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# WEAKLY CONFLUENT MAPPINGS AND ATRIODIC SUSLINIAN CURVES

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There are theorems in which some classes of topological spaces are characterized by means of properties of mappings of these spaces into a single space. For example, it is well known that a compactum $X$ is at most $n$-dimensional if and only if no mapping of $X$ into an $(n + 1)$-cube has a stable value [**5**, Theorems VI.1–2, pp. 75–77]. Also, a curve $X$ is tree-like if and only if no mapping of $X$ into a figure eight is homotopically essential [**1**, Theorem 1, pp. 74–75; **8**, p. 91]. By a *curve* we mean any at most 1-dimensional continuum; a *continuum* is a connected compactum; a *compactum* is a compact metric space, and a *mapping* is a continuous function. The aim of the present paper is to prove another theorem of this type. We distinguish a class of curves and show that it is characterized by imposing the condition that no weakly confluent mapping [**13**] can transform the given curve onto a simple triod (see 2.4). A related result is applied to a generalized branch-point covering theorem (see 3.2). In addition, two results are obtained in which we establish some characterizations of weakly confluent images and preimages of the product of the Cantor set and an arc (see 1.1 and 2.2). Continua that are such images turn out to be identical with regular curves (see 1.3).

**1. Paths in regular curves.** Let $X$ be a compactum. We denote by $C(X)$ the collection of all non-empty continua contained in $X$. We say that $X$ has *property S uniformly* provided, for each number $\epsilon > 0$, there exists a positive integer $k = k(\epsilon)$ such that if $K \in C(X)$, then $K$ is the union of a finite sequence of $k$ non-empty continua each of diameter less than $\epsilon$. Thus, if a continuum has property $S$ uniformly, it is hereditarily locally connected [**23**, (15.1), p. 20]. There exist, however, hereditarily locally connected continua which do not have property $S$ uniformly; such as, by 1.2 below, any hereditarily locally connected continuum which is not a regular curve [**11**, pp. 283-284]. By a *regular* curve we understand a continuum possessing a basis of open sets whose boundaries are finite. In particular, each dendrite is a regular curve [**11**, Theorem 4, p. 301].

We denote by $I = [0, 1]$ the unit closed interval of the real line. A *path* in a metric space $X$ is a mapping $\varphi : I \to X$. We say that a family $\Phi$ of paths in $X$ is *equicontinuous* provided, for each number $\epsilon > 0$, there exists a number $\delta = \delta(\epsilon) > 0$ such that if $\varphi \in \Phi$, $t_1, t_2 \in I$ and $|t_1 - t_2| < \delta$, then dist $(\varphi(t_1), \varphi(t_2))$

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$< \epsilon$. If $X$ and $Y$ are compacta, we say that a mapping $f : X \to Y$ is *weakly confluent* [**13**, p. 98] provided it induces a surjective function $f_C : C(X) \to C(Y)$, that is, for each continuum $L \in C(Y)$, there exists a continuum $K \in C(X)$ such that $f(K) = L$. Compositions of weakly confluent mappings are weakly confluent [**15**, 1.5, p. 1337]. By $C$ we denote the Cantor set in $I$.

1.1. THEOREM. *Let $X$ be a compactum. The following conditions are equivalent:*

(i) *$X$ has property $S$ uniformly,*

(ii) *there exists an equicontinuous family $\Phi$ of paths in $X$ such that $C(X) = \{\varphi(I) : \varphi \in \Phi\}$, and*

(iii) *there exists a weakly confluent mapping of $C \times I$ onto $X$.*

*Proof.* The fact that (i) implies (ii) has been stated implicitly in [**9**, Theorem 3.7, p. 323]. A classical theorem characterizes locally connected continua as those which are images of some paths. The proof of this theorem given by Sierpiński [**22**] can be adapted here to get an equicontinuous family of paths in a compactum $X$ under the assumption that $X$ has property $S$ uniformly. Indeed, for $n = 1, 2, \ldots$, let $k_n = k(n^{-1}) > 1$ be an integer such that each non-empty continuum contained in $X$ can be represented as the union of a finite sequence of $k_n$ non-empty continua each of diameter less than $n^{-1}$. Put

$$(1) \qquad h_0 = 1, \quad h_n = (k_1 \cdot \ldots \cdot k_n)^2 \quad (n = 1, 2, \ldots)$$

and consider a continuum $K \in C(X)$. We shall define, by induction on $n$, a finite sequence $\beta_n$ $(n = 0, 1, \ldots)$ of $h_n$ non-empty continua such that each pair of adjacent terms in $\beta_n$ has a non-empty intersection and the union of all terms in $\beta_n$ is $K$. Moreover, for $n = 1, 2, \ldots$, the terms in $\beta_n$ will be continua of diameters less than $n^{-1}$. Let $\beta_0 = (K)$ and suppose $\beta_{n-1}$ is defined, where $n > 0$. Then we have

$$\beta_{n-1} = (B_1, \ldots, B_{h_{n-1}}), \quad B_{i-1} \cap B_i \neq \emptyset \quad (i = 2, \ldots, h_{n-1})$$

and $K = B_1 \cup \ldots \cup B_{h_{n-1}}$. We select points $b_0 \in B_1$, $b_{i-1} \in B_{i-1} \cap B_i$ $(i = 2, \ldots, h_{n-1})$ and $b_{h_{n-1}} \in B_{h_{n-1}}$. Since $B_i \in C(X)$ $(i = 1, \ldots, h_{n-1})$, the continuum $B_i$ can be represented as the union $B_i = B_{i1} \cup \ldots \cup B_{ik_n}$ of a finite sequence of $k_n$ non-empty continua $B_{ij}$ each of diameter less than $n^{-1}$. Without loss of generality, we can assume that $b_{i-1} \in B_{i1}$ and $b_i \in B_{ik_n}$. Let $i$ be fixed for a while. Since $B_i$ is connected, each two of the continua $B_{ij}$ can be joined together by means of a finite chain of different $B_{ij}$'s in which any two adjacent links intersect. The number of links in such a chain does not exceed $k_n$, and the number of chains needed to connect all the continua in the sequence $(B_{i1}, \ldots, B_{ik_n})$ is $k_n - 1$. Thus we can rearrange $B_{ij}$'s, allowing some repetitions, to obtain a representation of $B_i$ as the union

