ON INTEGRATION OF VECTOR-VALUED FUNCTIONS

D. O. SNOW

1. Introduction. Among the variety of integrals which have been devised for integrating vector-valued functions the most widely used is that of Bochner **(2)**, perhaps because of the simplicity of its formulation. Other approaches, including one by Birkhoff **(1)**, have yielded more general integrals yet none of them seems to have supplanted the Bochner integral to a significant extent.

Another simple approach is that of Graves **(4)**. This is an adaptation of the Riemann definition, and the resulting integral has most of the properties of the ordinary Riemann integral. A noteworthy exception is that there exist functions which are everywhere discontinuous and yet are Graves integrable. In sharp contrast to the real variable case the Bochner (Lebesgue) integral does not include the Graves (Riemann) integral. Neither does the Graves integral include that of Bochner.

In the present paper we show that the Graves integral can be generalized in a simple way to produce an integral which includes the Bochner integral as a special case, and is equivalent to the Birkhoff integral for functions defined on a bounded Lebesgue measurable set in n-dimensional Euclidean space. This generalization stems from the fact that the Lebesgue measurability of a finite real-valued function f, on a measurable set E, is equivalent to the validity of the well-known Lusin condition **(9**, p. 72**)** for f. It has been pointed out by Hildebrandt **(6)** that a definition of the Lebesgue integral due to Hahn **(5)** is based on the Lusin property and that this suggests an alternate approach to the Bochner integral. Bourbaki **(3**, p. 180**)** gives a definition of measurability for a function f, defined on a locally compact space E with values in an arbitrary topological space F, which is also based on the Lusin condition in that f is required to be continuous on each of a collection of compact sets with total measure approximating that of E. It turns out that when the range space is a Banach space this definition is equivalent to Bochner measurability **(3**, Theorem 3, p. 189**)**. We notice, however, that there exist fairly simple functions which are Graves (Riemann) integrable but not measurable in the Bochner or Bourbaki senses, nor in any sense that implies the Lusin property. The classical example is that of Graves **(4**, p. 166**)** which involves the space M of bounded real functions $f(t)$ on $0 \leqslant t \leqslant 1$, with

$$||f(t)|| = \sup_{0\leqslant t\leqslant 1}|f(t)|.$$

Let $x(\alpha) = f_\alpha(t)$ where $f_\alpha(t) = 0$ on $0 \leqslant t \leqslant \alpha$, and $f_\alpha(t) = 1$ on $\alpha < t \leqslant 1$. Thus $x(\alpha)$ is defined on $0 \leqslant \alpha \leqslant 1$ and is everywhere discontinuous there.

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On the other hand, this function is integrable in the sense of Birkhoff and of Jeffery **(7)**. These facts suggest a weakening of the Lusin condition in which we replace the sets on which the function is required to be continuous **(5; 6; 3)** by sets over which the function is integrable in the Graves sense (generalized in a natural way so as to be defined on closed sets), and hence may in some cases be everywhere discontinuous on these sets. A definition of measurability is based on this weakened condition and the Hahn approach is then used in defining our generalized integral.

2. Notation. Throughout this paper X will denote an arbitrary linear normed complete space, or Banach space, R the space of real numbers, $x(\alpha)$, $y(\alpha)$ functions valued in X, and $f(\alpha)$, $g(\alpha)$ real-valued functions. The symbol $[a, b]$ will denote a closed interval on the real line, P, P', F closed subsets of $[a, b]$, and $|E|$ the Lebesgue measure of a measurable set E.

3. A Graves integral defined over a closed set.

Definition 3.1. Let $x(\alpha)$ be defined and bounded on P. Let π be a subdivision of $[a, b]$ into subintervals (α_{i-1}, α_i); let $\Delta\alpha_1$ denote the closed interval $[\alpha_0, \alpha_1]$ and $\Delta\alpha_i$, $i>1$, denote the half-open interval $(\alpha_{i-1}, \alpha_i]$. Let $N\pi$ be the maximum of the differences $\alpha_i - \alpha_{i-1}$, called the *norm* of π. If X contains an element L such that for every $\epsilon > 0$ there exists $\delta > 0$ with

$$\left|\left| \sum_{i=1}^{n} x(\xi_i)|P \cap \Delta\alpha_i| - L \right|\right| < \epsilon$$

for every subdivision with $N\pi < \delta$, and every choice of ξ_i in $P \cap \Delta\alpha_i (i = 1, 2, \ldots, n)$ then L is the Graves integral, or G-integral, of $x(\alpha)$ over P and we write

$$(\mathrm{G}) \int_P x(\alpha)\, d\alpha = L.$$

It is not difficult to see that when P is a closed interval the G-integral reduces to the original Graves integral.

Because of the frequency and importance of its uses in the remainder of this paper we state the following rssult, which has been proved in a variety of ways by several writers including Birkhoff **(1)**, Jeffery **(7)**, and Macphail **(8)**.

