ANALYTICAL EVALUATION OF ECONOMIC RISK CAPITAL FOR PORTFOLIOS OF GAMMA RISKS

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ABSTRACT

Based on the notions of value-at-risk and expected shortfall, we consider two functionals, abbreviated VaR and RaC, which represent the economic risk capital of a risky business over some time period required to cover losses with a high probability. These functionals are consistent with the risk preferences of profit-seeking (and risk averse) decision makers and preserve the stochastic dominance order (and the stop-loss order). Quantitatively, RaC is equal to VaR plus an additional stop-loss dependent term, which takes into account the average amount at loss. Furthermore, RaC is additive for comonotonic risks, which is an important extremal situation encountered in the modeling of dependencies in multivariate risk portfolios. Numerical illustrations for portfolios of gamma distributed risks follow. As a result of independent interest, new analytical expressions for the exact probability density of sums of independent gamma random variables are included, which are similar but different to previous expressions by Provost (1989) and Sim (1992).

KEYWORDS

Value-at-risk, expected shortfall, risk-adjusted capital, comonotonicity, additivity, supermodular order, stop-loss order, gamma convolutions.

1. ECONOMIC RISK CAPITAL USING VaR AND RaC

Suppose a firm is confronted with a risky business over some time period, and let the random variable X represent the potential *loss* or risk the firm incurs at the end of the period. To be able to cover any loss with a high probability, the firm borrows at the beginning of the time period on the capital market the amount ERC_0, called *economic risk capital*. At the end of the period, the firm has to pay interest on this at the interest rate i_R. To guarantee with certainty the value of the borrowed capital at the end of the period, the firm invests ERC_0 at the risk-free interest rate $i_f < i_R$. The value of the economic risk capital at the end of the period is thus $ERC = ERC_0 \cdot (1 + i_f - i_R)$. The risky business will

be successful at the end of the period provided the event $\{X > ERC\}$ occurs only with a small tolerance probability.

There exist several risk management principles applied to evaluate ERC. Two simple methods that have been considered so far are the value-at-risk and the expected shortfall approach (e.g. Arztner et al. (1997a/b), Arztner (1999), Embrechts (1995), Hürlimann (1998a), Schröder (1996), Wirch (1999)). According to the value-at-risk method one identifies the economic risk capital with the *value-at-risk* of the loss setting

$$(1.1) \qquad\qquad ERC = VAR_a[X] := Q_X(a),$$

where $Q_X(u) = \inf\{x \,|\, F_X(x) \geq u\}$ is a quantile function of X, with $F_X(x) = \Pr(X \leq x)$ the distribution of X. This quantile represents the maximum possible loss, which is not exceeded with the (high) probability a (called security level). According to the expected shortfall method one identifies the economic risk capital with the *risk-adjusted capital* of the loss setting

$$(1.2) \qquad\qquad ERC = RaC_a[X] := E[X | X > VaR_a[X]].$$

This value represents the conditional expected loss given the loss exceeds its value-at-risk. Clearly one has

$$(1.3) \qquad RaC_a[X] = Q_X(a) + m_X[Q_X(a)] = Q_X(a) + \frac{1}{\varepsilon}\pi_X[Q_X(a)],$$

where $m_X(x) = E[X - x | X > x]$ is the mean excess function, $\pi_X(x) = (1 - F_X(x)) \cdot m_X(x)$ is the stop-loss transform, and $\varepsilon = 1 - a$ is interpreted as loss probability (called loss tolerance level). In Arztner (1999) the expression (4.3) is called *tail conditional expectation* and abbreviated TailVaR there (for *tail value-at-risk*). Mathematically, VaR and RaC, which have been defined as functions of random variables, may be viewed as functionals defined on the space of probability distributions associated with these random variables. By abuse of language, we will use the terminology functionals when appropriate.

It is important to observe that both ERC functionals satisfy two important risk-preference criteria in the economics of insurance (see Denuit et al. (1999) for a recent review). They are consistent with the risk preferences of profit-seeking decision makers respectively profit-seeking risk averse decision makers. To see this let us first recall two partial orders of riskiness.

Definitions 1.1. A risk X is less dangerous than a risk Y in the *stochastic order*, written $X \leq_{st} Y$, if $Q_X(u) \leq Q_Y(u)$ for all $u \in [0, 1]$. A risk X is less dangerous than a risk Y in the *stop-loss order*, written $X \leq_{sl} Y$, if $\pi_X(x) \leq \pi_Y(x)$ for all x.

To compare economic risk capitals using criteria, which do not depend on the choice of the loss tolerance level, let us introduce two further partial orders of riskiness.

Definitions 1.2. A loss X is less dangerous than a loss Y in the *VaR order*, written $X \leq_{VaR} Y$, if the value-at-risk quantities satisfy $VaR_a[X] \leq VaR_a[Y]$, for

all $a \in [0, 1]$. A loss X is less dangerous than a loss Y in the *RaC order*, written $X \leq_{RaC} Y$, if the risk-adjusted capital quantities satisfy $RaC_a[X] \leq RaC_a[Y]$, for all $a \in [0, 1]$.