$$B_i = B'_{i1} \cup \ldots \cup B'_{ik^2_n},$$

where $B'_{i1} = B_{i1}$, $B'_{ik^2_n} = B_{ik_n}$, any two adjacent terms in this new sequence intersect, each $B_{ij}'$ is one of $B_{ij}$'s, and each $B_{ij}$ is taken at least once as one of

$B'_{ij}$'s. We define

$$\beta_n = (B'_{11}, \ldots, B'_{1k_n^2}, \ldots, B'_{i1}, \ldots, B'_{ik_n^2}, \ldots, B'_{h_{n-1}1}, \ldots, B'_{h_{n-1}k^2_n}).$$

The number of terms in $\beta_n$ is $h_{n-1} \cdot k_n^2 = h_n$, by (1), and we also have

$$b_{i-1} \in B_{i-1,k_n} \cap B_{i,1} = B'_{i-1,k^2_n} \cap B'_{i,1} \quad (i = 2, \ldots, h_{n-1}),$$

which implies that each pair of adjacent terms in $\beta_n$ has a non-empty intersection. The union of all $B'_{ij}$'s is the same as the union of all $B_{ij}$'s; hence it is $K$. The diameters of the continua $B'_{ij}$ from $\beta_n$ are all less than $n^{-1}$, since so were the diameters of the continua $B_{ij}$.

Now, take the partition of the unit closed interval $I$ into $h_n$ congruent closed subintervals, and denote by $\alpha_n$ $(n = 0, 1, \ldots)$ the finite sequence of all these closed intervals of length $h_n^{-1}$ ordered by the natural ordering of the real line. Let the $i$th term in $\alpha_n$ correspond to the $i$th term in $\beta_n$ $(i = 1, \ldots, h_n)$. For $t \in I$, denote by $B_n(t)$ the union of all terms in $\beta_n$ corresponding to those terms of $\alpha_n$ which contain $t$. There is only one such term in $\alpha_n$ or two adjacent ones. Consequently, $B_n(t)$ is a continuum of diameter less than $2/n$ $(n = 1, 2, \ldots)$. We notice that $\alpha_n$ is a refinement of $\alpha_{n-1}$, and if $B \in \beta_{n-1}$ and $B' \in \beta_n$ correspond to $A \in \alpha_{n-1}$ and $A' \in \alpha_n$, respectively, then $A' \subset A$ implies $B' \subset B$. It follows that

$$B_n(t) \subset B_{n-1}(t) \quad (n = 1, 2, \ldots; t \in I),$$

and, as a result, the intersection $B_0(t) \cap B_1(t) \cap \ldots$ is a single point. We set this point to be $\varphi(t)$. Clearly, $\varphi(I) = K$. Let $\Phi$ be the family of all functions $\varphi$ so obtained, one for each $K \in C(X)$. We show that $\Phi$ is an equicontinuous family of paths. In fact, if $\epsilon > 0$, there exists a positive integer $m$ such that $4/m < \epsilon$. We put $\delta = h_m^{-1}$ and observe that if $t_1, t_2 \in I$ and $|t_1 - t_2| < \delta$, then the points $t_1$ and $t_2$ are not separated by any interval of the sequence $\alpha_m$, as these intervals have all lengths equal to $\delta$. Thus there exist $A^*, A^{**} \in \alpha_m$ such that $t_1 \in A^*$, $t_2 \in A^{**}$ and either $A^* = A^{**}$ or $A^*, A^{**}$ are adjacent terms in $\alpha_m$. In any case, for each $K \in C(X)$, if $\beta_m$ is the sequence constructed for $K$, then the corresponding terms $B^*, B^{**}$ in $\beta_m$ have a non-empty intersection, whence $B_m(t_1) \cap B_m(t_2) \neq \emptyset$. Therefore the diameter of $B_m(t_1) \cup B_m(t_2)$ is less than $4/m$. But $\varphi(t_1) \in B_m(t_1)$ and $\varphi(t_2) \in B_m(t_2)$, which yields

$$\text{dist}\,(\varphi(t_1), \varphi(t_2)) < 4/m < \epsilon,$$

and we conclude that the compactum $X$ satisfies condition (ii).

To prove that (ii) implies (iii), we follow an idea due to Kelley [**6**]. Consider an equicontinuous family $\Phi$ of paths in $X$ such that $C(X) = \{\varphi(I) : \varphi \in \Phi\}$. Take $\Phi$ with the pointwise convergence topology, that is, interpret $\Phi$ as a subspace of the Cartesian product $\Pi$, where

$$\Pi = \mathop{\mathbf{P}}_{t \in I} X_t, \quad X_t = X \quad (t \in I).$$

Let $F$ be the closure of $\Phi$ in $\Pi$. Since $\Pi$ is a compact Hausdorff space, so is $F$. Also, $F$ is contained in the space $X^I$ of all paths and $F$ is equicontinuous [**3**, Lemma 3, p. 332], whence the pointwise convergence topology in $F$ coincides with the compact-open topology [**3**, Lemma 2, p. 332]. The space $F$ with the compact-open topology is metrizable [**3**, Theorem 6, p. 182] which means that $F$ is a compactum. Let $g : C \rightarrow F$ be a mapping of the Cantor set $C$ onto $F$. Then the function $f$ defined by the formula

$$(2) \qquad f(c, t) = [g(c)](t) \qquad (c \in C, t \in I)$$

is continuous [**11**, Theorem 1, p. 77]. Hence it is a mapping $f : C \times I \rightarrow X$. If $L \in C(X)$, there exists a path $\varphi \in \Phi$ such that $\varphi(I) = L$. We have $\varphi \in F$ and there is a point $c \in C$ with $g(c) = \varphi$. Setting $K = \{c\} \times I$, we obtain $f(K) = [g(c)](I) = \varphi(I) = L$, by (2). Consequently, $f$ is a weakly confluent mapping of $C \times I$ onto $X$.

