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Vector-Valued Modular Forms of Weight Two Associated With Jacobi-Like Forms

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Abstract. We construct vector-valued modular forms of weight 2 associated to Jacobi-like forms with respect to a symmetric tensor representation of Γ by using the method of Kuga and Shimura as well as the correspondence between Jacobi-like forms and sequences of modular forms. As an application, we obtain vector-valued modular forms determined by theta functions and by pseudodifferential operators.

1 Introduction

Jacobi-like forms are formal power series, whose coefficients are holomorphic functions defined on the Poincaré upper half plane, satisfying a certain functional equation with respect to an action of a discrete subgroup Γ of $SL(2,\mathbb{R})$. This functional equation is essentially one of the two equations that must be satisfied by Jacobi forms for Γ introduced systematically by Eichler and Zagier [4]. The same functional equation induces certain relations among the coefficients of a Jacobi-like form, which can be used to express each coefficient of a Jacobi-like form for Γ as a linear combination of derivatives of some modular forms for Γ. Thus each Jacobi-like form determines a certain family of modular forms. In fact, such modular forms can in turn be written in terms of derivatives of coefficients of the associated Jacobi-form, which allows us to establish a one-to-one correspondence between Jacobi-like forms and certain sequences of modular forms (see [1, 8]).

Vector-valued modular forms for a discrete subgroup Γ of $SL(2,\mathbb{R})$ generalize usual modular forms for Γ and are defined by using a representation of Γ in a complex vector space. Such modular forms play an important role in number theory. For example, vector-valued modular forms associated to a symmetric tensor representation can be used to establish the Eichler–Shimura correspondence between modular forms and cohomology of Γ, *cf.* [3, 7]. If ρ_m is a symmetric tensor representation of Γ of degree m, then certain types of vector-valued modular forms of weight 2 with respect to ρ_m correspond to usual modular forms for Γ of weight $m+2$. It was Kuga and Shimura [5] who constructed such vector-valued modular forms of weight 2 by using derivatives of a modular form.

In this paper we construct vector-valued modular forms of weight 2 associated to Jacobi-like forms with respect to a symmetric tensor representation of Γ by using the method of Kuga and Shimura as well as the correspondence between Jacobi-like

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forms and sequences of modular forms. As an application we obtain vector-valued modular forms determined by theta functions and by pseudodifferential operators.

2 Vector-Valued Modular Forms

In this section we review the method of Kuga and Shimura [5] of constructing vector-valued modular forms of weight two by using derivatives of a usual scalar-valued modular form.

Let $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ be the Poincaré upper half plane on which the group $SL(2,\mathbb{R})$ acts as usual by linear fractional transformations. Thus, if $z \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$, we have

$$\gamma z = \frac{az + b}{cz + d}.$$

For such z and γ, we set

$$J(\gamma, z) = cz + d, \tag{2.1}$$

so that the resulting map $J\colon\ SL(2,\mathbb{R}) \times \mathcal{H} \to \mathbb{C}$ is an automorphy factor satisfying the cocycle condition

$$J(\gamma\gamma', z) = J(\gamma, \gamma' z) J(\gamma', z)$$

for all $z \in \mathcal{H}$ and $\gamma, \gamma' \in SL(2,\mathbb{R})$.

Let Γ be a discrete subgroup of $SL(2,\mathbb{R})$, and let $\rho\colon \Gamma \to GL(\ell,\mathbb{C})$ be a representation of Γ in the complex vector space $\mathbb{C}^\ell$ for some positive integer ℓ. Given a nonnegative integer k, we now modify the usual definition of modular forms and vector-valued modular forms of weight k by suppressing the usual finiteness condition at the cusps.

Definition 2.1

(i) A *modular form of weight k for Γ* is a holomorphic function $f\colon \mathcal{H} \to \mathbb{C}$ satisfying

$$f(\gamma z) = J(\gamma, z)^k f(z)$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$.

(ii) A *vector-valued modular form of weight k for Γ with respect to ρ* is a holomorphic function $\Psi\colon \mathcal{H} \to \mathbb{C}^\ell$ satisfying

$$\Psi(\gamma z) = J(\gamma, z)^k \rho(\gamma)\Psi(z)$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$.