The value-at-risk and expected shortfall methods are consistent with ordering of risks in the sense that profit-seeking (risk averse) decision makers require higher VaR (RaC) by increasing risk, where risk is compared using the stochastic order $\leq_{st}$ (stop-loss order $\leq_{sl}$). Reciprocally, increasing VaR (RaC) is always coupled with higher risk. The following result expresses these ordering properties mathematically.

Theorem 1.1. If X and Y are two loss random variables, then $X \leq_{VaR} Y \Leftrightarrow X \leq_{st} Y$ and $X \leq_{RaC} Y \Leftrightarrow X \leq_{sl} Y$.

Proof. Since $X \leq_{st} Y \Leftrightarrow Q_X(u) \leq Q_Y(u)$ for all $a \in [0, 1]$, the first property is immediate by (1.1). Consider the Hardy-Littlewood transform defined by

$$(1.4) \qquad HL_X(u) = \begin{cases} \dfrac{1}{1-u} \cdot \displaystyle\int_u^1 Q_X(t)\,dt, & u < 1 \\[2mm] Q_X(1), & u = 1 \end{cases}$$

Its name stems from the Hardy-Littlewood (1930) maximal function and has been extensively used in both theoretical and applied mathematics (e.g. Blackwell and Dubins (1963), Dubins and Gilat (1978), Meilijson and Nàdas (1979), Kertz and Rösler (1990/92/93), Rüschendorf (1991), Hürlimann (1998b/c/d)). One knows that there exists a random variable X^H associated to X such that (e.g. Hürlimann (1998b), Theorem 2.1)

$$(1.5) \qquad HL_X(u) = Q_{X^H}(u) = Q_X(u) + m_X[Q_X(u)],$$

hence $RAC_a[X] = Q_{X^H}(a)$ by (1.3). The result follows from the fact that $X \leq_{sl} Y$ if and only if $X^H \leq_{st} Y^H$, where $\leq_{st}$ denotes the usual stochastic dominance order (e.g Kertz and Rösler (1992), Lemma 1.8, or Hürlimann (1998c), Theorem 2.3). For the convenience of the reader, an alternative perhaps more accessible proof should also be pointed out. Consider the so-called distortion function $g_a(x) = \min\left\{\frac{x}{1-a}, 1\right\}$. It is easy to show that

$$(1.6) \qquad RaC_a[X] = \int_0^\infty g_a[1 - F_X(x)]\,dx$$

identifies the RaC functional with a member of the class of distortion pricing principles in Wang (1996). The result follows by Dhaene et al. (2000), Theorem 3, which contains a proof of the stated equivalence. ◊

Finally, it is important to observe that, except for a world of elliptical linear portfolio losses (Embrechts et al. (1998), Fundamental Theorem of Risk Management), the VaR functional has several shortcomings. It is not subadditive

and not scalar multiplicative, and it cannot discriminate between risk-averse and risk-taking portfolios (examples 1 to 3 in Wirch (1999)). Some more details for the practitioner are in order. Recall that a risk measure $R[\cdot]$ acting on the set of all risks is *subadditive* provided $R[X+Y] \leq R[X] + R[Y]$ for all X, Y, that is merging two risks does not create extra risk. If a firm must meet a requirement of extra economic risk capital that did not satisfy this property, the firm might separate in two subunits requiring less capital, a matter of concern for the supervising authority. A risk measure is *scalar multiplicative* provided $R[cX] = cR[X]$ for all X, all constants $c \geq 0$. In situations where no diversification occurs capital requirement depends on the size of the risk. In contrast to this, the RaC functional, which is subadditve and scalar multiplicative, is a coherent risk measure in the sense of Arztner et al. (1997) and appears thus more suitable in general applications. A recent work devoted to the evaluation of economic risk capital in life-insurance using the VaR and RaC approaches is Ballmann and Hürlimann (2000).

2. THE MAXIMUM RaC FOR THE AGGREGATE RISK OF PORTFOLIOS

An important but complex problem is the evaluation of RaC for the aggregate risk of portfolios. Let $X = (X_1, ..., X_n)$ be a portfolio of multivariate risks, where the marginal risks X_i have distributions $F_i(x)$, $i = 1, ..., n$. In a first step, one is interested in the maximum RaC for the aggregate risk $S(X) = X_1 + ... + X_n$ whenever $X \in D(F_1, ..., F_n)$, the set of all multivariate risks with given marginals $F_i(x)$. It will be shown below that the maximum RaC is attained when the margins X_i show the strongest possible dependence structure, an extremal situation for which one says that $X_1, ..., X_n$ are mutually comonotonic.