Finally, (iii) implies (i). To this end, assume that $f : C \times I \rightarrow X$ is a weakly confluent mapping and $\epsilon > 0$ is a number. Let $\mathbf{G}$ denote the collection of all open subsets of $X$ of diameters less than $\epsilon$. Let $\lambda > 0$ be a Lebesgue number of the open cover of $C \times I$ consisting of the sets $f^{-1}(G)$, where $G \in \mathbf{G}$. There exists a positive integer $k$ such that $k^{-1} < \lambda$. Suppose $L \in C(X)$ is a non-degenerate continuum. Since $f$ is weakly confluent, there exists a continuum $K$ contained in $C \times I$ such that $f(K) = L$. Thus $K = \{c_0\} \times I_0$, where $c_0 \in C$ and $I_0$ is a closed subinterval of $I$. We take the partition

$$(3) \qquad I_0 = I_1 \cup \ldots \cup I_k$$

of $I_0$ into $k$ congruent closed subintervals. The diameter of each set $\{c_0\} \times I_i$ $(i = 1, \ldots, k)$ does not exceed $k^{-1}$, so it is less than $\lambda$ and, consequently, the image $L_i = f(\{c_0\} \times I_i)$ is a subset of an element of $\mathbf{G}$. Hence each $L_i$ $(i = 1, \ldots, k)$ is a non-empty continuum of diameter less than $\epsilon$, and (3) implies that

$$L = f(K) = f(\{c_0\} \times I_0) = f(\{c_0\} \times (I_1 \cup \ldots \cup I_k)) = L_1 \cup \ldots \cup L_k.$$

This completes the proof of Theorem 1.1.

1.2. *A continuum $X$ is a regular curve if and only if $X$ has property $S$ uniformly.*

*Proof.* Let $X$ be a non-degenerate regular curve and let $\epsilon > 0$ be a number. There exist open sets $G_i \subset X$ $(i = 1, \ldots, n)$, each of diameter less than $\epsilon$, such that the boundary $\bar{G}_i \backslash G_i$ consists of exactly $k_i$ points, where $k_i$ is a positive integer, and $X = G_1 \cup \ldots \cup G_n$. We show that $k = k_1 + \ldots + k_n$ can serve as an integer needed to have $X$ possess property $S$ uniformly. Let $K \in C(X)$. It is enough to prove that each set $\bar{G}_i \cap K$ $(i = 1, \ldots, n)$ has a finite number, $h_i$, of components and $0 \leqq h_i \leqq k_i$. If $K \subset \bar{G}_i$, then $h_i = 1$. If $K \not\subset \bar{G}_i$, then $\bar{G}_i \cap K$ is a closed proper subset of the continuum $K$ and, consequently, the number of components of $\bar{G}_i \cap K$ does not exceed the number of components of the set $\operatorname{Fr}(\bar{G}_i \cap K)$ [**11**, Theorem 3, p. 173]. Thus $h_i \leqq k_i$, since we have

$$\begin{aligned}\operatorname{Fr}(\bar{G}_i \cap K) &= (\bar{G}_i \cap K) \cap \overline{K \backslash (\bar{G}_i \cap K)} \\ &= \bar{G}_i \cap \overline{K \backslash \bar{G}_i} \subset \bar{G}_i \cap \overline{X \backslash G_i} = \bar{G}_i \backslash G_i.\end{aligned}$$

Assume now that $X$ is a continuum and $X$ has property $S$ uniformly. Then $X$ is locally connected. Let $\epsilon > 0$, $p \in X$, and let $U$ be the open ball in $X$ having radius $\epsilon$ and center $p$. To prove the regularity of $X$, it is sufficient to show that a finite set separates $p$ from $X \backslash U$ in $X$. Let $k$ be a positive integer such that each continuum in $C(X)$ is the union of a finite sequence of $k$ non-empty continua of diameters less than $\epsilon/2$. We show that a $(k - 1)$-point set separates $p$ from $X \backslash U$. Suppose, on the contrary, that such a set does not exist. It follows [**19**, p. 216] that there exist $k$ arcs in $X$ each joining the point $p$ and a point of $X \backslash U$ such that by removing the end-points one obtains pairwise disjoint sets. The distance between $p$ and any point of $X \backslash U$ is at least $\epsilon$. Cutting off small parts of these arcs at the end-points in $X \backslash U$, we can get arcs $A_i (i = 1, \dots, k)$ with end-points $p$ and $p_i$, respectively, such that

$$(4) \qquad \epsilon/2 < \operatorname{dist}(p, p_i), \quad A_i \cap A_j = \{p\} \quad (i, j = 1, \dots, k; i \neq j).$$

The set $B = A_1 \cup \dots \cup A_k$ is a continuum and $p \in B$. Let $B = B_1 \cup \dots \cup B_k$, where $B_i$'s are continua of diameters less than $\epsilon/2$. For $i = 1, \dots, k$, let $m_i$ be a subscript such that $p_i \in B_{m_i}$. We have $p \notin B_{m_i}$ and $A_i \backslash \{p\}$ is a component of $B \backslash \{p\}$, by (4). Hence $B_{m_i} \subset A_i \backslash \{p\}$ and $m_i \neq m_j$ for $i \neq j$. As a result, $(m_1, \dots, m_k)$ is a permutation of $(1, \dots, k)$ and

$$B = B_{m_1} \cup \dots \cup B_{m_k} \subset B \backslash \{p\},$$

which contradicts the fact that $p \in B$. We have shown that $X$ is a regular curve.