THEOREM 3.1. *Let $e_1, e_2, \ldots, e_n$ be any n disjoint Lebesgue measurable sets on a measurable set E, $|E| < \infty$, ξ_i any point on e_i, and $S = \sum x(\xi_i)|e_i|$ where x is a bounded function on E with values in a space X. Let*

$$e_{i1}, e_{i2}, \ldots, e_{ik_i}$$

be a subdivision of e_i into disjoint measurable sets, and ξ_{ij}, ξ'_{ij} any points on e_{ij}. Then

$$\left\|\sum_{i=1}^{n}\sum_{j=1}^{k_i}\{x(\xi_{ij}) - x(\xi'_{ij})\}|e_{ij}|\right\| \leqslant \sup\left\|\sum_{i=1}^{n}\{x(\xi_i) - x(\xi'_i)\}|e_i|\right\|$$

and

$$\left\|\sum_{i=1}^{n}\sum_{j=1}^{k_i} x(\xi_{ij})|e_{ij}| - S\right\| \leqslant \sup\left\|\sum_{i=1}^{n}\{x(\xi_i) - x(\xi'_i)\}|e_i|\right\|.$$

One consequence of this result is the following theorem.

THEOREM 3.2. *Let $x(\alpha)$ be defined and bounded on P. A necessary and sufficient condition for the existence of the* G*-integral of $x(\alpha)$ over P is that there exist a sequence of subdivisions π_n of $[a, b]$ such that*

$$\lim_{n\to\infty}\sum_{\pi_n} x(\xi_{ni})|P\cap\Delta\alpha_{ni}|$$

exists, ξ_{ni} any point on $P\cap\Delta\alpha_{ni}$.

Proof. The necessity is obvious in view of definition 3.1. To prove the sufficiency choose any $\epsilon > 0$ and consider a sequence $\{\pi_n\}$ which yields a limit J. Then there exists an n_0 such that for $n > n_0$ we have $||\Sigma_{\pi_n} - J|| < \frac{1}{2}\epsilon$. If $n > n_0$ is fixed and ξ_{ni}, ξ'_{ni} are allowed to be any points in $P \cap \Delta\alpha_{ni}$ it follows that

$$\left\|\sum\{x(\xi_{ni}) - x(\xi'_{ni})\}|P\cap\Delta\alpha_{ni}|\right\| < \epsilon.$$

Then let π_k be any subdivision of $[a, b]$, not necessarily in the sequence, with $N\pi_k$ sufficiently small to insure that the total length of the intervals of π_k which contain points of subdivision of π_n is less than ϵ/M, where $M = \sup ||x(\alpha)||$, α on P. Suitable applications of Theorem 3.1 show that $||\Sigma_{\pi_k} - J||$ is less than a fixed multiple of ϵ.

In the next theorem we list several properties of the G-integral which we shall use in making our generalization. The proofs follow from the definition and Theorem 3.1 by standard arguments and are omitted.

THEOREM 3.3. (a) *If x, y, and f are* G*-integrable over P, and $||x(\alpha)|| \leqslant f(\alpha)$ for α on P, then*

(i) $$(G)\int_P (x + y)d\alpha = (G)\int_P xd\alpha + (G)\int_P yd\alpha,$$

(ii) $$\left\|(G)\int_P xd\alpha\right\| \leqslant (G)\int_P fd\alpha.$$

(b) *If $P \cap P' = 0$ and $x(\alpha)$ is* G*-integrable over the sets P and P' then it is integrable over $P \cup P'$ and*

$$(G)\int_{P\cup P'} xd\alpha = (G)\int_P xd\alpha + (G)\int_{P'} xd\alpha.$$

(c) *If P' is contained in P then $x(\alpha)$ is* G-*integrable over P' if it is* G-*integrable over P and*

$$\left|\left|(\mathrm{G})\int_P x d\alpha - (\mathrm{G})\int_{P'} x d\alpha\right|\right| \leqslant M|P - P'|,\ M = \sup_{\alpha \epsilon P}||x(\alpha)||.$$

(d) *If $x_n(\alpha)$ $(n = 1, 2, \ldots)$ is* G-*integrable over P and if $x_n(\alpha)$ converges uniformly to $x(\alpha)$ on P then $x(\alpha)$ is* G-*integrable over P and*

$$(\mathrm{G})\int_P x_n(\alpha)d\alpha \to (\mathrm{G})\int_P x(\alpha)d\alpha.$$

4. The generalized Graves integral. In this and the remaining sections E, E_i will denote bounded Lebesgue measurable sets of the real line.

Definition 4.1. A function $x(\alpha)$ defined on a set E with values in X is *P_ϵ-measurable* on E if for every $\epsilon > 0$ there exists a closed set P, contained in E, such that (i) $|E - P| < \epsilon$, and (ii) $x(\alpha)$ is G-integrable over P.

This is a generalization of the classical Lusin condition, to which it is equivalent when $X = R$. For arbitrary X we observe that if $x(\alpha)$ is continuous on P contained in E, with $|E - P| < \epsilon$, it is Graves integrable on P and therefore P_ϵ-measurable. Conversely, if a real-valued function $x(\alpha)$ is P_ϵ-measurable on E and hence G-integrable on $P' \supset E$, $|E - P'| < \frac{1}{2}\epsilon$, a standard argument shows that the measure of the set of its discontinuities on P' is zero. Then there exists a set P in E, on which $x(\alpha)$ is continuous with $|E - P| < \epsilon$.

On the other hand, the function cited in the introduction is P_ϵ-measurable without satisfying the Lusin condition.