We shall denote by $M_k(\Gamma)$ and $\mathbf{M}_k(\Gamma, \rho)$ the space of modular forms of weight k for Γ and the space of vector-valued modular forms of weight k for Γ with respect to ρ, respectively. Throughout this paper, we shall also use $(\,\cdot\,)^T$ to denote the transpose of the matrix $(\,\cdot\,)$. In particular, if $x_1, \dots, x_\ell \in \mathbb{C}$, the corresponding column vector belonging to $\mathbb{C}^\ell$ will be denoted by $(x_1, \dots, x_\ell)^T$.

If m is a positive integer, we denote by $\rho_m\colon SL(2,\mathbb{R}) \to GL(m+1,\mathbb{C})$ the m-th symmetric tensor power of the standard representation of $SL(2,\mathbb{R})$ in $\mathbb{C}^2$. Thus, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$, then we have

$$\begin{aligned}\rho_m(\gamma)&(u^m, u^{m-1}v, \dots, uv^{m-1}, v^m)^T \\ &= ((au+bv)^m, (au+bv)^{m-1}(cu+dv), \dots, (au+bv)(cu+dv)^{m-1}, (cu+dv)^m)^T\end{aligned}$$

for all $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in \mathbb{C}^2$. By restricting ρ_m to Γ we obtain a representation of Γ in $\mathbb{C}^{m+1}$, which we also denote by ρ_m.

Definition 2.2 We define the matrix-valued function $\widehat{\rho}_m\colon \mathcal{H} \to GL(m+1,\mathbb{C})$ on $\mathcal{H}$ associated to ρ_m by

$$\widehat{\rho}_m(z) = \rho_m\left(\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right)\right)$$

for all $z \in \mathcal{H}$.

Let $\alpha, \beta \in \mathbb{Z}$ be even with $\alpha > 0$ and

$$-(\alpha-2) \le \beta \le \alpha + 2, \tag{2.2}$$

and set

$$\delta = \frac{\alpha + 2 - \beta}{2}.$$

For each nonnegative integer $k \le \delta$ we denote by $\eta_{k,\alpha,\beta}$ the rational number defined by

$$\eta_{k,\alpha,\beta} = \begin{cases} 0 & \text{if } k < 1-\beta, \\ \frac{(k+\alpha-\delta)!}{k!(\beta+k-1)!} & \text{if } k \ge 1-\beta. \end{cases} \tag{2.3}$$

Given a holomorphic function $f\colon \mathcal{H} \to \mathbb{C}$, we use its derivatives as well as the numbers $\eta_{k,\alpha,\beta}$ to define the finite sequence $\{\phi_{\ell,\alpha,\beta}\}_{\ell=0}^{\alpha}$ of functions on $\mathcal{H}$ by

$$\phi_{\ell,\alpha,\beta}(z) = \begin{cases} 0 & \text{if } \ell < \alpha - \delta, \\ \eta_{\ell-\alpha+\delta,\alpha,\beta} f^{(\ell-\alpha+\delta)}(z) & \text{if } \ell \ge \alpha - \delta \end{cases}$$

for $z \in \mathcal{H}$ and $0 \le \ell \le \alpha$.

Definition 2.3 We define the vector-valued function $\Phi_f\colon \mathcal{H} \to \mathbb{C}^{\alpha+1}$ associated to $f\colon \mathcal{H} \to \mathbb{C}$ by

$$\Phi_f(z) = \widehat{\rho}_\alpha(z)(\phi_{0,\alpha,\beta}(z), \phi_{1,\alpha,\beta}(z), \dots, \phi_{\alpha,\alpha,\beta}(z))^T \tag{2.4}$$

for all $z \in \mathcal{H}$.

Theorem 2.4 *If $f \in M_\beta(\Gamma)$, then the associated $\mathbb{C}^{\alpha+1}$-valued function Φ_f given by* (2.4) *is a vector-valued modular form belonging to $\mathbf{M}_2(\Gamma, \rho_\alpha)$.*

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Proof This follows from [5, Theorem 3]. ∎

Remark 2.5 Let $\Psi\colon \mathcal{H} \to \mathbb{C}^{\alpha+1}$ be a vector-valued holomorphic function which can be written in the form

$$\Psi(z) = f(z)(z^{2n}, z^{2n-1}, \dots, z, 1)^T$$

for all $z \in \mathcal{H}$, where f is a holomorphic function on $\mathcal{H}$. Then it can be easily shown that Ψ is a vector-valued modular form belonging to $\mathbf{M}_2(\Gamma, \rho_\alpha)$ if and only if f is a modular form belonging to $M_{\alpha+2}(\Gamma)$.

