set. $\mathcal{L}_c(E,F)$ cannot be locally compact unless F is also, but as we shall see in Corollary 2.5 and in Lemma 3.2, $\mathcal{L}_w(E,F)$ is locally compact whenever F is a reflexive Banach space.

Corollary 2.3. *The following identity maps are continuous*:

$$\begin{array}{ccc}
\mathcal{L}_c(E,F) & \longrightarrow & \mathcal{L}_w(E,F)\\
\big\downarrow & & \big\downarrow\\
\mathcal{L}_s(E,F) & \longrightarrow & \mathcal{L}_{wo}(E,F)
\end{array}$$

We examine the relationship between the c-structure and w-structure on $\mathcal{L}(E,F)$. Write F_c for F considered as a subspace of $\mathcal{L}_c(F')$. If F is a reflexive Banach space, then $F_c=\mathcal{L}_c(F')$.

Proposition 2.4. $\mathcal{L}_w(E,F)=\mathcal{L}_c(E,F_c)$.

Proof. Since it is immediate from the definitions that the structures $\mathcal{L}_w(E,F)$ and $\mathcal{L}_c(E,F_c)$ coincide, we show only that $\mathcal{L}(E,F)=\mathcal{L}(E,F_c)$. Since the identity map $i: F \to F_c$ is continuous, it is clear that $\mathscr{L}(E, F) \subset \mathscr{L}(E, F_c)$. So suppose $t \in \mathscr{L}(E, F_c)$. Then $t(\mathscr{U}) \to 0$ in $L_c(F')$, i.e. $t(\mathscr{U})(\mathscr{V}') \to 0$. Let V, V' be the unit balls in F, F' respectively. Hence for some n and some $U \in \mathscr{U}$,

$$t((1/n)U)(V') \subset [-1, 1],$$

i.e.

$$t(U)((1/n)V') \subset [-1, 1].$$

Hence $t(U) \subset ((1/n)V')^0$ so that $((1/n)V')^0 \in t(\mathscr{U})$. Therefore $nV \in t(\mathscr{U})$. Thus $t(\mathscr{U})$ is finer than $\mathscr{V}$ so that $t \in \mathscr{L}(E, F)$.

COROLLARY 2.5. *If F is a reflexive Banach space, then*

$$\mathscr{L}_w(E, F) = \mathscr{L}_c(E, \mathscr{L}_c(F')).$$

The above characterization of $\mathscr{L}_w(E, F)$ will enable us to compute the dual space of $\mathscr{L}_c(E, F)$ in the next section.

§3. **The dual space of $\mathscr{L}_c(E, F)$.** From now on, we assume E is locally convex. With this assumption, we shall show that $\mathscr{L}_c(E, F)$ and $\mathscr{L}_w(E, F)$ have the same dual space, and if, in addition, F is a reflexive Banach space, this dual space is $E \hat{\otimes} F'$, the completion of the projective tensor product of E and F'.

LEMMA 3.1. $\mathscr{L}\mathscr{L}_c(E, F) = \mathscr{L}\mathscr{L}_w(E, F)$.

Proof. Since, by Corollary 2.3, the c-structure on $\mathscr{L}(E, F)$ is finer than the w-structure, $\mathscr{L}\mathscr{L}_w(E, F) \subset \mathscr{L}\mathscr{L}_c(E, F)$. By Corollary 2.2, a linear functional φ on $\mathscr{L}_c(E, F)$ is continuous if and only if, for each equicontinuous subset H of $\mathscr{L}(E, F)$, φ restricted to H is s-continuous, i.e. continuous with respect to the topology of pointwise convergence. In fact, we may replace the family of all equicontinuous subsets of $\mathscr{L}(E, F)$ by the cofinal collection of pointwise closed, convex equicontinuous sets $\{H_U : U$ a closed convex neighbourhood of 0 in $E\}$, where $H_U = \{t \in \mathscr{L}(E, F) : \|t(x)\| \leq 1$, all $x \in U\}$. Thus, since φ is c-continuous, each $\varphi^{-1}[-1, 1] \cap H_U$ is s-closed. But the closed convex sets of $\mathscr{L}_s(E, F)$ are the same as those of $\mathscr{L}_{wo}(E, F)$, since by [5, Corollary 4, p. 139], $\mathscr{L}\mathscr{L}_s(E, F) = \mathscr{L}\mathscr{L}_{wo}(E, F)$. Hence, $\varphi^{-1}[-1,1] \cap H_U$ is wo-closed and therefore φ is continuous on $\mathscr{L}_w(E, F)$.