Definition 4.2. If $x(\alpha)$ is P_ϵ-measurable on E and if there is an element I in X such that given $\eta > 0$ there exists $\epsilon > 0$ with

$$\left|\left|(\mathrm{G})\int_P x(\alpha)d\alpha - I\right|\right| < \eta$$

for every P contained in E, $|E - P| < \epsilon$, over which $x(\alpha)$ is G-integrable, then we say that I is the G*-*integral* of $x(\alpha)$ over E and denote it by (G^*) $\int_E x(\alpha)d\alpha$.

THEOREM 4.1. *If $x(\alpha)$ is P_ϵ-measurable on E a necessary and sufficient condition for*

$$(\mathrm{G}^*)\int_E x(\alpha)d\alpha$$

to exist is that for every $\eta > 0$ there exist $\delta > 0$ such that if P and P' are two closed sets in E, with measures greater than $|E| - \delta$, for which

$$(\mathrm{G})\int_P x d\alpha,\ (\mathrm{G})\int_{P'} x d\alpha$$

exist then

$$\left\| (\mathrm{G}) \int_P x d\alpha - (\mathrm{G}) \int_{P'} x d\alpha \right\| < \eta. \tag{4.1}$$

The proof is easily obtained by a standard argument and will be omitted.

THEOREM 4.2. *If $x(\alpha)$ is* G**-integrable over E it is* G**-integrable over every measurable subset e contained in E.*

Proof. Given $\epsilon > 0$, let P be a closed set with $|E - P| < \frac{1}{2}\epsilon$ and such that $x(\alpha)$ is G-integrable over P. Let e be any measurable subset of E. Then eP is measurable and so it contains a closed set P' with $|eP - P'| < \frac{1}{2}\epsilon$. Moreover

$$|e - eP| < |E - P| < \tfrac{1}{2}\epsilon.$$

Then $|e - P'| < \epsilon$. Since $x(\alpha)$ is G-integrable over P' by Theorem 3.3(c) our conclusion follows.

We next prove the analogue of the fact, basic in real variable theory, that every bounded Lebesgue measurable function on a set E is Lebesgue integrable.

THEOREM 4.3. *If $x(\alpha)$ is bounded and P_ϵ-measurable on E it is* G**-integrable over E.*

Proof. Let $M = \sup\|x(\alpha)\|$ for α in E. Let $\eta > 0$ be given and let P and P' be closed sets, contained in E, on which $x(\alpha)$ is G-integrable and such that the measure of each set is greater than $|E| - (\eta/2M)$. Then $x(\alpha)$ is G-integrable over $P \cap P'$ by Theorem 3.3(c). Moreover

$$|P' - (P \cap P')| < \frac{\eta}{2M}, \ |P - (P \cap P')| < \frac{\eta}{2M}.$$

Hence, by the second part of Theorem 3.3(c),

$$\begin{aligned} &\left\| (\mathrm{G}) \int_P x d\alpha - (\mathrm{G}) \int_{P'} x d\alpha \right\| \\ &\leqslant \left\| (\mathrm{G}) \int_P x d\alpha - (\mathrm{G}) \int_{P \cap P'} x d\alpha \right\| + \left\| (\mathrm{G}) \int_{P \cap P'} x d\alpha - (\mathrm{G}) \int_{P'} x d\alpha \right\| \\ &< M \cdot \frac{\eta}{2M} + M \cdot \frac{\eta}{2M} = \eta. \end{aligned}$$

THEOREM 4.4. *If $E_1 \cap E_2 = 0$, and if $x(\alpha)$ is* G**-integrable over E_1 and E_2 then it is* G**-integrable over $E_1 \cup E_2$ and*

$$(\mathrm{G}^*) \int_{E_1 \cup E_2} x d\alpha = (\mathrm{G}^*) \int_{E_1} x d\alpha + (\mathrm{G}^*) \int_{E_2} x d\alpha. \tag{4.2}$$

Proof. Let $\{P_n\}$, $\{P_n'\}$ be sequences of sets over each of which $x(\alpha)$ is G-integrable and with $P_n \subset E_1$, $P_n' \subset E_2$, for all n. Suppose that

$$|P_n| \to |E_1|, \ |P_n'| \to |E_2| \qquad n \to \infty.$$

Then $|P_n + P_n'| \to |E_1 + E_2|$. For each n, by Theorem 3.3(b),

$$(\mathrm{G})\int_{P_n} x d\alpha + (\mathrm{G})\int_{P_n'} x d\alpha = (\mathrm{G})\int_{P_n \cup P_n'} x d\alpha.$$

If $x(\alpha)$ is G*-integrable on $E_1 \cup E_2$ it follows that (4.2) holds.