A multivariate loss $(X_1, ..., X_n)$ is called *comonotonic* whenever an increase of a single loss $X_i(\omega_1) < X_i(\omega_2)$ for two events ω_1, ω_2 implies a nondecrease of all other losses $X_j(\omega_1) \leq X_j(\omega_2)$, $j \neq i$ (Schmeidler (1986), Yaari (1987)). For $X \in D(F_1, ..., F_n)$ this is exactly the case when $X = \left(F_1^{-1}(U), ..., F_n^{-1}(U)\right)$ with U a uniform $(0,1)$ random variable noting that $F_i^{-1}(U)$ has distribution F_i and F_i^{-1} is increasing for all i. The distribution F of a comonotonic random vector is determined by its marginal distributions F_i through the relationship $F(x_1, ..., x_n) = \min_{1 \leq i \leq n} \{F_i(x_i)\}$. Mathematically, four equivalent defining conditions of comonotonicity can be given.

Definition 2.1. (Bäuerle and Müller (1998)) The components of a random vector $X = (X_1, ..., X_n) \in D(F_1, ..., F_n)$ are called mutually *comonotonic* if any of the following equivalent conditions hold:

(C1) The multivariate distribution $F(x_1, ..., x_n)$ of $(X_1, ..., X_n)$ identifies with the so-called Fréchet upper bound $F(x_1, ..., x_n) = \min_{1 \leq i \leq n} \{F_i(x_i)\}$.

(C2) There exists a random variable Z and non-decreasing real functions $u_1, ..., u_n$ such that $(u_1(Z), ..., u_n(Z))$ has the distribution F.

(C3) The random vector $\left(F_1^{-1}(U),...,F_n^{-1}(U) \right)$, where U is uniformly distributed on [0,1], has distribution F.

(C4) There is a random vector X, distributed as F, such that $X_i(\omega_1) < X_i(\omega_2)$ implies $X_j(\omega_1) \le X_j(\omega_2)$ for all $j \ne i$.

We need further the notion of supermodular order.

Definition 2.2. A random vector X precedes Y in the *supermodular order*, written $X \le_{sm} Y$, if $E[f(X)] \le E[f(Y)]$ for all supermodular functions f such that the expectations exist, where f is called *supermodular* if

$$(2.1) \qquad f(x \wedge y) + f(x \vee y) \ge f(x) + f(y) \text{ for all } x, y, \in \mathbf{R}^n,$$

with the notation $(x_1,..., x_n) \wedge (y_1,...,y_n) = (x_1 \wedge y_1,..., x_n \wedge y_n)$, $\wedge$ the minimum operator, and $(x_1,..., x_n) \vee (y_1,...,y_n) = (x_1 \vee y_1,..., x_n \vee y_n)$, $\vee$ the maximum operator.

Intuitively the notion of supermodular function can be grasped as follows. Let $x_1, ..., x_n$ be n individual losses in a portfolio, and let $f(x_1, ..., x_n)$ be the aggregate loss caused by these losses. Then supermodularity of the function f means that the influence on the aggregate loss of an increase of a single loss is greater, the higher the other losses are. In the literature supermodular functions are also called superadditive, and have been originally studied in applied mathematics and operations research (e.g. Marshall and Olkin (1979)). They have been extensively applied in economics (e.g. Topkis (1998)). The related supermodular order allows for a comparison of the strength of dependence between random vectors. Its origin in the statistical literature can be traced back to Block and Sampson (1988), Joe (1990), Meester and Shanthikumar (1993), Szekli et al. (1994), Shaked and Shanthikumar (1997). Actuarial applications of this order are discussed in Müller (1997), Bäuerle and Müller (1998), Goovaerts and Dhaene (1999).

To compare the riskiness of portfolios, one says that a portfolio $X = (X_1, ..., X_n)$ is less risky than a portfolio $Y = (Y_1, ..., Y_n)$ if the corresponding aggregate risks $S(X) = X_1 + ... + X_n$ and $S(Y) = Y_1 + ... + Y_n$ are stop-loss ordered, that is $S(X) \le_{sl} S(Y)$. A sufficient condition for this is the supermodular order.

Theorem 2.1. Let $X = (X_1, ..., X_n)$ and $Y = (Y_1, ..., Y_n)$ be random vectors in $D(F_1, ..., F_n)$ such that $X \le_{sm} Y$, then one has $S(X) \le_{sl} S(Y)$.

Proof. This is shown in Müller (1997), Theorem 3.1. ◊

The significance of the supermodular order for economic risk capital calculations is now immediate. Given two portfolios $X, Y \in D(F_1, ..., F_n)$ such that $X \le_{sm} Y$, it is possible to compare the RaC of the aggregate risk $S(X)$ with the RaC of the aggregate risk $S(Y)$.

Corollary 2.1. Let $X = (X_1, ..., X_n)$ and $Y = (Y_1, ..., Y_n)$ be random vectors in $D(F_1, ..., F_n)$ such that $X \le_{sm} Y$, then one has $RaC_a[S(X)] \le RaC_a[S(Y)]$ for all $a \in [0,1]$.

Proof. This is an immediate consequence of Theorem 2.1 and Theorem 1.1. ◊

Even more, one obtains that the portfolio $X^c = \left(F_1^{-1}(U), ..., F_1^{-1}(U) \right) \in D(F_1, ..., F_n)$ with mutually comonotonic margins yields the maximum RaC.