*Remarks.* The present paper was referred to in [**14**, 2.8, p. 53] as containing another theorem on weakly confluent mappings. Specifically, our "Theorem 1.2" was supposed to be the following result: each weakly confluent image of an acyclic curve is a curve. This result was obtained by the first author right after the Oklahoma Topology Conference, in April, 1972 (see [**13**, p. 102]). Its stronger version, however, was published in [**15**, Theorem 5.5, p. 1347]. In the meantime, the original result with exactly the same proof was obtained independently by Krasinkiewicz [**7**, Theorem 2, p. 481]. Also in April, 1972, we established some results which are included in the present paper (in particular, a part of Theorem 2.4 of the next section).

1.3. COROLLARY. *A continuum $X$ is a regular curve if and only if $X$ satisfies (any) one of conditions (i)–(iii).*

1.4. COROLLARY. *Each dendrite is a weakly confluent image of $C \times I$.*

**2. Subsets of non-Suslinian compacta.** We say that a continuum $T$ is a *triod* [**20**, p. 218] provided there exists a continuum $C_0$, called a *core* of $T$, and three continua $C_1$, $C_2$, $C_3$ such that $C_0$ is a proper subcontinuum of $C_i$ $(i = 1, 2, 3)$ and

$$C_0 = C_1 \cap C_2 = C_1 \cap C_3 = C_2 \cap C_3, \quad T = C_1 \cup C_2 \cup C_3.$$

If, in addition, the continua $C_1$, $C_2$, $C_3$ are arcs and $C_0$ is a one-point set, $C_0 = \{v_0\}$, such that $v_0$ is an end-point of $C_i$ $(i = 1, 2, 3)$, then the triod $T$ is said to be a *simple triod* and $v_0$ is the *vertex* of $T$. A compactum $X$ is called *atriodic* provided $X$ contains no triod. We say that a compactum $X$ is *Suslinian* [**12**, p. 131] provided each collection of pairwise disjoint non-degenerate continua contained in $X$ is countable. We shall give several characterizations of atriodic Suslinian compacta (see 2.4). Before doing so, we need to establish some facts concerning the structure of non-Suslinian compacta.

2.1. *A compactum $X$ is non-Suslinian if and only if there exist a closed subset $A \subset X$ and a number $\epsilon_0 > 0$ such that the components of $A$ are all of diameters greater than or equal to $\epsilon_0$, the space of components of $A$ is a Cantor set, and the decomposition of $A$ into components is continuous.*

*Proof.* The condition is obviously sufficient for $X$ to be non-Suslinian. We prove it is also necessary. Let $X$ be a non-Suslinian compactum, and let $\mathbf{C} \subset C(X)$ be an uncountable collection of pairwise disjoint non-degenerate continua. The elements of $\mathbf{C}$ have positive diameters. Thus there exist a number $\epsilon_0 > 0$ and an uncountable sub-collection $\mathbf{C}_0 \subset \mathbf{C}$ such that $\epsilon_0 \leqq \operatorname{diam} K$ for $K \in \mathbf{C}_0$.

We consider $C(X)$ (and also the collection of all non-empty closed subsets of $X$) to be space equipped with the Hausdorff distance [**10**, p. 214]. So metrized, $C(X)$ is a compactum [**11**, Theorem 1, p. 45, p. 47, Theorem 14, p. 139]. Hence there exists a uncountable subcollection $\mathbf{C}_1 \subset \mathbf{C}_0$ such that $\mathbf{C}_1$ as a subspace of $C(X)$ is dense in itself [**10**, p. 253]. For each finite sequence $(k_1, \ldots, k_n)$ ,where $k_i = 0, 1$ $(i = 1, \ldots, n)$, we shall define, by induction on $n$, a continuum $K_{k_1\ldots k_n} \in \mathbf{C}_1$ and open set $W_{k_1\ldots k_n} \subset X$ such that

(5) $\quad \operatorname{dist}(K_{k_1\ldots k_n l}, K_{k_1\ldots k_n}) < 2^{-n} \quad (l = 0, 1; n = 1, 2, \ldots),$

(6) $\quad \operatorname{dist}(K_{k_1\ldots k_n}, \bar{W}_{k_1\ldots k_n}) < 2^{-n} \quad (n = 1, 2, \ldots),$

(7) $\quad K_{k_1\ldots k_n} \subset W_{k_1\ldots k_n} \quad (n = 1, 2, \ldots),$

(8) $\quad \bar{W}_{k_1\ldots k_n l} \subset W_{k_1\ldots k_n} \quad (l = 0, 1; n = 1, 2, \ldots),$

(9) $\quad \bar{W}_0 \cap \bar{W}_1 = \emptyset = \bar{W}_{k_1\ldots k_n 0} \cap \bar{W}_{k_1\ldots k_n 1} \quad (n = 1, 2, \ldots).$