It remains to be shown that $x(\alpha)$ is G*-integrable on $E_1 \cup E_2$. Let I denote the right side of (4.2). Given $\eta > 0$ there exists $\epsilon > 0$ such that if P and P' are closed sets contained in E_1, E_2 respectively, with $|E_1 - P| < \epsilon$, $|E_2 - P'| < \epsilon$, and if $x(\alpha)$ is G-integrable on P and P', we have

$$\left|\left|(\mathrm{G})\int_P x d\alpha - (\mathrm{G}^*)\int_{E_1} x d\alpha\right|\right| < \tfrac{1}{3}\eta, \quad \left|\left|(\mathrm{G})\int_{P'} x d\alpha - (\mathrm{G}^*)\int_{E_2} x d\alpha\right|\right| < \tfrac{1}{3}\eta.$$

Now, suppose F is any closed set contained in $E_1 \cup E_2$, with $|(E_1 \cup E_2) - F| < \frac{1}{2}\epsilon$, and on which $x(\alpha)$ is G-integrable. It follows that

$$|E_1 - (F \cap E_1)| = |E_1 - F| < |(E_1 \cup E_2) - F| < \tfrac{1}{2}\epsilon.$$

Similarly $|E_2 - (F \cap E_2)| < \frac{1}{2}\epsilon$. Then let P, P' contained in $F \cap E_1$, $F \cap E_2$ respectively, be such that

$$|E_1 - P| < \epsilon, \; |E_2 - P'| < \epsilon, \; |F - (P \cup P')| < \frac{\eta}{3M},$$

where $M = \sup||x(\alpha)||$ on F. $x(\alpha)$ is G-integrable on the sets P and P' by Theorem 3.3(c). Hence

$$\begin{aligned}\left|\left|(\mathrm{G})\int_F x d\alpha - I\right|\right| & \\ \leqslant \left|\left|(\mathrm{G})\int_F x d\alpha - (\mathrm{G})\int_{P\cup P'} x d\alpha\right|\right| &+ \left|\left|(\mathrm{G})\int_P x d\alpha - (\mathrm{G}^*)\int_{E_1} x d\alpha\right|\right| \\ &+ \left|\left|(\mathrm{G})\int_{P'} x d\alpha - (\mathrm{G}^*)\int_{E_2} x d\alpha\right|\right| \\ < \eta. &\end{aligned}$$

Definition 4.3. If a set function ν is defined on the class of Lebesgue measurable subsets of E with values in X, and if for every $\epsilon > 0$ there exists $\delta > 0$ such that $||\nu(e)|| < \epsilon$ for every subset e of E with $|e| < \delta$, then we say that ν is *absolutely continuous* over the measurable subsets of E.

THEOREM 4.5. *Suppose $x(\alpha)$ is* G*-*integrable over a set* E. *Then the* G*-*integral is an absolutely continuous function of the measurable sets e contained in* E.

Proof. Let $\eta > 0$ be given. Fix $\delta > 0$ so that condition (4.1) of Theorem 4.1 holds. Now consider any set e contained in E with $|e| < \delta$. Let P be any closed set contained in e such that

$$\left|\left|(\mathrm{G}^*)\int_e x d\alpha - (\mathrm{G})\int_P x d\alpha\right|\right| < \eta.$$

Also let F be a closed set contained in E-e, with $|E - F| < \epsilon$, and on which $x(\alpha)$ is G-integrable. Then

$$\left\|(G)\int_P xd\alpha\right\| = \left\|(G)\int_{P\cup F} xd\alpha - (G)\int_F xd\alpha\right\| < \eta$$

and

$$\left\|(G^*)\int_e xd\alpha\right\| \leqslant \left\|(G^*)\int_e xd\alpha - (G)\int_P xd\alpha\right\| + \left\|(G)\int_P xd\alpha\right\| < 2\eta.$$

Since η is arbitrary our conclusion follows.

THEOREM 4.6. *If $x(\alpha)$ is G*-integrable over E, and e represents a measurable subset of E, then*

$$(G^*)\int_e x(\alpha)\, d\alpha$$

is a completely additive set function over E.

Proof. Let $e_1, e_2, \ldots$ be a sequence of disjoint measurable sets on E and let $e_1 + \ldots + e_n = E_n$ and

$$\sum_{i=1}^{\infty} e_i = e.$$

Then $x(\alpha)$ is G-integrable over e and E_n by Theorem 4.2, and by Theorem 4.4 we have

$$\sum_{i=1}^{n} (G^*)\int_{e_i} xd\alpha = (G^*)\int_{E_n} xd\alpha$$

and

$$(G^*)\int_{E_n} xd\alpha + (G^*)\int_{e-E_n} xd\alpha = (G^*)\int_e xd\alpha.$$

Now $|e - E_n| \to 0$ as $n \to \infty$. Hence, by Theorem 4.5,

$$\sum_{i=1}^{\infty} (G)\int_{e_i} xd\alpha = \lim_{n\to\infty} (G^*)\int_{E_n} xd\alpha = (G^*)\int_e xd\alpha.$$

Although we have restricted ourselves to sets on the real line for the sake of simplicity in writing, it is clear that the above definitions and theorems can be extended to a function with its values in X and defined on any bounded Lebesgue measurable set in an n-dimensional Euclidean space. In addition, the usual procedure of taking limits would lead to a definition of the G*-integral in cases where E is not bounded and $|E|$ is not finite.

5. Sequences of P_ϵ-measurable functions. We now consider two important properties of sequences of P_ϵ-measurable functions which generalize corresponding results in the Lebesgue theory.

Definition 5.1. A sequence of functions $\{x_n(\alpha)\}$ defined on E converges to $x(\alpha)$ in E *almost uniformly* if, given $\epsilon > 0$, there exists a set E' contained in E such that $|E - E'| < \epsilon$ and $x_n(\alpha)$ converges uniformly to $x(\alpha)$ in E'.