3 Jacobi-Like Forms

In this section we construct vector-valued modular forms of weight 2 associated with Jacobi-like forms. As an application we also obtain vector-valued modular forms that can be expressed in terms of derivatives of powers of a scalar-valued modular form.

Let R be the ring of holomorphic functions on the Poincaré upper half plane $\mathcal{H}$, and denote by $R[[X]]$ the space of formal power series in X with coefficients in R. Let Γ be a discrete subgroup of $SL(2,\mathbb{R})$ as in Section 2. We now define Jacobi-like forms without the usual holomorphy conditions at the cusps.

Definition 3.1 Given a nonnegative integer ℓ, a formal power series $\Phi(z, X) \in R[[X]]$ is a *Jacobi-like form of weight ℓ for Γ* if

$$\Phi(\gamma z, J(\gamma, z)^{-2}X) = J(\gamma, z)^{\ell} \exp(J(\gamma, z)^{-1}\gamma_{2,1}X)\Phi(z, X) \tag{3.1}$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$, where $J(\gamma, z)$ is as in (2.1) and $\gamma_{2,1}$ denotes the $(2, 1)$-entry of the matrix γ. We shall denote by $\mathcal{J}_\ell(\Gamma)$ the space of Jacobi-like forms of weight ℓ for Γ.

Proposition 3.2 *Let $\Psi(z, X) = \sum_{k=1}^{\infty} \psi_k(z)X^k$ be a formal power series in $R[[X]]$.*

(i) *The power series $\Psi(z, X)$ is a Jacobi-like form belonging to $\mathcal{J}_\ell(\Gamma)$ if and only if*

$$\psi_k = \sum_{r=0}^{k-1} \frac{1}{r!(2k + \ell - r - 1)!} h_{k-r}^{(r)} \tag{3.2}$$

for all $k \geq 1$, where $h_j \in M_{2j+\ell}(\Gamma)$ for each $j \geq 1$.

(ii) *The system of relations (3.2) is equivalent to the condition*

$$h_j = (2j + \ell - 1) \sum_{r=0}^{j-1} (-1)^r \frac{(2j + \ell - r - 2)!}{r!} \psi_{k-r}^{(r)} \tag{3.3}$$

for all $j \geq 1$.

Proof The proposition follows from the results in [8, p. 62]. The case of weight zero is contained in [1, Proposition 2]. ∎

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Theorem 3.3 *Let n and σ be positive integers with $n \leq \sigma \leq 2n$, and let $\Phi(z, X) = \sum_{k=1}^{\infty} \phi_k(z) X^k$ be a Jacobi-like form belonging to $\mathcal{J}_{2w}(\Gamma)$ for some positive integer w. Let $\Lambda_\Phi \colon \mathcal{H} \to \mathbb{C}^{2n+1}$ be the vector-valued function defined by*

$$\Lambda_\Phi(z) = \widehat{\rho}_{2n}(z)(\lambda_0(z), \lambda_1(z), \dots, \lambda_{2n}(z))^T$$

for all $z \in \mathcal{H}$, where

$$\lambda_\ell(z) = \begin{cases} 0 & \text{if } 0 \leq \ell < \sigma, \\ \displaystyle\sum_{r=0}^{\sigma-n-w} \frac{(-1)^r \ell!(2\sigma - 2n - r)!}{(\ell - \sigma)!(\ell + \sigma + 1 - 2n)!r!} \phi_{\sigma+1-n-w-r}^{(\ell-\sigma+r)}(z) & \text{if } \sigma \leq \ell \leq 2n. \end{cases}$$

Then Λ_Φ is a vector-valued modular form belonging to $\mathbf{M}_2(\Gamma, \rho_{2n})$.