Theorem 2.2. The maximum RaC for the aggregate risk of a portfolio with fixed marginal risks is attained at the portfolio with mutually comonotonic components, that is one has

$$(2.2) \qquad \max_{X \in D(F_1, ..., F_n)} \{ RaC_a[S(X)] \} = RaC_a[S(X^c)].$$

Proof. By the inequality of Lorentz (1953) (e.g. Theorem 5 in Tchen (1980)), one knows that $X \leq_{sm} X^c$ for all $X \in D(F_1, ..., F_n)$. The result follows by Corollary 2.1. Alternatively, it is possible to prove directly that $X \leq_{sl} X^c$ for all $X \in D(F_1, ..., F_n)$ as shown by Goovaerts et al. (2000) (see also Dhaene et al. (2000), Corollary 6). Then Theorem 1.1 implies the result. ◊

This result means that comonotonicity, which displays the strongest possible dependence structure, corresponds to the riskiest portfolio under all portfolios with the same marginal risks and requires the maximum RaC under all these portfolios. It is further remarkable that under a simple regularity condition the maximum RaC is an additive functional.

Theorem 2.3. Let $X^c = (X_1, ..., X_n)$ be a portfolio of mutually comonotonic risks with absolutely continuous marginal distributions $F_i(x)$, $i = 1, ..., n$. Then the RaC functional satisfies the additive property

$$(2.3) \qquad RaC_a[S(X^c)] = \sum_{i=1}^{n} RaC_a[X_i].$$

Proof. Denote by $F_s(x)$ the distribution of $S(X^c)$. Consider the quantiles $d = Q_S(a)$, $d_i = Q_{X_i}(a)$, $i = 1, ..., n$, and the stop-loss transforms $\pi_s(x)$, $\pi_i(x) := \pi_{x_i}(x)$, $i = 1, ..., n$. By the comonotonic assumption, the quantiles and stop-loss transforms behave additively, that is one has $d = \sum_{i=1}^{n} d_i$ (e.g. Landsberger and Meilijson (1994), Denneberg (1994), Kaas et al. (2000)) and $\pi_s(d) = \sum_{i=1}^{n} \pi_i(d_i)$ (Dhaene et al. (2000), Theorem 8, special case of absolutely continuous distributions, or Kaas et al. (2000)). The assertion follows from (1.3) using the relationship $(1 - F_X(x)) \cdot m_X(x) = \pi_X(x)$ between the mean excess function and the stop-loss transform by means of the equalities

$$RaC_a[S(X^c)] = d + \frac{1}{\varepsilon} \cdot \pi_s(d) = \sum_{i=1}^{n} \left\{ d_i + \frac{1}{\varepsilon} \cdot \pi_i(d_i) \right\} = \sum_{i=1}^{n} RaC_a[X_i]. \quad ◊$$

Remark 2.1.

As pointed out by a referee, the additive relation (2.3) is a special case of a more general result due to Dellacherie (1970) and quoted in Schmeidler

(1986). Let A be a σ-algebra of subsets of a set S, and χ the set of all bounded real-valued A-measurable functions on S. For a monotone set function v on S such that $v(\varnothing) = 0$, $v(S) = 1$, and a non-negative real valued function $X \in \chi$, consider the Choquet integral $H_v[X] = \int_S X dv = \int_0^\infty v(X > x) dx$. Dellacherie's result states that if $X, Y \in \chi$ are comonotonic, then $H_v[X + Y] = H_v[X] + H_v[Y]$. In the special case of a probability space (Ω, P, A), consider the distortion function $g_a(x) = \min\left\{\frac{x}{1-a}, 1\right\}$ and the set function $v = g_a \circ P$. With (1.6) one obtains

$$RaC_a[X] = \int_0^\infty g_a[P(X > x)]\,dx = H_v[X].$$ The additivity (2.3) for comonotonic risks

follows from Dellacherie's result. However, note that our Theorem 2.3 is not restricted to bounded random variables, an essential assumption in Schmeidler's paper.

An interesting problem concerns the impact of various "positive" dependence structures between risks $X_1, ..., X_n$ on the evaluation of RaC for the aggregate risk $S(X) = X_1 + ... + X_n$. Independent risks with an aggregate denoted by $S^i = X_1 \oplus ... \oplus X_n$ and comonotonic risks with an aggregate $S^c = X_1 +_c ... +_c X_n$ are two extreme cases of primary importance. Let us motivate this assertion. In virtue of Corollary 2.1 and Theorem 2.2 it seems reasonable to restrict the attention to positive dependent portfolios $X \in D(F_1, ..., F_n)$ satisfying the supermodular inequality $X^i \leq_{sm} X \leq_{sm} X^c$, which implies $S^i \leq_{sl} S(X) \leq_{sl} S^c$ and $S^i \leq_{RaC} S(X) \leq_{RaC} S^c$. As an example, the family of multivariate elliptically contoured distributions is increasing in the supermodular order as the correlation increases (Block and Sampson (1988)). Portfolios satisfying only the stop-loss inequality $S^i \leq_{sl} S(X) \leq_{sl} S^c$, which by Theorem 1.1 is sufficient to imply $S^i \leq_{RaC} S(X) \leq_{RaC} S^c$, might also be of interest (e.g Bäuerle and Müller (1998), Section 4).