LEMMA 5.1. *Let $\{x_n(\alpha)\}$ be a sequence of P_ϵ-measurable functions defined on E with values in X. If $x_n(\alpha)$ converges to $x(\alpha)$ in E almost uniformly then $x(\alpha)$ is P_ϵ-measurable on E.*

Proof. Suppose $\epsilon > 0$ is given. Then there exists a set E', contained in E, on which $x_n(\alpha)$ converges to $x(\alpha)$ uniformly and such that $|E - E'| < \epsilon/2$. Now for each n there exists a closed set P_n contained in E with $|E - P_n| < \epsilon/2^{n+2}$ and such that $x_n(\alpha)$ is G-integrable over P_n. Furthermore, there exists a closed set P_n' in the measurable set E' such that $|E' - P_n'| < \epsilon/2^{n+2}$. Hence we may set $F_n = P_n \cap P_n'$ for each n and $x_n(\alpha)$ will be G-integrable on F_n, by Theorem 3.3(c), with $|E' - F_n| < \epsilon/2^{n+1}$. Then the intersection of the sequence of sets $\{F_n\}$ is a closed set F such that $|E - F| < \epsilon$ and each $x_n(\alpha)$ is G-integrable on F. Hence (G) $\int_F x(\alpha)\, d\alpha$ exists by Theorem 3.3(d). Since ϵ is arbitrary it follows that $x(\alpha)$ is P_ϵ-measurable on E.

LEMMA 5.2. *If $x(\alpha)$ is G*-integrable over a set E and if $f(\alpha)$ is a real-valued summable function over E, with $||x(\alpha)|| \leqslant f(\alpha)$ for every α in E, then*

$$\left|\left|(\mathrm{G}^*)\int_E x(\alpha)d\alpha\right|\right| \leqslant (\mathrm{G}^*)\int_E f(\alpha)d\alpha.$$

Proof. This follows immediately from the second part of Theorem 3.3(a) and the definition of the G*-integral.

THEOREM 5.1. *Let $\{x_n(\alpha)\}$ be a sequence of P_ϵ-measurable functions defined on E with values in X. Suppose $||x_n(\alpha)|| \leqslant f(\alpha)$ for all values of n, and all α in E, where $f(\alpha)$ is a real-valued summable function over E. Suppose also that $x_n(\alpha)$ converges to $x(\alpha)$ in E almost uniformly. Then $x_n(\alpha)$ is* G**-integrable over E for each n, $x(\alpha)$ is* G**-integrable over E, and*

$$\lim_{n\to\infty} (\mathrm{G}^*)\int_E x_n(\alpha)d\alpha = (\mathrm{G}^*)\int_E x(\alpha)d\alpha. \tag{5.1}$$

Proof. $x(\alpha)$ is P_ϵ-measurable on E by Lemma 5.1. Furthermore, given $\eta > 0$ there exists $\delta > 0$ such that for e contained in E and $|e| < \delta$,

$$(\mathrm{G}^*)\int_e f(\alpha)d\alpha < \tfrac{1}{4}\eta.$$

Let P and P' be any two closed sets in E, on each of which $x(\alpha)$ is G-integrable, and such that $|E - P| < \delta$, $|E - P'| < \delta$. Then

$$\begin{aligned}
&\left|\left|(\mathrm{G})\int_P x d\alpha - (\mathrm{G})\int_{P'} x d\alpha\right|\right| \\
&= \left|\left|(\mathrm{G}^*)\int_P x d\alpha - (\mathrm{G}^*)\int_{P'} x d\alpha\right|\right| \\
&= \left|\left|(\mathrm{G}^*)\int_{P-P'} x d\alpha - (\mathrm{G}^*)\int_{P'-P} x d\alpha\right|\right| \\
&\leqslant \left|\left|(\mathrm{G}^*)\int_{P-P'} x d\alpha\right|\right| + \left|\left|(\mathrm{G}^*)\int_{P'-P} x d\alpha\right|\right| < \eta.
\end{aligned}$$

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Hence $x(\alpha)$ is G*-integrable over E by Theorem 4.1, and the same argument shows that for each n, $x_n(\alpha)$ is G*-integrable over E.

Finally, let $x(\alpha) = x_n(\alpha) - y_n(\alpha)$. By our hypothesis there exists a set E' contained in E, such that $|E - E'| < \delta$, and a fixed positive integer N such that for $n > N$ we have

$$||x(\alpha) - x_n(\alpha)|| < \frac{\eta}{2|E|}$$

on E'. Then

$$\begin{aligned} &\left|\left|(\mathrm{G}^*)\int_E \{x(\alpha) - x_n(\alpha)\}d\alpha\right|\right| \\ &= \left|\left|(\mathrm{G}^*)\int_{E'} y_n(\alpha)d\alpha + (\mathrm{G}^*)\int_{E-E'} y_n(\alpha)d\alpha\right|\right| \\ &\leqslant \left|\left|(\mathrm{G}^*)\int_{E'} y_n(\alpha)d\alpha\right|\right| + \left|\left|(\mathrm{G}^*)\int_{E-E'} y_n(\alpha)d\alpha\right|\right| \\ &\leqslant \left|\left|(\mathrm{G}^*)\int_{E'} y_n(\alpha)d\alpha\right|\right| + (\mathrm{G}^*)\int_{E-E'} 2f(\alpha)d\alpha \\ &< \frac{\eta}{2|E|}\cdot |E'| + \tfrac{1}{2}\eta \leqslant \eta. \end{aligned}$$

This completes the proof.