Proof Given a Jacobi-like form $\Phi(z, X) = \sum_{k=1}^{\infty} \phi_k(z) X^k \in \mathcal{J}_{2w}(\Gamma)$ and an integer $j \geq 1$, using (3.3), we see that the function $f_j \colon \mathcal{H} \to \mathbb{C}$ defined by

$$f_j = \sum_{r=0}^{j-1} (-1)^r \frac{(2j + 2w - r - 2)!}{r!} \phi_{j-r}^{(r)}$$

is an element of $M_{2j+2w}(\Gamma)$. In particular, for $j = \sigma + 1 - n - w$, we obtain an element $f_{\sigma+1-n-w} \in M_{2\sigma+2-2n}(\Gamma)$ given by

$$f_{\sigma+1-n-w} = \sum_{r=0}^{\sigma-n-w} (-1)^r \frac{(2\sigma - 2n - r)!}{r!} \phi_{\sigma+1-n-w-r}^{(r)}. \tag{3.4}$$

We now apply Theorem 2.4 for

$$\alpha = 2n, \quad \beta = 2\sigma + 2 - \alpha = 2\sigma + 2 - 2n.$$

Then, from the condition $n \leq \sigma \leq 2n$ we obtain

$$2 \leq \beta \leq \alpha + 2;$$

hence the integers α and β satisfy (2.2), and we can apply Theorem 2.4 by using

$$\delta = (\alpha + 2 - \beta)/2 = \alpha - \sigma = 2n - \sigma.$$

Since $1 - \beta = 2n - 2\sigma - 1 \leq -1$, it follows from (2.3) that

$$\eta_{k,2n,2\sigma+2-2n} = \frac{(k + \sigma)!}{k!(k + 1 + 2\sigma - 2n)!} \tag{3.5}$$

for $0 \leq k \leq 2n - \sigma$. Then from (3.4) and (3.5), we obtain

$$\begin{aligned} \eta_{\ell-\sigma,2n,2\sigma+2-2n} f_{\sigma+1-n-w}^{(\ell-\sigma)} &= \eta_{\ell-\sigma,2n,2\sigma+2-2n} \sum_{r=0}^{\sigma-n-w} (-1)^r \frac{(2\sigma - 2n - r)!}{r!} \phi_{\sigma+1-n-w-r}^{(\ell-\sigma+r)} \\ &= \frac{\ell!}{(\ell - \sigma)!(\ell + \sigma + 1 - 2n)!} \sum_{r=0}^{\sigma-n-w} (-1)^r \frac{(2\sigma - 2n - r)!}{r!} \phi_{\sigma+1-n-w-r}^{(\ell-\sigma+r)}. \end{aligned}$$

Hence the theorem follows from this and Theorem 2.4. ∎

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As an application of Theorem 3.3 we now discuss vector-valued modular forms associated to a scalar-valued modular form. We shall express a vector-valued modular form in terms of derivatives of the given modular form.

Theorem 3.4 *Let $f \in M_{2w}(\Gamma)$ for some positive integer w, and set*

$$\Xi_f(z) = \widehat{\rho}_{2n}(z)(\xi_0(z), \xi_1(z), \dots, \xi_{2n}(z))^T$$

for all $z \in \mathcal{H}$, where $\xi_\ell(z) = 0$ for $0 \le \ell < \sigma$ and

$$\xi_\ell = \sum_{r=0}^{\sigma-n-w} \sum_{s=2}^{\lfloor(\sigma+1-n-r)/w\rfloor} \frac{(-1)^r \ell!(2\sigma-2n-r)!}{(\ell-\sigma)!(\ell+\sigma+1-2n)!r!} \times \frac{(f^{s-1})^{(\ell+1-n-sw)}}{(\sigma+1-n-r-sw)!(\sigma-n-r+(s-2)w)!}$$

for $\sigma \le \ell \le 2n$; here $\lfloor k/w \rfloor$ denotes the largest integer less than or equal to the rational number k/w. Then Ξ_f is a vector-valued modular form belonging to $\mathbf{M}_2(\Gamma, \rho_{2n})$.