It is well-known that the stop-loss order relation $S^i \leq_{sl} S^c$ implies a considerable difference between the corresponding stop-loss premiums. However, the quantitative impact of this relation on the evaluation of RaC has not yet been examined. The additive property of Theorem 2.3 is of evident help for the quantitative analysis of the property $RaC_a[S^i] \leq RaC_a[S^c]$. Since insurance risks are often quite well approximated by gamma distributed risks or translations thereof (e.g. Seal (1977), Dufresne et al. (1991), Dickson and Waters (1993)), we will restrict ourselves in the present paper to a quantitative evaluation of this inequality for gamma risks. Since the exact distribution of sums of independent gamma random variables is not very well-known among actuaries, the next Section is of additional independent interest.

3. SUMS OF INDEPENDENT GAMMA RANDOM VARIABLES

Gamma distributions, which include the exponential, Erlang and chi-square distributions, are among the most important distributions widely used in applications. They are also of great importance in theoretical work. Thorin

(1977) introduced the class of generalized gamma convolutions, defined as the smallest class of distributions on the positive real line that contains the gamma distributions and is closed with respect to convolution and weak limits, to prove the infinite divisibility of many distributions. The class of generalized gamma convolutions is surprisingly rich and has a remarkable structure. It has been extensively studied in the last century by Bondesson (1992).

Though not noticed in actuarial science (e.g. one misses them in Panjer and Willmot (1992)), expressions for the exact probability density of sums of *independent* gamma random variables are known from the statistical literature. For example, Johnson et al. (1994), pp. 384-85, refers to Mathai (1982), Moschopoulos (1985) and Sim (1992). One can add Provost (1989), which determines the exact density applying the inverse Mellin transform. The result by Sim (1992) uses the following direct elementary approach. Let $X_i \sim \Gamma(a_i, \beta_i)$, $i = 1, ..., n$, be n independent gamma random variables with densities

$$(3.1) \qquad f_i(x) = g(\beta_i x; a_i) := \frac{(\beta_i x)^{a_i}}{\Gamma(a_i)} \cdot \frac{e^{-\beta_i x}}{x}, \quad x > 0, \ a_i, \ \beta_i > 0.$$

The special case of identical scale parameters being well-known, one assumes that $\beta_1 > \beta_2 > ... > \beta_n$. The density of the independent sum $S_n = X_1 \oplus ... \oplus X_n$ can be obtained from the convolution formula

$$(3.2) \qquad f_{S_n}(s) = \int_0^s f_{S_{n-1}}(t) \cdot f_n(s - t) dt$$

applying mathematical induction. A calculation yields the result by Sim (1992) (see also Johnson et al. (1994), formula (17.110)):

$$(3.3) \ f_{S_n}(s) = \frac{1}{\Gamma(a^{(n)})} \cdot \left[\prod_{i=1}^n \beta_i^{a_i} \right] \cdot s^{a^{(n)}-1} \cdot e^{-\beta_1 s} \cdot \sum_{k=0}^\infty C_k^n \cdot \frac{\left(a^{(n-1)}\right)_k}{k! \cdot \left(a^{(n)}\right)_k} \cdot \left[(\beta_1 - \beta_2)s\right]^k,$$

where $a^{(k)} = a_1 + ... + a_k$, $(c)_k = \dfrac{\Gamma(c + k)}{\Gamma(c)}$, and

$$(3.4) \qquad C_k^i = \begin{cases} 1, & i = 2 \\ \displaystyle\sum_{j=0}^k C_j^{i-1} \cdot \binom{k}{j} \cdot \frac{\left(a^{(i-2)}\right)_j}{\left(a^{(i-1)}\right)_j} \cdot \left[\dfrac{\beta_{n-i+2} - \beta_{n-i+3}}{\beta_{n-i+1} - \beta_{n-i+2}} \right]^j, & i = 3, ..., n. \end{cases}$$

A rearrangement shows that (3.3) is an infinite linear combination of gamma densities with the same scale parameter β_1, a property already observed by Provost (1989). Applying another elementary approach, we obtain below a new similar representation of the exact probability density, which differs from the results by Provost (1989) and Sim (1992).