Let $K_0, K_1 \in \mathbf{C}_1$ be two distinct continua and let $W_0, W_1 \subset X$ be open sets such that $K_j \subset W_j$, $\operatorname{dist}(K_j, \bar{W}_j) < 2^{-1}$ $(j = 0, 1)$ and $\bar{W}_0 \cap \bar{W}_1 = \emptyset$. The existence of such $W_j$'s follows from the fact that $K_j$'s being elements of $\mathbf{C}_1 \subset \mathbf{C}$ are disjoint. Suppose that $K_{k_1\ldots k_n}$ and $W_{k_1\ldots k_n}$ are defined, where $n > 0$, and that each of conditions (5)–(9) is satisfied whenever applicable. Since $\mathbf{C}_1$ is dense in itself, there exist, by (7), distinct continua $K_{k_1\ldots k_n 0}, K_{k_1\ldots k_n 1} \in \mathbf{C}_1$

such that condition (5) holds and

$$K_{k_1\ldots k_n l} \subset W_{k_1\ldots k_n}$$

for $l = 0, 1$. Again, since these two continua are disjoint, it is possible to find open neighborhoods $W_{k_1\ldots k_n 0}$ and $W_{k_1\ldots k_n 1}$ of them in $X$, respectively, such that conditions (8) and (9) hold, and

$$\text{dist}\,(K_{k_1\ldots k_n l}, \bar{W}_{k_1\ldots k_n l}) < 2^{-n-1}$$

for $l = 0, 1$. Therefore conditions (6) and (7) are also satisfied.

We claim that the set $A$ defined by the formula

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k_i} \bar{W}_{k_1\ldots k_n}$$

has all the properties required in 2.1. Clearly, $A$ is a closed subset of $X$. To check the components of $A$, we first interpret the Cantor set $C$ as the Cartesian product

$$C = \mathop{\mathbf{P}}_{i=1}^{\infty} \{0, 1\}_i$$

of countably many copies of the discrete 2-point space $\{0, 1\}$. Then, for each point $c \in C$, where $c = (k_1, k_2, \ldots)$ and $k_i = 0, 1$ $(i = 1, 2, \ldots)$, we define a set $F(c) \subset A$ by

$$F(c) = \bigcap_{n=1}^{\infty} \bar{W}_{k_1\ldots k_n}.$$

It will be shown that the collection of the sets $F(c)$ $(c \in C)$ is that of components of $A$. Since

$$\text{(10)} \quad F(c) = \operatorname*{Lim}_{n\to\infty} \bar{W}_{k_1\ldots k_n} = \operatorname*{Lim}_{n\to\infty} K_{k_1\ldots k_n} \quad (c = (k_1, k_2, \ldots)),$$

by (6) and (8) [**10**, (8), p. 339; **11**, p. 49], it follows that $F(c)$ is a non-empty continuum [**11**, Theorem 6, p. 171]. The continua $K_{k_1\ldots k_n}$ being elements of $\mathbf{C}_1 \subset \mathbf{C}_0$ have diameters greater than or equal to $\epsilon_0$, and so does their limit $F(c)$. The definitions of $A$ and $F(c)$ imply, by (8) and (9), that every point of $A$ is a point of some $F(c)$ $(c \in C)$. Thus $A$ is the union of the continua $F(c)$ $(c \in C)$. If $c, c' \in C$ and $c \neq c'$, the points $c$ and $c'$ differ on at least one coordinate, say $k_m \neq k'_m$, where $c' = (k'_1, k'_2, \ldots)$. Then

$$F(c) \subset \bar{W}_{k_1\ldots k_m}, \quad F(c') \subset \bar{W}_{k'_1\ldots k'_m},$$

and these $\bar{W}$'s are disjoint sets, by (9). Hence $F(c) \cap F(c') = \emptyset$. Moreover, the common part of $A$ with each of these $\bar{W}$'s is a closed-open subset of $A$, by (8) and (9). This means that the compactum $A$ is not connected between $F(c)$ and $F(c')$. Consequently, the continua $F(c)$ $(c \in C)$ do, indeed, coincide with the components of $A$.

Let $f$ be the function defined on $A$ by setting $f(x) = c$ if and only if $x \in F(c)$. In other words, we have $f^{-1}(c) = F(c)$ for $c \in C$, and $f(A) = C$. Let the points $x \in A$ and $c \in C$, with $f(x) = c = (k_1, k_2, \ldots)$, be fixed for a while. We denote by $V_n$ $(n = 1, 2, \ldots)$ the subset of $C$ consisting of all the points of $C$ whose first $n$ coordinates coincide with those of $c$, that is, are equal to $k_1, \ldots, k_n$, respectively. Suppose $x' \in A$ is an arbitrary point and $f(x') = c' = (k'_1, k'_2, \ldots)$. If $k_i \neq k'_i$ for at least one subscript $i \leqq n$, then we have

$$x' \in f^{-1}(c') = F(c') \subset \bar{W}_{k'_1\ldots k'_n} \subset X \backslash \bar{W}_{k_1\ldots k_n} \subset X \backslash W_{k_1\ldots k_n},$$

by (8) and (9). Hence $x' \notin W_{k_1\ldots k_n}$. Thus if $x' \in W_{k_1\ldots k_n}$, then $k_i = k'_i$ for $i = 1, \ldots, n$, which means that $c' \in V_n$. Put $U_n = A \cap W_{k_1\ldots k_n}$. We have just proved that $f(U_n) \subset V_n$ $(n = 1, 2, \ldots)$. Moreover, the sets $U_n$ are open in $A$ and

$$x \in f^{-1}(c) = F(c) \subset A \cap \bar{W}_{k_1\ldots k_{n+1}} \subset A \cap W_{k_1\ldots k_n} = U_n,$$

by (8). This implies that the function $f$ is continuous at $x$, since the sets $V_n$ $(n = 1, 2, \ldots)$ constitute standard basic neighborhoods of $c$ in $C$. On the other hand, the function $F$ transforms the Cantor set $C$ into the compactum $C(X)$ metrized by the Hausdorff distance. If $c' = (k_1', k_2', \ldots)$ is any point of $V_n$, then $k_i' = k_i$ for $i = 1, \ldots, n$, and we obtain