Proof Given $f \in M_{2w}(\Gamma)$, since $f^p \in M_{2pw}(\Gamma)$ for each $p \ge 1$, we have a sequence $(h_{j,w,f})_{j=0}^{\infty}$ of modular forms for Γ with $h_{j,w,f} \in M_{2j}(\Gamma)$ such that

$$h_{j,w,f} = \begin{cases} f^p & \text{if } j = pw \text{ for some } p \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, using Proposition 3.2(i), we obtain a Jacobi-like form $\Phi(z, X) = \sum_{k=1}^{\infty} \psi_k(z) X^k$ belonging to $\mathcal{J}_0(\Gamma)$, where

$$\psi_k = \sum_{p=1}^{\lfloor k/w \rfloor} \frac{1}{(k-pw)!(k+pw-1)!} (f^p)^{(k-pw)}$$

for all $k \ge 1$. For $\sigma \le \ell \le 2n$, we set

$$\begin{aligned} \xi_\ell &= \sum_{r=0}^{\sigma-n-w} \frac{(-1)^r \ell!(2\sigma-2n-r)!}{(\ell-\sigma)!(\ell+\sigma+1-2n)!r!} \psi_{\sigma+1-n-w-r}^{(\ell-\sigma+r)} \\ &= \sum_{r=0}^{\sigma-n-w} \sum_{p=1}^{\lfloor(\sigma+1-n-w-r)/w\rfloor} \frac{(-1)^r \ell!(2\sigma-2n-r)!}{(\ell-\sigma)!(\ell+\sigma+1-2n)!r!} \\ &\quad \times \frac{(f^p)^{(\ell+1-n-w-pw)}}{(\sigma+1-n-w-r-pw)!(\sigma-n-w-r+pw)!} \\ &= \sum_{r=0}^{\sigma-n-w} \sum_{s=2}^{\lfloor(\sigma+1-n-r)/w\rfloor} \frac{(-1)^r \ell!(2\sigma-2n-r)!}{(\ell-\sigma)!(\ell+\sigma+1-2n)!r!} \\ &\quad \times \frac{(f^{s-1})^{(\ell+1-n-sw)}}{(\sigma+1-n-r-sw)!(\sigma-n-r+(s-2)w)!}. \end{aligned}$$

Hence the theorem follows from this and Theorem 3.3. ■

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4 Theta Functions

In this section we construct vector-valued modular forms of weight 2 by applying Theorem 3.3 to Jacobi-like forms associated to certain theta functions of the type studied by Dong and Mason [2].

We fix a positive integer w, an element v of $\mathbb{C}^{2w}$, and a symmetric positive definite integral $2w \times 2w$ matrix A whose diagonal entries are even. For each nonnegative integer k, we define the theta function $\theta_k\colon \mathcal{H} \to \mathbb{C}$ by

$$\theta_k(z) = \sum_{\ell \in \mathbb{Z}^{2w}} (v^T A \ell)^k e^{\pi i (\ell^T A \ell) z} \tag{4.1}$$

for all $z \in \mathcal{H}$. Given integers σ and ℓ with $n \le \sigma \le \ell \le 2n$, we set

$$\begin{aligned} \lambda_\ell(z) = \sum_{r=0}^{\sigma-n-w} \sum_{j=0}^{\sigma+1-n-w-r} \sum_{s=0}^{\ell-\sigma+r} \Big(\frac{2\pi i}{v^T A v} \Big)^{\sigma+1-n-w-r} \\ \times \frac{(-1)^r \ell! (2\sigma-2n-r)! (\ell-\sigma+r)!}{(\ell-\sigma)!(\ell+\sigma+1-2n)! r! s! (\ell-\sigma+r-s)!} \\ \times \frac{\theta_{2j}^{(s)}(z) \theta_{2\sigma+2-2n-2w-2r-2j}^{(\ell-\sigma+r-s)}(z)}{(2j)!(2\sigma+2-2n-2w-2r-2j)!} \end{aligned} \tag{4.2}$$

for all $z \in \mathcal{H}$.

Let N be the smallest positive integer such that NA^{-1} is an integral matrix with even diagonal entries, and let $\Gamma_0(N) \subset SL(2,\mathbb{Z})$ be the associated congruence subgroup given by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod N \right\}.$$

Theorem 4.1 *Let $\Psi\colon \mathcal{H} \to \mathbb{C}^{2n+1}$ be the vector-valued function on $\mathcal{H}$ defined by*

$$\Psi(z) = \widehat{\rho}(z)(0, \dots, 0, \lambda_\sigma(z), \dots, \lambda_{2n}(z))^T,$$

where $\lambda_\ell(z)$ for $\sigma \le \ell \le 2n$ is as in (4.2). Then Ψ is a vector-valued modular form belonging to $\mathbf{M}_2(\Gamma_0(N), \rho_{2n})$.