Denoting by $\mathscr{B}_c(E \times F, \mathbf{R})$ the set of continuous bilinear functionals on $E \times F$ with the c-structure, we now show

LEMMA 3.2. (i) $\mathscr{B}_c(E \times F, \mathbf{R}) = \mathscr{L}_c(E, \mathscr{L}_c(F))$.

(ii) $\mathscr{B}_c(E \times F, \mathbf{R}) = \mathscr{L}_c(E \otimes F)$.

Proof. (i) It is straightforward to show that the map $t \to \hat{t}$ from $\mathscr{B}_c(E \times F, \mathbf{R})$ to $\mathscr{L}_c(E, \mathscr{L}_c(F))$ is a homeomorphism where $\hat{t}$ is given by $\hat{t}(x) = t(x, \cdot)$.
(ii) From [5, p. 93], $\mathscr{B}(E \times F, \mathbf{R})$ and $\mathscr{L}(E \hat{\otimes} F)$ are algebraically isomorphic.

The definition of the projective tensor product assures that they are homeomorphic.

It is clear that the mapping $b:\mathcal{L}(E, F)\times E\tilde{\otimes}F'\to\mathbf{R}$ given by $b(t, \alpha)=t(\alpha)$ for every $t\in\mathcal{L}(E, F)$ and every $\alpha\in E\tilde{\otimes}F'$ is a point-separating bilinear mapping. In this way, the pair

$$\langle\mathcal{L}(E, F), E\tilde{\otimes}F'\rangle$$

forms a dual pair of vector spaces. Since it is known (see e.g. [5, p. 93]) that the dual space of $E\otimes F'$ is $\mathcal{B}(E\times F', \mathbf{R})=\mathcal{L}(E, F)$, then $\mathcal{L}(E, F)$ is also the dual space of $E\tilde{\otimes}F'$. As the following theorem shows, if F is reflexive, $E\tilde{\otimes}F'$ is itself the dual space of $\mathcal{L}_c(E, F)$

THEOREM 3.3. *Let F be a reflexive Banach space. Then*

$$\mathcal{L}\mathcal{L}_c(E, F)=\mathcal{L}\mathcal{L}_w(E, F)=E\tilde{\otimes}F'.$$

Proof. By Lemma 3.1, only the second equality remains to be proven. But

$$\begin{aligned}\mathcal{L}\mathcal{L}_w(E, F)&=\mathcal{L}\mathcal{L}_c(E, \mathcal{L}_c(F')) \text{ by Corollary 2.5}\\&=\mathcal{L}\mathcal{B}_c(E\times F', \mathbf{R}) \text{ by Lemma 3.2 (i)}\\&=\mathcal{L}\mathcal{L}_c(E\otimes F') \text{ by Lemma 3.2 (ii)}\end{aligned}$$

But $\mathcal{L}\mathcal{L}_c(E\otimes F')=E\tilde{\otimes}F'$ since for any convex topological vector space E, it is known [2, p. 101], that $\mathcal{L}_c\mathcal{L}_c(E)=\tilde{E}$, the completion of E. Thus $\mathcal{L}\mathcal{L}_c(E, F)=\mathcal{L}\mathcal{L}_w(E, F)=E\tilde{\otimes}F'$.

Loosely speaking, we have shown that the c-structure on $\mathcal{L}(E, F)$ is consistent with the dual pairing

$$\langle\mathcal{L}(E, F), E\tilde{\otimes}F'\rangle$$

and that

$$\langle\mathcal{L}_c(E, F), E\tilde{\otimes}F'\rangle$$

is a dual pair of cvs, each of which is the dual space of the other.

The foregoing results should be compared to the classical results concerning the compact open (co-) topology. For, it is an immediate consequence of [4, Corollary 1, p. 52] that, if E is a Fréchet space, then $\mathcal{B}_{co}(E\times F, \mathbf{R})=\mathcal{L}_{co}(E,\mathcal{L}_{co}(F))=\mathcal{L}_{co}(E\tilde{\otimes}F)$. Also, it is known (cf. [4, Proposition 22, p. 114] and [5, Problem 30, p. 198]) that if E is a Fréchet space, the dual space of $\mathcal{L}_{co}(E, F)$ can be identified with a quotient of $E\tilde{\otimes}F'$ and, if F is reflexive, with $E\tilde{\otimes}F'$ itself.

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