$$F(c') = \operatorname*{Lim}_{m\to\infty} K_{k'_1\ldots k'_m} = \operatorname*{Lim}_{m\to\infty} K_{k_1\ldots k_n k'_{n+1}\ldots k'_{n+m}},$$

by (10). It follows from (5) that

$$\text{dist}\,(K_{k_1\ldots k_n k'_{n+1}\ldots k'_{n+m}}, K_{k_1\ldots k_n}) < 2^{-(n+m-1)} + \ldots + 2^{-n} < 2^{1-n}$$

for $m = 1, 2, \ldots$. We conclude that

$$\text{dist}\,(F(c'), K_{k_1\ldots k_n}) \leqq 2^{1-n} \quad (c' \in V_n),$$

and, in particular, the latter inequality holds for $c' = c$ as $c \in V_n$. Thus

$$\text{dist}\,(F(c), F(c')) \leqq 2^{2-n} < 2^{3-n} \quad (c' \in V_n),$$

which means that $F$ transforms the set $V_n$ $(n = 1, 2, \ldots)$ into the open ball in $C(X)$ with center $F(c)$ and radius $2^{3-n}$. Since these balls form basic neighborhoods of $F(c)$ in $C(X)$, and the sets $V_n$ are open in $C$, the function $F$ is continuous at $c$.

As a result, the functions $f : A \to C$ and $F : C \to C(X)$ are continuous. The components of $A$ are the sets $F(c) = f^{-1}(c)$ $(c \in C)$. The continuity of $f$ implies that the (quotient) space of components of $A$ is homeomorphic to $f(A) = C$ [**3**, Theorem 3, p. 84]. The continuity of $F = f^{-1}$ implies that the decomposition of the compactum $A$ into its components is continuous [**10**, Theorem 2, p. 173, Theorem 4, p. 174; **11**, Theorem 1, p. 68]. The proof of 2.1 is now complete.

2.2. THEOREM. *A compactum $X$ is non-Suslinian if and only if there exist a closed subset $A \subset X$ and a weakly confluent mapping of $A$ onto $C \times I$.*

*Proof.* Let $A \subset X$ be a closed subset such that a weakly confluent mapping transforms $A$ onto $C \times I$. Then, for each point $c \in C$, a continuum contained in $A$ is mapped onto $\{c\} \times I$. These continua are non-degenerate and pairwise disjoint, whence $X$ is non-Suslinian.

Let us assume that $X$ is non-Suslinian, and let $A \subset X$ be a closed subset which satisfies the conditions from 2.1. Let $p : A \to p(A)$ denote the natural projection of $A$ onto the (quotient) space $p(A)$ of components of $A$. Then $p(A)$ is a Cantor set. Let us take a point $y_0 \in p(A)$ and observe that $p^{-1}(y_0)$ is non-degenerate. Select two points $x_0, x_1 \in p^{-1}(y_0)$, $x_0 \neq x_1$. There exists a continuous real-valued function $g : A \to R$ such that $g(x_0) = -1$ and $g(x_1) = 2$. The sets

$$(11) \quad U_0 = g^{-1}(\{t : t < 0\}), \quad U_1 = g^{-1}(\{t : t > 1\})$$

are open subsets of $A$ and $x_i \in U_i$ $(i = 0, 1)$. Hence $y_0 = p(x_i) \in p(U_i)$ $(i = 0, 1)$. Because the decomposition of $A$ into its components $p^{-1}(y)$ is continuous, the mapping $p$ is open [**11**, Theorem 1, p. 68]. Thus the sets $p(U_0)$ and $p(U_1)$ are open subsets of $p(A)$. Their common part is non-empty since it contains $y_0$. Consequently, there exists a topological copy $C'$ of the Cantor set $C$ such that $C' \subset p(U_0) \cap p(U_1)$. Let $h : C' \to C$ be a homeomorphism of $C'$ onto $C$. Since $p(A)$ is also a Cantor set, there exists a retraction $r_1 : p(A) \to C'$ of $p(A)$ onto $C'$. Let $r_2 : R \to I$ be the retraction of the real line $R$ onto $I$ defined by the formula

$$r_2(t) = \begin{cases} 0 & t \leqq 0, \\ t & t \in I, \\ 1 & t \geqq 1. \end{cases}$$

A mapping $f : A \to C \times I$ is now defined by

$$(12) \quad f(x) = (hr_1p(x), r_2g(x)) \quad (x \in A),$$

and we prove that $f$ is weakly confluent. If $L \subset C \times I$ is a non-empty continuum, there is a point $c \in C$ such that $L \subset \{c\} \times I$. Then the point $y = h^{-1}(c)$ belongs to $C'$, whence $y \in p(U_0) \cap p(U_1)$. Let $a_i \in U_i$ $(i = 0, 1)$ be points such that $p(a_i) = y$. We have $g(a_0) < 0$ and $g(a_1) > 1$, by (11), and thus $r_2g(a_0) = 0$ and $r_2g(a_1) = 1$. If $x \in p^{-1}(y)$, then $hr_1p(x) = hr_1(y) = h(y) = c$. Therefore $f(p^{-1}(y)) \subset \{c\} \times I$, by (12). Since the continuum $p^{-1}(y)$ contains both points $a_0$ and $a_1$, its image under $f$ contains the end-points $f(a_0) = (c, 0)$ and $f(a_1) = (c, 1)$ of the arc $\{c\} \times I$, by (12). We get $f(p^{-1}(y)) = \{c\} \times I$, so that $f|p^{-1}(y)$ is a weakly confluent mapping of $p^{-1}(y)$ onto $\{c\} \times I$ [**21**, p. 236]. It means there exists a continuum $K \subset p^{-1}(y)$ such that $f(K) = L$, i.e., $f$ is weakly confluent, and 2.2 is proved.