COROLLARY. *If* $\{x_n(\alpha)\}$ *is uniformly bounded on* E, *and* $x_n(\alpha)$ *converges to* $x(\alpha)$ *in* E *almost uniformly, then* (5.1) *holds.*

For a sequence of real-valued Lebesgue measurable functions on a bounded set E, convergence almost everywhere is equivalent to convergence almost uniformly. It is clear then that Theorem 5.1 is a generalization of the well-known dominated convergence theorem of Lebesgue. However, because of the failure of a theorem of the Egoroff type to hold, in general, for P_ϵ-measurable functions the traditional hypothesis of the Lebesgue theorem cannot be retained, and we must require specifically that the functions converge almost uniformly.

6. The equivalence of the G*-integral and the Birkhoff integral. It is easy to show directly that the G*-integral includes the Bochner integral. However, the Birkhoff integral also includes that of Bochner and is more general than the latter. For this reason we shall compare the G*-integral with that of Birkhoff.

Macphail **(8)** points out that if $x(\alpha)$ is bounded on E the infinite partitions of Birkhoff may be replaced by finite partitions. If $\mathfrak{P}$ is a finite partition of E into measurable subsets e_i, and if we define

$$S(\mathfrak{P}) = \textstyle\sum_i x(\xi_i)|e_i|, \; D(\mathfrak{P}) = \sum_i [x(\xi_i') - x(\xi_i'')]|e_i|,$$

and $\omega(\mathfrak{P}) = \sup||D(\mathfrak{P})||$, where ξ_i, ξ_i', ξ_i'' are arbitrary points in e_i the diameter of the integral range which appears in Birkhoff's definition **(1**, p. 367**)** is precisely $\omega(\mathfrak{P})$. Then a bounded function $x(\alpha)$ is Birkhoff integrable if and only if there exists **(1**, Theorem 13**)** a sequence of partitions $\{\mathfrak{P}_n\}$ such that $\omega(\mathfrak{P}_n) \to 0$. The following lemmas are consequences of the above definitions, with the aid of Theorem 3.1.

LEMMA 6.1. *If $\mathfrak{P}$ is any finite partition of E on which a bounded function $x(\alpha)$ is (Bk)-integrable then*

$$\left|\left|(Bk)\int_E xd\alpha - S(\mathfrak{P})\right|\right| \leqslant \omega(\mathfrak{P}).$$

LEMMA 6.2. *If Q is a set consisting of a selection $\mathfrak{Q}$ of the subsets comprising $\mathfrak{P}$ then*

$$\left|\left|(Bk)\int_Q xd\alpha - S(\mathfrak{Q})\right|\right| \leqslant \omega(\mathfrak{Q}) \leqslant \omega(\mathfrak{P}).$$

In **(1)** the function $x(\alpha)$ is considered to be defined on an abstract domain on which a measure is defined. However, as in the previous sections, we restrict the present discussion to a function defined on a bounded Lebesgue measurable linear set, observing that the same proofs hold for n-dimensional sets. The Birkhoff integral of $x(\alpha)$ over E will be denoted by $(Bk)\int_E x(\alpha)d\alpha$.

THEOREM 6.1. *If $x(\alpha)$ is Birkhoff integrable over a set E then it is also* G*-*integrable over E to the same value.*

Proof. Suppose first that $x(\alpha)$ is bounded on E, and let $M > 0$ be such that $||x(\alpha)|| < M$ for all α in E. Let $\{\mathfrak{P}_n\}$ be a sequence of finite partitions of the set E which yield a Birkhoff integral. That is, for each n, $E = e_{n1} + e_{n2} + \ldots + e_{nk}$. Then, given $\epsilon > 0$ we can choose closed sets e^c_{ni} contained in the e_{ni} such that the measure of $E^c_n = e^c_{n1} + e^c_{n2} + \ldots + e^c_{nk}$ differs from the measure of E by less than $\epsilon/2^n$.

Let $F = E^c_1 \cap E^c_2 \cap \ldots$. This is a closed set and its measure differs from that of E by less than ϵ. We show that

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} x(\xi_i)|F \cap e^c_{ni}| = (Bk)\int_F x(\alpha)d\alpha.$$

First of all **(1**, Theorem 14**)**, $x(\alpha)$ is (Bk)-integrable on F. Also, the sets $F \cap e^c_{ni}$ form a partition of F which we may denote by $\mathfrak{P}_{n(F)}$. Clearly $\mathfrak{P}_{n(F)}$ consists of a selection of sets from $\mathfrak{P}_n'$, the partition of E formed by the sets

$$F \cap e^c_{ni},\ e^c_{ni} - F, \text{ and } e_{ni} - e^c_{ni}, \qquad i = 1, 2, \ldots, k_n.$$