Theorem 3.1. Let $X_i \sim \Gamma(a_i, \beta_i)$ be n independent gamma random variables such that $\beta_1 > \beta_2 > \ldots > \beta_n$. Then the density of the independent sum $S_n = X_1 \oplus \ldots \oplus X_n$ is analytically described by the infinite series

$$(3.5) \qquad f_{S_n}(s) = A_n \cdot \sum_{k=0}^{\infty} C_k^n \cdot g(\beta_1 s; a^{(n)} + k), \text{ with}$$

$$a^{(n)} = a_1 + \ldots + a_n, \quad A_n = \prod_{j=2}^{n} \left[\frac{\beta_j}{\beta_1} \right]^{a_j}, \quad n \ge 2, \quad A_1 = 1,$$

$$(3.6) \qquad C_k^i = \begin{cases} \left[1 - \dfrac{\beta_2}{\beta_1} \right]^k \cdot \dfrac{(a_2)_k}{k!}, & i = 2 \\[3ex] \displaystyle\sum_{j=0}^{k} C_j^{i-1} \cdot \left[1 - \dfrac{\beta_i}{\beta_1} \right]^{k-j} \cdot \dfrac{(a_i)_{k-j}}{(k-j)!}, & i = 3, \ldots, n \end{cases}$$

$$C_k^1 = 0 \quad k = 1, 2, \ldots, \quad C_0^i = 1 \quad i = 1, \ldots, n.$$

Proof. This is shown through induction. Clearly, the series representation holds for n=1. By induction, assume the representation holds for the index n and show it for the index n+1. For convenience set

$$(3.7) \qquad S_n = X_n \oplus S_{n-1} \in (0, \infty), \quad R_n = \frac{X_n}{S_n} \in (0, 1).$$

Applying the standard method of transformation of random variables based on Jacobians (e.g. Fisz (1973), p. 77), the density of the sum S_n is determined recursively by the formulas

$$(3.8) \qquad f_{S_n}(s) = \int_0^1 f_{(S_n, R_n)}(s, r) dr,$$

$$f_{(S_n, R_n)}(s, r) = s \cdot f_{X_n}(rs) \cdot f_{S_{n-1}}((1 - r)s).$$

Using this and the induction assumption, one obtains

$$(3.9) \qquad f_{S_{n+1}}(s) = A_n \cdot \sum_{k=0}^{\infty} C_k^n \cdot s^{a^{(n+1)}+k-1} \cdot e^{-\beta_1 s} \cdot \frac{\beta_1^{a^{(n)}+k}}{\Gamma(a^{(n)} + k)} \cdot I(n, k),$$

with

$$(3.10) \qquad \begin{aligned} I(n, k) &= \int_0^1 r^{a_{n+1}-1} \cdot (1 - r)^{a^{(n)}+k-1} \cdot e^{-(\beta_{n+1}-\beta_1)rs} dr \\ &= \sum_{j=0}^{\infty} \frac{(\beta_1 - \beta_{n+1})^j}{j!} \cdot s^j \cdot \frac{\Gamma(a^{(n)} + k)\Gamma(a_{n+1} + j)}{\Gamma(a^{(n+1)} + k + j)}. \end{aligned}$$

Through rearrangement it follows that

$$(3.11) \quad f_{S_{n+1}}(s) = A_n \cdot \left[\frac{\beta_{n+1}}{\beta_1}\right]^{a_{n+1}} \cdot \sum_{k=0}^{\infty} C_k^n \cdot \left\{ \sum_{j=0}^{\infty} \frac{\left(1-\frac{\beta_{n+1}}{\beta_1}\right)^j \Gamma(a_{n+1}+j)}{j! \Gamma(a_{n+1})} \cdot g\left(\beta_1 s; a^{(n+1)}+k+j\right) \right\}$$

$$= A_{n+1} \cdot \sum_{k=0}^{\infty} C_k^{n+1} \cdot g\left(\beta_1 s; a^{(n+1)}+k\right).$$

The analytical formula (3.5) is shown. ◊

Remark 3.1.

Though the coefficients of $g(\beta_1 s; a^{(n)}+k)$ in (3.3) and (3.5) are evaluated using different expressions, they are identical. However, the formulas (3.6) are more symmetric and simpler, and for this reason they should be preferred.

Using the incomplete gamma function defined by

$$(3.12) \quad G(\beta x; a) = \frac{1}{\Gamma(a)} \cdot \int_0^{\beta x} t^{a-1} e^{-t} dt,$$

the distribution function of an independent gamma sum is through integration of (3.5) equal to

$$(3.13) \quad F_{S_n}(s) = A_n \cdot \sum_{k=0}^{\infty} C_k^n \cdot G\left(\beta_1 s; a^{(n)}+k\right).$$

The evaluation of RaC for portfolios of independent gamma risks requires an analytical expression for the stop-loss transform of S_n.

Corollary 3.1. The stop-loss transform $\pi_{S_n}(d) = E[(S_n - d)_+]$ of a sum $S_n = X_1 \oplus \ldots \oplus X_n$ of n independent gamma random variables $X_i \sim \Gamma(a_i, \beta_i)$, $i = 1, \ldots, n$, such that $\beta_1 > \beta_2 > \ldots > \beta_n$, is determined by the analytical formula

$$(3.14) \quad \pi_{S_n}(d) = E[S_n] - d \cdot \overline{F}_{S_n}(d) - \frac{1}{\beta_1} \cdot A_n \cdot \sum_{k=0}^{\infty} (a^{(n)}+k) \cdot C_k^n \cdot G(\beta_1 d; a^{(n)}+k+1)$$

Proof. This follows without difficulty noting that $\pi_{S_n}(d) = E[S_n] - d \cdot \overline{F}_{S_n}(d) - \int_0^d s f_{S_n}(s) ds$, and $\int_0^d s g(\beta s; a) ds = \frac{a}{\beta} G(\beta d; a+1)$. ◊

Remark 3.2.