2.3. *If $X$ is a non-Suslinian compactum and $D$ is a dendrite, then there exists a weakly confluent mapping of $X$ onto $D$.*

*Proof.* By 2.2, we have a closed subset $A \subset X$ and a weakly confluent mapping $f : A \rightarrow C \times I$ of $A$ onto $C \times I$. By 1.4, there exists a weakly confluent mapping $g : C \times I \rightarrow D$ of $C \times I$ onto $D$. The composite $gf$ is a weakly confluent mapping of $A$ onto $D$. Let $f^* : X \rightarrow D$ be a continuous extension of $gf$ over $X$ [**11**, Theorem 16, p. 344]. Clearly, $f^*$ is also weakly confluent.

*Remark.* Several earlier results of other authors [**4**, Examples 1-2; **18**, Example (5.20)] are jointly generalized in 2.3, since the compacta considered by them are all non-Suslinian.

2.4. THEOREM. *Let $X$ be a compactum. The following conditions are equivalent:*

(I) *$X$ is atriodic and Suslinian,*

(II) *each weakly confluent image of $X$ is atriodic and Suslinian,*

(III) *each weakly confluent image of $X$ is atriodic, and*

(IV) *no mapping of $X$ onto a simple triod is weakly confluent.*

*Consequently, a continuum is an atriodic Suslinian curve if and only if no mapping of it onto a simple triod is weakly confluent.*

*Proof.* Let $X$ satisfy (I) and let $f$ be a weakly confluent mapping of $X$. Evidently, $f(X)$ is Suslinian. That $f(X)$ is atriodic has been proved in [**4**, Theorem 5] under the assumption of $X$ being a continuum. The connectedness of $X$, however, has not been used there, and the identical proof works in our slightly more general situation with $X$ being a compactum. Thus (I) implies (II). Furthermore, (II) implies (III), and (III) implies (IV) trivially.

The following argument proves that (IV) implies (I). Suppose condition (I) is violated, i.e., $X$ is either non-atriodic or non-Suslinian. If $X$ is non-Suslinian, the existence of a weakly confluent mapping of $X$ onto a simple triod is guaranteed by 2.3, so that condition (IV) does not hold. We can then assume that $X$ is non-atriodic, and let $T \subset X$ be a triod with a core $C_0$. Let $S$ be a simple triod with a vertex $v_0$. We have three arcs $A_1$, $A_2$, $A_3$ which form this simple triod, i.e., $S$ is their union, and $v_0$ is the only common point of any two of them and an end-point of each. Let $a_i$ $(i = 1, 2, 3)$ denote the end-point of the arc $A_i$ different from $v_0$. We also have three continua $C_1$, $C_2$, $C_3$ which form the triod $T$, i.e., $T$ is their union, and $C_0$ is the common part of any two of them and a proper subcontinuum of each. For $i = 1, 2, 3$, let $x_i \in C_i \backslash C_0$ be a point and let $f_i : C_i \rightarrow A_i$ be a mapping such that $f_i(x_i) = a_i$ and $f_i^{-1}(v_0) = C_0$. We define a mapping $f : T \rightarrow S$ by setting $f(x) = f_i(x)$ for $x \in C_i$ $(i = 1, 2, 3)$. Since $f_i$ and $f_j$ coincide on $C_0 = C_i \cap C_j$ $(i \neq j)$, the mapping $f$ is well-defined. Since $C_i$ is a continuum and $f(C_i)$ contains both end-points of the arc $A_i$, we obtain $f(C_i) = A_i$ $(i = 1, 2, 3)$, and thus $f_i$ is a weakly confluent mapping of $C_i$ onto $A_i$ [**21**, p. 236]. We show that $f$ is a weakly confluent mapping of $T$ onto $S$. Let $L \subset S$ be a non-empty continuum. If $v_0 \notin L$, then $L \subset A_i$ for some $i = 1, 2, 3$, and because $f_i$ is weakly confluent, there exists a continuum $K \subset$

$C_i \subset T$ such that $f(K) = f_i(K) = L$. If $v_0 \in L$, then $L_i = A_i \cap L$ $(i = 1, 2, 3)$ is a non-empty subcontinuum of $A_i$, and again there exists a continuum $K_i \subset C_i$ such that $f_i(K_i) = L_i$. In this case, define

$$K = C_0 \cup K_1 \cup K_2 \cup K_3$$

and observe that $C_0 = f_i^{-1}(v_0)$ meets $K_i$ $(i = 1, 2, 3)$, since $v_0 \in f_i(K_i)$. Therefore $K$ is a continuum and

$$f(K) = f(C_0) \cup f(K_1) \cup f(K_2) \cup f(K_3) = \{v_0\} \cup L_1 \cup L_2 \cup L_3 = L.$$

Hence $f : T \to S$ is a weakly confluent mapping. Let $f^* : X \to S$ be a continuous extension of $f$ over $X$. Obviously, $f^*$ is also weakly confluent and $f^*(X) = S$, which means that condition (IV) does not hold, completing the proof of 2.4.

*Remarks.* According to Theorem 2.4, an arc cannot be mapped by a weakly confluent mapping onto a simple triod [**2**, Corollary II.3]. Also, there is an analogue of the implication (I) $\Rightarrow$ (II) for hereditarily decomposable continua instead of Suslinian ones [**18**, Theorem (5.16)].

## 3. Approaching branch-continua and branch-points.

A core of any triod contained in a compactum $X$ is called a *branch-continuum* of $X$. Similarly, the vertex of any simple triod contained in $X$ is called a *branch-point* of $X$.