Proof Given a nonnegative integer k, we set

$$\phi_k(z) = \sum_{j=0}^{k} \Big(\frac{2\pi i}{v^T A v} \Big)^k \frac{\theta_{2j}(z) \theta_{2k-2j}(z)}{(2j)!(2k-2j)!} \tag{4.3}$$

for all $z \in \mathcal{H}$. Then by [6, Lemma 4.1] the formal power series

$$\vartheta(z, X) = \sum_{k=1}^{\infty} \phi_k(z) X^k$$

is an element of $\mathcal{J}_{2w}(\Gamma_0(N))$. From (4.3) we obtain

$$\phi_{\sigma+1-n-w-r}^{(\ell-\sigma+r)} = \sum_{j=0}^{\sigma+1-n-w-r} \Big(\frac{2\pi i}{v^T A v}\Big)^{\sigma+1-n-w-r} \frac{(\theta_{2j}\theta_{2\sigma+2-2n-2w-2r-2j})^{(\ell-\sigma+r)}}{(2j)!(2\sigma+2-2n-2w-2r-2j)!}$$
$$= \sum_{j=0}^{\sigma+1-n-w-r} \sum_{s=0}^{\ell-\sigma+r} \Big(\frac{2\pi i}{v^T A v}\Big)^{\sigma+1-n-w-r} \binom{\ell-\sigma+r}{s} \times \frac{\theta_{2j}^{(s)}\theta_{2\sigma+2-2n-2w-2r-2j}^{(\ell-\sigma+r-s)}}{(2j)!(2\sigma+2-2n-2w-2r-2j)!}.$$

Hence the theorem follows from this and Theorem 3.3. ■

5 Pseudodifferential Operators

Jacobi-like forms for a discrete subgroup of $SL(2,\mathbb{R})$ are known to be in one-to-one correspondence with certain pseudodifferential operators that are invariant under the same discrete group (see [1, 8]). In this section we combine this fact with the results in Section 3 to construct another type of vector-valued modular form.

Let R be the ring of holomorphic functions on $\mathcal{H}$ as in Section 3, and denote by ∂ the differential operator d/dz acting on R. Then a pseudodifferential operator over R is a formal series of the form

$$\psi(z) = \sum_{k=-\infty}^{k_0} \xi_k(z)\partial^k$$

with $z \in \mathcal{H}$ for some $k_0 \in \mathbb{Z}$, where $\xi_k \in R$ for all $k \leq k_0$. Let ΨDO denote the set of all pseudodifferential operators over R. Then ΨDO is an algebra over R whose multiplication is given by

$$\Big(\sum_{k=-\infty}^{k_0} \xi_k(z)\partial^k\Big)\Big(\sum_{m=-\infty}^{m_0} \eta_m(z)\partial^m\Big) = \sum_{k=-\infty}^{k_0}\sum_{m=-\infty}^{m_0}\sum_{r=0}^{\infty}\binom{k}{r}\xi_k(z)\eta^{(r)}(z)\partial^{k+m-r},$$

where $\eta^{(r)}$ denotes the derivative of η of order r, $\binom{k}{0} = 1$, and

$$\binom{k}{r} = \frac{k(k-1)\cdots(k-r+1)}{r!}$$

for $k \in \mathbb{Z}$ and $r \geq 1$.

If $\Xi = \sum_{k=-\infty}^{k_0} \xi_k(z)\partial^k$ is an element of ΨDO, we define $\Xi \cdot \gamma$ to be the element of ΨDO that is obtained from Ξ by the coordinate change $z \mapsto \gamma z$. Thus we have

$$\Xi\cdot\gamma = \sum_{k=-\infty}^{k_0} \xi_k(\gamma z)\partial_{\gamma z}^k = \sum_{k=-\infty}^{k_0} \xi_k(\gamma z)\Big(\frac{d}{d(\gamma z)}\Big)^k$$
$$= \sum_{k=-\infty}^{k_0} \xi_k(\gamma z)\Big(\Big(\frac{d(\gamma z)}{dz}\Big)^{-1}\partial\Big)^k$$