The special case $\beta_1 = ... = \beta_n = \beta$ of identical scale parameters is well-known. In this situation, the above formulas are replaced by the very simple ones

$$f_{s_n}(s) = g(\beta s; a^{(n)}), \quad F_{s_n}(s) = G(\beta s; a^{(n)}),$$

(3.15)

$$\pi_{s_n}(d) = \left(\frac{a^{(n)}}{\beta} - d\right) \cdot \overline{F}_{s_n}(d) + \frac{d}{\beta} \cdot f_{s_n}(s),$$

where the last one is obtained through partial integration.

4. NUMERICAL EXAMPLES

First, let us calculate RaC for portfolios of independent gamma risks. Given the loss tolerance level ε, first determine using (3.13) the solution d_ε of the equation $F_{s_n}(d_\varepsilon) = 1 - \varepsilon$. Inserting the obtained value in (3.14), one obtains the formula

$$RaC_{1-\varepsilon}\left[S_n^i\right] = d_\varepsilon + \frac{1}{\varepsilon} \cdot \pi_{s_n}(d_\varepsilon)$$

(4.1)

$$= \frac{1}{\varepsilon} \cdot \left[E[S_n] - \frac{1}{\beta_1} \cdot A_n \cdot \sum_{k=0}^{\infty} (a^{(n)} + k) \cdot C_k^n \cdot G(\beta_1 d_\varepsilon; a^{(n)} + k + 1)\right].$$

In the special case $\beta_1 = ... = \beta_n = \beta$ of Remark 3.2, the quantile d_ε is solution of $G(\beta d_\varepsilon; a^{(n)}) = 1 - \varepsilon$ and (4.1) simplifies to

(4.2) $$RaC_{1-\varepsilon}\left[S_n^i\right] = E[S_n] \cdot \left\{1 + \frac{1}{\varepsilon} \cdot \frac{d_\varepsilon}{a^{(n)}} \cdot g(\beta d_\varepsilon; a^{(n)})\right\}.$$

Second, let us calculate RaC for portfolios of comonotonic gamma risks. The evaluation uses the additive property of Theorem 2.2. For $i = 1, ..., n$, determine the solution $d_{i,\varepsilon}$ of $G(\beta_i d_{i,\varepsilon}; a_i) = 1 - \varepsilon$. Replacing $a^{(n)}$ by a_i in (3.15) one gets

$$\pi_i(d_{i,\varepsilon}) = \varepsilon \cdot (E[X_i] - d_{i,\varepsilon}) + \frac{d_{i,\varepsilon}}{\beta_i} \cdot g(\beta_i d_{i,\varepsilon}; a_i).$$

It follows that

$$RaC_{1-\varepsilon}\left[S_n^c\right] = \sum_{i=1}^{n} RaC_{1-\varepsilon}[X_i] = \sum_{i=1}^{n} \left\{d_{i,\varepsilon} + \frac{1}{\varepsilon} \cdot \pi_i(d_{i,\varepsilon})\right\}$$

(4.3)

$$= E[S_n] + \frac{1}{\varepsilon} \cdot \sum_{i=1}^{n} \frac{d_{i,\varepsilon}}{\beta_i} \cdot g(\beta_i d_{i,\varepsilon}; a_i).$$

Example 4.1: independence versus comonotonic assumption

In the special case $\beta_1 = ... = \beta_n = \beta$ of Remark 3.2, a comparison of (4.2) and (4.3) yields the difference formula for RaC:

$$(4.4) \quad \beta\varepsilon \cdot \left(RaC_{1-\varepsilon}\left[S_n^c\right] - RaC_{1-\varepsilon}\left[S_n^i\right]\right) = \sum_{i=1}^{n} \frac{d_{i,\varepsilon}}{\beta_i} \cdot g\left(\beta_i d_{i,\varepsilon}; a_i\right) - d_\varepsilon \cdot g\left(\beta d_\varepsilon; a^{(n)}\right),$$

where $G(\beta d_\varepsilon; a^{(n)}) = 1-\varepsilon$ and $G(\beta_i d_{i,\varepsilon}; a_i) = 1-\varepsilon$, $i = 1,..., n$. A numerical illustration for the exponential case $\beta = 1$, $\alpha_1 = ... = \alpha_n = 1$, is summarized in Table 4.1 below. In this situation $RaC_{1-\varepsilon}\left[S_n^c\right] = n \cdot \{1 - \ln(\varepsilon)\}$ depends *linearly* on the number of exponential risks. The difference increases non-linearly according to the formula

$$(4.5) \qquad RaC_{1-\varepsilon}\left[S_n^c\right] - RaC_{1-\varepsilon}\left[S_n^i\right] = \frac{1}{\varepsilon} \cdot \left\{ n \cdot \varepsilon \cdot [-\ln(\varepsilon)] - \frac{d_\varepsilon^n e^{-d_\varepsilon}}{(n-1)!} \right\},$$

where $G(d_\varepsilon; n) = 1-\varepsilon$. As an interesting observation, one notes for increasing n a *decreasing percentage increase* of $RaC_{1-\varepsilon}\left[S_n^i\right]$ over the ε-range between 0.05 and 0.001.