3.1. *If $X$ is a compactum, $X_0$ is a branch-continuum of $X$, and $U \subset X$ is an open subset such that $X_0 \subset U$, then there exists a triod contained in $U$ whose core is $X_0$.*

*Proof.* Without loss of generality, it can be assumed that $X$ is a triod with a core $X_0$. Let $X_1, X_2, X_3$ be the remaining three continua which form the triod $X$. We take an open subset $V \subset X$ such that $X_0 \subset V$ and $\bar{V} \subset U$. Let $C_i$ $(i = 1, 2, 3)$ be the component of $\bar{V} \cap X_i$ which contains the continuum $X_0$. If $X_i \subset \bar{V}$, then $C_i = X_i$. If $X_i \not\subset \bar{V}$, then $\bar{V} \cap X_i$ is a closed proper subset of the continuum $X_i$ and, consequently, $C_i$ contains a point of the set

$$\overline{X_i \backslash (\bar{V} \cap X_i)} = \overline{X_i \backslash \bar{V}} \subset \overline{X_i \backslash V} = X_i \backslash V \subset X_i \backslash X_0$$

[**11**, Theorem 1, p. 172]. In both cases, $X_0$ is a proper subcontinuum of $C_i$ $(i = 1, 2, 3)$. Since

$$\begin{aligned} C_i \cap C_j \subset (\bar{V} \cap X_i) \cap (\bar{V} \cap X_j) \subset X_i \cap X_j \\ = X_0 \quad (i, j = 1, 2, 3; i \neq j). \end{aligned}$$

we get $C_i \cap C_j = X_0$ for $i, j = 1, 2, 3$ and $i \neq j$. Thus the union $C_1 \cup C_2 \cup C_3$ is a triod contained in $\bar{V} \subset U$ whose core is $X_0$.

3.2. THEOREM. *Let $f : X \to Y$ be a weakly confluent mapping of a Suslinian compactum $X$ onto a compactum $Y$. If $Y_0$ is a branch-continuum of $Y$ and*

*$U \subset Y$ is an open set containing $Y_0$, then there exists a triod $T \subset X$ such that $f(T) \subset U$.*

*Consequently, if $y_0$ is a branch-point of $Y$, then there exists an infinite sequence $T_1, T_2, \ldots$ of triods in $X$ such that*

$$\operatorname*{Lim}_{n\to\infty} f(T_n) = \{y_0\}.$$

*Proof.* Let $V \subset Y$ be an open subset such that $Y_0 \subset V$ and $\bar{V} \subset U$. Then $f|f^{-1}(\bar{V}) : f^{-1}(\bar{V}) \rightarrow \bar{V}$ is a weakly confluent mapping of the Suslinian compactum $f^{-1}(\bar{V})$ onto $\bar{V}$. By 3.1, $\bar{V}$ contains a triod, whence $f^{-1}(\bar{V})$ is not atriodic, by 2.4. We conclude that $f^{-1}(\bar{V})$ contains a triod $T$, and $f(T) \subset \bar{V} \subset U$. This completes the proof of 3.2.

By a *hereditarily arcwise connected* compactum we mean any compactum such that each continuum contained in it is arcwise connected.

3.3. *If $T$ is a hereditarily arcwise connected triod, then there exists a simple triod $T' \subset T$ such that the vertex of $T'$ belongs to a core of $T$.*

*Proof.* Denote by $C_0$ a core of $T$ and by $C_1, C_2, C_3$ the other three continua which form the triod $T$. Select points $c_i \in C_i\backslash C_0$ $(i = 1, 2, 3)$. Since $C_1 \cup C_2$ is a subcontinuum of $T$, it is arcwise connected, and let $A \subset C_1 \cup C_2$ be an arc with end-points $c_1$ and $c_2$. Then $A$ must meet the set $C_1 \cap C_2 = C_0$, and let $c_0 \in A \cap C_0$ be a point. Hence $c_0 \neq c_3$ and $c_0 \in C_3$. Let $B \subset C_3$ be an arc with end-points $c_0$ and $c_3$. Since $c_0 \in A \cap B$, the closed set $A \cap B$ is nonempty. Let $v_0$ be the last point of the set $A \cap B$ on the arc $B$ linearly ordered from $c_0$ to $c_3$. Since

$$v_0 \in A \cap B \subset (C_1 \cup C_2) \cap C_3 = (C_1 \cap C_3) \cup (C_2 \cap C_3) = C_0,$$

we have $v_0 \neq c_i$ $(i = 1, 2, 3)$. Let $B'$ be the subarc of $B$ with end-points $v_0$ and $c_3$. The set $T' = A \cup B'$ is a simple triod and $v_0$ is the vertex of $T'$, proving 3.3.

3.4. COROLLARY. *If $f : X \rightarrow Y$ is a weakly confluent mapping of a Suslinian hereditarily arcwise connected compactum $X$ onto a compactum $Y$, then $Y$ is Suslinian and hereditarily arcwise connected, and the set of branch-points of $Y$ is contained in the closure of the image under $f$ of the set of branch-points of $X$.*

*Remarks.* Because all hereditarily locally connected continua are Suslinian and hereditarily arcwise connected (although not conversely), a previous result [**2**, Theorem II.1] is generalized in 3.4. The property of being a point of the closure of a set is a local one, and thus locally weakly confluent mappings [**17**] could be taken in 3.4 instead of weakly confluent mappings. A result concerning such mappings of hereditarily locally connected continua [**17**, Theorem (3.2), p. 232] also follows from 3.4. Since all dendroids are hereditarily arcwise connected compacta, the conclusion of 3.4 holds for weakly confluent mappings of Suslinian dendroids (cf. [**16**, Theorem 5.6, p. 263]). We note that, by 1.4, the condition of $X$ being Suslinian cannot be removed from either 3.2 or 3.4 (cf. [**2**, Example III.1, p. 413]).

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