

CONVECTION AND ITS STABILITY IN THE EQUATORIAL REGIONS OF THE CONVECTION ZONE

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The problem of the existence, evolution, and stability of spatial structures in convection is of considerable importance to astrophysics as well as to geophysical hydrodynamics. The Boussinesq approximation will be used because the considered motions in stars are sufficiently slow. The system of hydrodynamic equations describing convection in a rotating inhomogeneous medium has the form:

$$\begin{aligned} D_t \mathbf{U} + 2\boldsymbol{\Omega} \times \mathbf{U} &= -\rho_m^{-1} \nabla P + (\beta_c C - \beta_T T) \mathbf{g} + \nu_m \Delta \mathbf{U} \\ D_t T + \mathbf{U} \cdot (\nabla T_b - \nabla T_{ad}) &= \chi_m \Delta T \\ D_t C + \mathbf{U} \cdot \nabla C_b &= \mathcal{D}_m \Delta C \\ \operatorname{div} \mathbf{U} &= 0 \end{aligned} \tag{1}$$

Here D_t is the total time derivative, \mathbf{U} the velocity, P , T , and C the deviations of the pressure, temperature, and helium abundance (by mass) from the basic equilibrium values, ρ_m , ν_m , χ_m , and \mathcal{D}_m the values averaged over the considered layer of the density, viscosity, thermal and helium diffusivities, β_T and β_c the averaged coefficients of the thermal and helium expansions, \mathbf{g} and $\boldsymbol{\Omega}$ the gravitational acceleration and angular velocity, ∇T_b and ∇C_b the values of the basic equilibrium temperature and helium gradients, and ∇T_{ad} the adiabatic temperature gradient.

An analysis of finite-amplitude convection in a rotating non-uniform star has been performed by Dolginov and Klyachkin (1987, 1988, 1989) using bifurcation theory methods applied to Eq. (1). It was shown that for a wide range of physical parameters such as rotation rate and degree of chemical inhomogeneity (∇C_b) convection arises when the initial temperature gradient exceeds a critical value (supercritical bifurcation). Furthermore the existence of convective motions for values of the basic temperature gradient less than the critical value (subcritical bifurcation) was found to be possible in the case of slow rotation and a sufficiently large downwards directed helium gradient. In both cases the convection occurs as waves in the form of banana cells elongated along the meridian with an equatorial amplitude maximum. These waves are propagating only azimuthally, both along and opposite to the direction of rotation. The presence of a chemical inhomogeneity in the considered

layer is very important for the physics of the convection processes. It leads not only to a change of the critical temperature gradient, but also to the appearance of slow gyroscopic waves (of Rossby type), with two double-diffusion modes determined by the competition between the thermal and helium diffusions.

It has however not been known whether the spatial structures are stable and how their evolution may be influenced by nonlinear interaction and energy transfer between the different generated modes. For a more detailed analysis of convection in the weakly nonlinear regime we will therefore use the method of amplitude equation, which is based on multiple scalings in space and time. As the propagation of waves is one-dimensional, only the azimuthal coordinate φ is scaled. Let us choose the scaling

$$\tau_1 = \epsilon t, \quad \tau_2 = \epsilon^2 t, \quad \varphi_1 = \epsilon \varphi, \quad \varphi_2 = \epsilon^2 \varphi,$$

where ϵ is a measure of supercriticality defined by $R_T = R_0 + \epsilon^2 R_2$. Here R_T is the thermal Rayleigh number, which is proportional to the basic temperature gradient, while R_0 is the critical value of R_T , as obtained with a linear approximation. We seek a solution of Eq. (1) in the form of an expansion in ϵ , which is a measure of the convection amplitude.

$$(V, W, T, C) = \sum_{n=1} \epsilon^n (V_n, W_n, T_n, C_n),$$

where V and W are the poloidal and toroidal velocity potentials ($\mathbf{U} = [\text{curl}^2 \mathbf{r}V + \text{curl} \mathbf{r}W]r_0^{-1}$, with $r_0 = R/h$ being the inner radius of the considered spherical shell in units of its depth). According to DiPrima et al. (1971), the multi-scaling method can be used for a Hopf bifurcation in the bounded spatial domain, if the wave number spectrum is sufficiently dense: $\Delta k \lesssim \epsilon^2 < 1$. For a spherical shell, $\Delta k = r_0^{-1}$, and therefore the amplitude equation can correctly describe the convection process in a thin ($r_0 > 1$) spherical layer. Physically it means that in spite of small supercriticality a great number of different spatial harmonics take part in the nonlinear interaction. As a result of the condition that Eq. system (1) has solutions in the $O(\epsilon^3)$ approximation (for details, see Klyachkin, 1989), the amplitude equation follows:

$$\partial A / \partial \tau_2 = \alpha A - \gamma |A|^2 A + \beta \partial^2 A / \partial x_1^2 \quad (2)$$