TABLE 4.1
RaC FOR EXPONENTIAL RISK PORTFOLIOS UNDER INDEPENDENCE (I)
AND COMONOTONIC (C) ASSUMPTION

n	$\varepsilon = 0.05$		$\varepsilon = 0.01$		$\varepsilon = 0.001$	
	(i)	(c)	(i)	(c)	(i)	(c)
1	4	4	5.6	5.6	7.9	7.9
2	5.9	8	7.8	11.2	10.3	15.8
3	7.6	12	9.6	16.8	12.4	23.7
4	9.2	16	11.4	22.4	14.3	31.6
5	10.7	20	13	28	16.1	39.5
10	17.6	40	20.5	56.1	24.2	79.1
20	30.3	79.9	34	112.1	38.6	158.2
50	65.7	199.8	70.9	280.3	77.3	395.4
100	121.7	399.6	128.7	560.5	137.2	790.8

Example 4.2: sums of independent gamma risks versus gamma and normal approximations

Suppose an insurer desires to calculate VaR and RaC for a portfolio of n independent risks, which follow a classical risk model. Each risk $X_i = Y_{i,1} + ... + Y_{i,N_i}$ has a compound Poisson distribution, where N_i is Poisson distributed

and the $Y_{i,j}$'s are the individual claims. Assume that one knows the expected number of claims $\lambda_i = E[N_i]$, as well as the first and second moments $v_i = E[Y_{i,1}]$, $m_{i,2} = E[Y_{i,1}^2]$ of the severity distributions, $i = 1, ..., n$. Let $k_i = c_i \cdot \lambda_i^{-\frac{1}{2}}$, with $c_i = \sqrt{m_{2,i} \cdot v_i^{-1}}$, be the coefficient of variation of X_i. As mentioned previously, it is often possible to assume that X_i is gamma distributed with parameters

$$(4.6) \qquad a_i = \frac{1}{k_i^2} = \frac{\lambda_i}{c_i^2}, \quad \beta_i = \frac{1}{c_i^2} \cdot \frac{1}{v_i}, \quad i = 1, ..., n.$$

The risk $S_n = X_1 \oplus ... \oplus X_n$ of the portfolio is again compound Poisson distributed with corresponding parameters

$$(4.7) \qquad \lambda = \sum_{i=1}^{n} \lambda_i, \quad v = \sum_{i=1}^{n} \left[\frac{\lambda_i}{\lambda} \right] \cdot v_i, \quad k = c \cdot \lambda^{-\frac{1}{2}}, \quad c^2 = \sum_{i=1}^{n} \left[\frac{\lambda_i}{\lambda} \right] \cdot \left[\frac{v_i}{v} \right]^2 \cdot c_i^2.$$

Now, it is possible to approximate S_n either by a sum of n independent gamma risks with the parameters a_i, β_i in (4.6) or by a gamma risk with parameters $a = \frac{\lambda}{c^2}$, $\beta = \frac{1}{c^2} \cdot \frac{1}{v}$ as defined in (4.7). To illustrate, we compare the VaR and RaC values of these approximations with the values obtained from a normal approximation for portfolios of 5 risks with parameters (typical for the aggregate claims of collectives of life insurance policies):

$$(4.8) \qquad \begin{aligned} (\lambda_1, ..., \lambda_5) &= m \cdot (1, 1, 1, 1, 1), \quad m = 1, 2, 5, 10, 20, 50, \\ (v_1, ..., v_5) &= (2, 2, 1, 3, 2), \quad (c_i, ..., c_5) = (1.25, 1.75, 2.5, 1.5, 2). \end{aligned}$$

The parameters for the overall gamma approximation are by (4.7) equal to $\lambda = 5m$, $v = 2$, $c = 1.74642$. Table 4.2 shows that the VaR and RaC values of both gamma approximations differ only slightly, but the normal approximation underestimates systematically these values, especially for small λ_i's and more considerably for RaC than for VaR.

TABLE 4.2

VaR AND RaC COMPARISONS BY FIXED $\varepsilon = 0.05$

m	VaR			RaC		
	normal approx.	gamma approx.	sum of ind. gamma	normal approx.	gamma approx.	sum of ind. gamma
1	22.8	25.3	25.3	26.1	32.1	32.4
2	38.2	40.9	41.0	42.8	49.1	49.5
5	78.7	81.8	81.9	86.0	92.6	93.0
10	140.6	143.8	144.0	150.9	157.6	158.1
20	257.5	260.7	260.9	272.0	278.8	279.3
50	590.8	594.2	594.4	613.9	620.7	621.2

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