Moreover, $\mathfrak{P}_n'$ is a refinement of the partition $\mathfrak{P}_n$ of E. Then, by Lemma 6.2,

$$\begin{aligned}\left|\left|(Bk)\int_F xd\alpha - \sum_{i=1}^{k_n} x(\xi_i)|F \cap e^c_{ni}|\right|\right| &= \left|\left|(Bk)\int_F xd\alpha - S(\mathfrak{P}_{n(F)})\right|\right| \\ &\leqslant \omega(\mathfrak{P}_{n(F)}) \leqslant \omega(\mathfrak{P}_n'),\end{aligned}$$

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and by Theorem 3.1, $\omega(\mathfrak{P}_n') \leqslant \omega(\mathfrak{P}_n)$, where $\omega(\mathfrak{P}_n)$ is associated with the original sequence $\{\mathfrak{P}_n\}$ and approaches zero as $n \to \infty$. This leads to the desired conclusion. Then, given $\eta > 0$ we can find and fix an m, depending on η, such that

$$\left|\left| \sum_{i=1}^{k_m} x(\xi_i)|F \cap e^c_{mi}| - (Bk)\int_F x d\alpha \right|\right| \leqslant \omega(\mathfrak{P}_m) < \eta.$$

Now the closed sets e^c_{mi} are disjoint and finite in number. Let d be the minimum of the distances between any two of these closed sets. Then, taking an interval $[a, b]$ containing E divide it into subintervals of length less than d, that is,

$$a = \alpha_0 < \alpha_1 < \ldots < \alpha_n = b.$$

Consider sets of the form $F \cap \Delta\alpha_i$ where the $\Delta\alpha_i$ are as described in definition 3.1. We see that each set $F \cap \Delta\alpha_i (i = 1, 2, \ldots, n)$ is equivalent to a set $F \cap e^c_{mj} \cap \Delta\alpha_i$ for some j, and the collection of such sets forms a refinement of the partition $\mathfrak{P}_{m(F)}$. Then it follows that

$$\begin{aligned} &\left|\left| \sum_{i=1}^{n} x(\xi_i)|F \cap \Delta\alpha_i| - (Bk)\int_F x(\alpha) d\alpha \right|\right| \\ &\leqslant \left|\left| \sum_{i=1}^{n} x(\xi_i)|F \cap \Delta\alpha_i| - \sum_{i=1}^{k_m} x(\xi_i)|F \cap e^c_{mi}| \right|\right| \\ &\qquad + \left|\left| \sum_{i=1}^{k_m} x(\xi_i)|F \cap e^c_{mi}| - (Bk)\int_F x(\alpha) d\alpha \right|\right| \\ &\leqslant 2\omega(\mathfrak{P}_m) < 2\eta. \end{aligned}$$

Thus $x(\alpha)$ is G-integrable on the closed set F where $|E - F| < \epsilon$. Hence $x(\alpha)$ is P_ϵ-measurable over E and being bounded it is G*-integrable over E by Theorem 4.3. Further, the G-integral equals the Birkhoff integral on F. Hence, as $|E - F| \to 0$, we have (G) $\int_F x d\alpha$ approaching the limit (Bk) $\int_E x d\alpha$ since the Birkhoff integral is absolutely continuous and a completely additive set function on E. Therefore,

$$(\mathrm{G}^*)\int_E x d\alpha = (Bk)\int_E x d\alpha.$$

If $x(\alpha)$ is unbounded on E but is Birkhoff integrable there it is also Birkhoff integrable on every measurable set e contained **(1)** in E and it follows that as $|e| \to |E|$,

$$(Bk)\int_e x d\alpha \to (Bk)\int_E x d\alpha.$$

Now, given $\epsilon > 0$, if E' is a measurable subset of E over which $x(\alpha)$ is bounded and such that $|E - E'| < \frac{1}{2}\epsilon$ then $(Bk)\int_{E'} x d\alpha$ exists and hence $(G^*)\int_{E'} x d\alpha$ exists. Then there is a closed set P contained in E' with $|E' - P| < \frac{1}{2}\epsilon$ and such that (G) $\int_P x d\alpha$ exists. Hence $x(\alpha)$ is P_ϵ-measurable on E. Then, given

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$\eta > 0$ there exists $\delta > 0$ such that for P, P' contained in E, with $|E - P| < \delta$, $|E - P'| < \delta$, and on each of which $x(\alpha)$ is G-integrable,

$$\left\| (\mathrm{G}) \int_P x d\alpha - (\mathrm{G}) \int_{P'} x d\alpha \right\|$$
$$= \left\| (Bk) \int_P x d\alpha - (Bk) \int_{P'} x d\alpha \right\|$$
$$\leqslant \left\| (Bk) \int_{P-P'} x d\alpha \right\| + \left\| (Bk) \int_{P'-P} x d\alpha \right\| < \eta.$$

Thus $x(\alpha)$ is G*-integrable on E.

Finally, since $x(\alpha)$ is P_ϵ-measurable on E there is a sequence of closed sets $\{P_n\}$ with $|P_n| \to |E|$ and on each of which $x(\alpha)$ is G-integrable. Hence

$$(Bk) \int_E x d\alpha = \lim_{|P_n| \to |E|} (Bk) \int_{P_n} x d\alpha = \lim_{|P_n| \to |E|} (\mathrm{G}) \int_{P_n} x d\alpha = (\mathrm{G}^*) \int_E x d\alpha.$$

Corollary. *If $x(\alpha)$ is Bochner integrable over E then it is also* G**-integrable over E to the same value.*

This follows at once from our theorem and the proof that Birkhoff's integral includes that of Bochner (**1**, p. 377).