It is a complex ($\beta = \beta_r + i\beta_I$, $\gamma = \gamma_r + i\gamma_I$), one-dimensional Ginzburg-Landau equation. Here $A = A(\tau_2, x_1, x_2)$ is a slowly varying complex amplitude, $x_1 = \epsilon(\varphi - c_g t)$, $x_2 = \epsilon^2(\varphi - c_g t)$, and c_g is the group speed of the most unstable mode. The velocity, temperature, and helium concentration depend both on the slow variables (reflecting the mutual interdependence of the modes) and on the rapid variables (reflecting the spatial and temporal dependence of the most unstable mode). For example

$$T_1 = \text{Re} \hat{T}_1 A(\tau_2, x_1, x_2) P_\ell^{\ell}(\cos \theta) \exp [i(\ell \varphi + \omega t)] \sin \pi(r - r_0),$$

where \hat{T}_1 is a constant, chosen from the normalization condition. The coefficients α , β , and γ are uniquely connected with the controlling parameters describing the rotation, degree of chemical inhomogeneity of the medium, and its physical characteristics: viscosity, thermal and helium diffusivities. These expressions for α , β , and γ have been given in Klyachkin (1989). The obvious solutions of Eq. (2) are monochromatic, finite-amplitude rotating waves. It is known that chaotic solutions of Eq. (2) are possible. The condition for such solutions

not to appear is also the condition for asymptotically reaching the monochromatic regime (Doering et al., 1988), as well as the condition for modulational stability of the spatially homogeneous solutions ($A = A(\tau_2)$). This condition has the form $\beta_r \gamma_r + \beta_I \gamma_I > 0$. It is satisfied for given values of α , β , and γ when rotation is not very rapid, and in the parameter domain corresponding to supercritical bifurcation in convection. For such parameter values convective motions appearing in the form of slow gyroscopic waves with retrograde propagation represent stable solutions. It is thus not surprising that gyroscopic waves have been found in numerical treatments of convection in a rotating, chemically homogeneous spherical shell in the mean-field approximation (Durney, 1970).

In the domain where subcritical bifurcation takes place, the coefficient $\gamma_r < 0$, and according to Doering et al. (1988) an explosive growth of the solution in a finite time is possible in this case. Phenomenologically this growth can be limited by adding a term with a higher degree of nonlinearity (Landau and Lifshits, 1986):

$$\partial A / \partial \tau_2 = \alpha A - \gamma |A|^2 A - \mu |A|^4 A + \beta \partial^2 A / \partial x_1^2. \quad (3)$$

The evident solutions of Eq. (3), finite-amplitude waves, can possess both small and large amplitudes when the wave number is fixed. The analysis shows that all small-amplitude solutions are modulationally unstable. The spatially homogeneous ($A = A(\tau_2)$) large-amplitude solutions are modulationally stable if

$$(\beta_r \mu_r + \beta_I \mu_I)(\gamma_r^2 + 4\mu_r \alpha)^{1/2} + \beta_I(\gamma_I \mu_r - \gamma_r \mu_I) > 0.$$

For spatially inhomogeneous solutions $A = a \exp[i(kx_1 + \omega\tau_2)]$, the neutral curves of modulational stability relative to perturbations with wave number k have the form (for details, see Klyachkin, 1989)

$$4k^2 \beta_r^2 = \left[|\beta|^2 \tilde{k}^2 + 2a^2(\beta_r \gamma_r + \beta_I \gamma_I) + 4a^4(\beta_r \mu_r + \beta_I \mu_I) \right] \left\{ 1 + [\beta_I \tilde{k}^2 + \gamma_I a^2 + 2\mu_I a^4]^2 [\beta_r \tilde{k}^2 + \gamma_r a^2 + 2\mu_r a^4]^{-2} \right\}^{-1}. \quad (4)$$

Here

$$a^2 = \left[-\gamma_r \pm \sqrt{\gamma_r^2 + 4\mu_r(\alpha - \beta_r k^2)} \right] (2\mu_r)^{-1}.$$

Criterion (4) is identical for $\mu = 0$ with the analogous one obtained by Doering et al. (1988).

Let us finally point out that Eqs. (2) and (3) do not describe the processes taking place near the multiple bifurcation line of codimension 2 or the multiple bifurcation point of codimension 3, which occur when a definite relation between rotation, external heating, and degree of chemical inhomogeneity exists.

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