for all $\Xi = \sum_k \xi_k(z)\partial^k \in \Psi\mathrm{DO}$ and $\gamma \in SL(2,\mathbb{R})$. In fact, it can be shown that

$$(5.1) \qquad \Xi \cdot \gamma = \sum_{k=-\infty}^{k_0} \sum_{m=0}^{\infty} m! \binom{k}{m} \binom{k-1}{m} \gamma_{2,1}^m J(\gamma, z)^{2k-m} \xi_k(\gamma z) \partial^{k-m}$$

for $\gamma \in SL(2,\mathbb{R})$ and that (5.1) determines a right action of $SL(2,\mathbb{R})$ on $\Psi\mathrm{DO}$. Let $\Psi\mathrm{DO}^\Gamma$ denote the set of elements of $\Psi\mathrm{DO}$ that are invariant under the Γ-action given by (5.1), that is,

$$\Psi\mathrm{DO}^\Gamma = \left\{ \Xi \in \Psi\mathrm{DO} \;\middle|\; \Xi \cdot \gamma = \Xi \text{ for all } \gamma \in \Gamma \right\}.$$

We consider a pseudodifferential operator of the form

$$(5.2) \qquad \Xi = \sum_{k=-\infty}^{-1} \xi_k(z) \partial^k \in \Psi\mathrm{DO},$$

and set

$$(5.3) \qquad \xi_j^*(z) = \frac{(-1)^j \xi_{-j}(z)}{j!(j-1)!}$$

for each $j \geq 1$.

Lemma 5.1 *Let $\Xi \in \Psi\mathrm{DO}$ be as in (5.2), and let $\Phi(z, X)$ be the formal power series given by*

$$\Phi(z, X) = \sum_{j=1}^{\infty} \xi_j^*(z) X^j,$$

where $\xi_j^*(z)$ is as in (5.3). Then $\Xi \in \Psi\mathrm{DO}^\Gamma$ if and only if $\Phi(z, X) \in \mathcal{J}_0(\Gamma)$.

Proof This follows from [1, Proposition 2]. ■

Given integers σ and ℓ with $n \leq \sigma \leq \ell \leq 2n$, we set

$$(5.4) \quad \widetilde{\lambda}_\ell(z) = \sum_{r=0}^{\sigma-n-w} \frac{(-1)^{\sigma+1-n-w} \ell! (2\sigma - 2n - r)!}{(\ell - \sigma)!(\ell + \sigma + 1 - 2n)!} \times \frac{\xi_{n+w+r-\sigma-1}^{(\ell-\sigma+r)}(z)}{r!(\sigma + 1 - n - w - r)!(\sigma - n - w - r)!}$$

for all $z \in \mathcal{H}$.

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Theorem 5.2 *Let $\Xi \in \Psi\mathrm{DO}$ be the pseudodifferential operator in (5.2), and let $\Psi\colon \mathcal{H} \to \mathbb{C}^{2n+1}$ be the vector-valued function on $\mathcal{H}$ defined by*

$$\Psi(z) = \widehat{\rho}(z)(0, \ldots, 0, \widetilde{\lambda}_\sigma(z), \ldots, \widetilde{\lambda}_{2n}(z))^T,$$

where $\lambda_\ell(z)$ for $\sigma \le \ell \le 2n$ is as in (5.4). If Ξ is Γ-invariant, then Ψ is a vector-valued modular form belonging to $\mathbf{M}_2(\Gamma, \rho_{2n})$.

Proof Since $\Xi \in \Psi\mathrm{DO}$ is Γ-invariant, by Lemma 5.1 the formal power series

$$\Phi(x, X) = \sum_{k=1}^{\infty} \xi_k^*(z) X^k = \sum_{k=1}^{\infty} \frac{(-1)^k \xi_{-k}(z)}{k!(k-1)!} X^k$$

is a Jacobi-like form belonging to $\mathcal{J}_0(\Gamma)$. Thus the theorem follows from Theorem 3.3 and the relation

$$(\xi^*_{\sigma+1-n-w-r})^{(\ell-\sigma+r)} = \frac{(-1)^{\sigma+1-n-w-r} \xi^{(\ell-\sigma+r)}_{n+w+r-\sigma-1}}{(\sigma+1-n-w-r)!(\sigma-n-w-r)!}$$

for $0 \le r \le \sigma - n - w$. ■

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