The fact that the everywhere discontinuous function given in the introduction is G*-integrable, being P_ϵ-measurable and bounded, but is not Bochner (strongly) measurable, shows that the converse does not hold.

Theorem 6.2. *If $x(\alpha)$ is* G**-integrable over E then it is also Birkhoff integrable over E to the same value.*

Proof. First suppose that $x(\alpha)$ is bounded on E, $||x(\alpha)|| < M$ for all α in E. Given $\epsilon > 0$ there exists a closed set P, with

$$|E - P| < \frac{\epsilon}{4M},$$

on which $x(\alpha)$ is G-integrable. By a suitable subdivision π we can partition this set into sets $e_i = P \cap \Delta\alpha_i$ and have

$$\left\| \sum_\pi [x(\xi_i') - x(\xi_i'')]|e_i| \right\| < \frac{\epsilon}{2},$$

for all ξ_i', ξ_i'' in e_i. Now by taking a set J, the complement of P in E, plus the sets e_i, we have a partition $\mathfrak{P}$ of the whole set E and can form the sum

$$D(\mathfrak{P}) = \sum_\pi [x(\xi_i') - x(\xi_i'')]|e_i| + \{[x(\xi_1) - x(\xi_2)]|J|, \xi_1, \xi_2 \text{ in } J\}.$$

Then

$$\sup \left\| D(\mathfrak{P}) \right\| \leqslant \frac{\epsilon}{2} + \frac{\epsilon}{4M} \cdot 2M = \epsilon.$$

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By starting with a sequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0$ as $n \to \infty$ we can construct a sequence of partitions $\mathfrak{P}_n$ such that

$$\omega(\mathfrak{P}_n) = \sup ||D(\mathfrak{P}_n)|| \to 0.$$

Thus the Birkhoff integral over E exists and is clearly equal to (G*) $\int_E x d\alpha$.

Next, suppose that $x(\alpha)$ is unbounded on E. Then, given $\epsilon > 0$, we have to find a partition under which $\sum x(\xi_i)|e_i|$ is unconditionally summable and the diameter of the integral range is less than ϵ.

Let $E_1, E_2, \ldots$ be non-overlapping sets in E such that $\sum|E_j| = |E|$ and $x(\alpha)$ is bounded on each E_j. For each E_j let $\mathfrak{E}_j$ be a partition into sets e_{ji} such that $\omega(\mathfrak{E}_j) < \epsilon_j$, $\sum \epsilon_j = \frac{1}{2}\epsilon$. Let $\delta > 0$ be such that for a measurable set e with $|e| < \delta$ we have

$$\left|\left|(\mathrm{G}^*)\int_e x(\alpha)d\alpha\right|\right| < \tfrac{1}{2}\epsilon.$$

Choose N such that $|E_N| + |E_{N+1}| + \ldots$ is less than δ. Next, choose any finite set of the e_{ji}, with $j > N$, and denote it by $\mathfrak{E}$: $(e_1, e_2, \ldots, e_k)$. Let

$$e = e_1 + e_2 + \ldots + e_k.$$

Then

$$(\mathrm{G}^*)\int_e x(\alpha)d\alpha$$

exists and hence

$$(Bk)\int_e x(\alpha)d\alpha$$

exists by the first part of the proof, and by Lemma 6.1 we have

$$\left|\left|(Bk)\int_e xd\alpha - S(\mathfrak{E})\right|\right| \leqslant \omega(\mathfrak{E}) \leqslant \sum \omega(\mathfrak{E}_j) < \sum \epsilon_j = \tfrac{1}{2}\epsilon.$$

Also, since e is contained in $E_N + E_{N+1} + \ldots$ we have

$$\left|\left|(\mathrm{G}^*)\int_e x(\alpha)d\alpha\right|\right| = \left|\left|(Bk)\int_e x(\alpha)d\alpha\right|\right| < \tfrac{1}{2}\epsilon.$$

Then

$$\begin{aligned}||S(\mathfrak{E})|| &= \left|\left|(Bk)\int_e xd\alpha - (Bk)\int_e xd\alpha + S(\mathfrak{E})\right|\right| \\ &\leqslant \left|\left|(Bk)\int_e xd\alpha\right|\right| + \left|\left|(Bk)\int_e xd\alpha - S(\mathfrak{E})\right|\right| \\ &< \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon.\end{aligned}$$

Hence $\sum_{ij} x(\xi_{ji})|e_{ji}|$ is unconditionally summable. Finally, the diameter of the integral range corresponding to the set of partitions $\mathfrak{E}_j$, which we shall denote by $\mathscr{D}\{\sum_{ij} x(\xi_{ji})\ |e_{ji}|\}$, satisfies the condition

$$\mathscr{D}\{\textstyle\sum_{ij} x(\xi_{ji})\ |e_{ji}|\} \leqslant \Sigma\ \omega\ (\mathfrak{E}_j) < \Sigma\ \epsilon_j = \tfrac{1}{2}\epsilon.$$

Since these are the conditions for Birkhoff integrability (**1**, Theorem 13) the proof of the theorem is complete.

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